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Kernel and Core of replicated Market Games

Ulrich Klein

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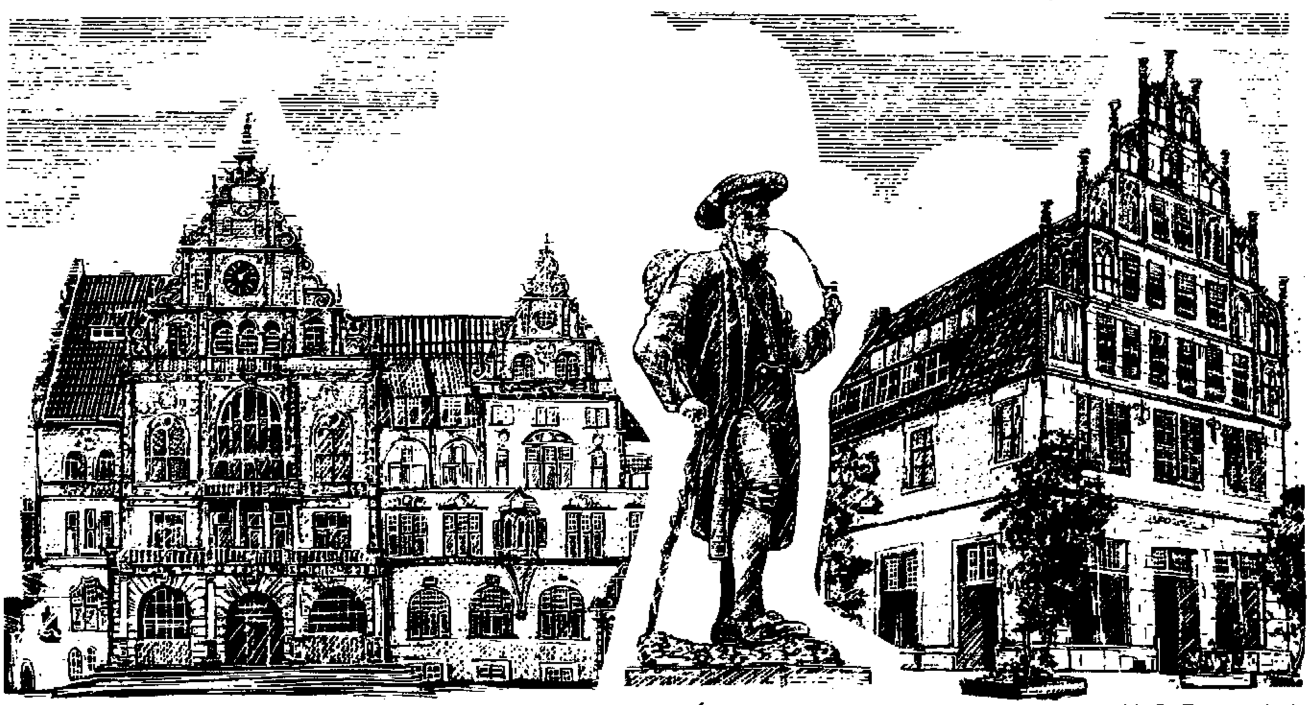


H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

1. Preface

Considering some solution concepts for market games one can study the properties of these concepts under replication. So, for example, the core is known to be "convergent" to the equilibrium payoff; or in other words, a payoff which lies in the core of a market game for every replication, represents an equilibrium payoff (cf. Debreu-Scarf [1963]).

Now the solution concepts nucleolus and lexicographic kernel are included in the core (cf. Justman [1977], Schmeidler [1969], and Yorom [1981]) and thus for market games they "converge" in the above sense to the equilibrium payoffs. On the other hand both concepts are contained in the kernel (cp. Davis, Maschler [1965]). So the following question arises: Does the kernel behave similarly under replication of markets?

Guided by two examples of L.P.-Games (Linear Production Games) we shall see that in general the kernel does not converge to the equilibrium payoff. On the contrary we can observe a kind of pulsation: Starting at a given set of players the kernel runs cyclically through a finite system of sets of payoffs. In a periodical change it coincides with the core and contains points outside the core respectively (chapter 6).

Furthermore, we find that for every market game belonging to a replicated market the kernel includes the core (chapter 4).

2. Some definitions

In this chapter we want to give a short presentation of the main objects we shall use in this paper. Most of the definitions and notations are taken from ROSENMÜLLER [1971] or [1981].

Let $\Omega = \{1,\dots,n\}$ be the set of players, $\mathcal{P}(\Omega)$ the set of all coalitions and $v : \mathcal{P}(\Omega) \to \mathbb{R}^+$ a real-valued set-function defined on $\mathcal{P}(\Omega)$ with $v(\emptyset) = 0$. Then the tripel $\Gamma = (\Omega, \mathcal{P}(\Omega), v)$ is called a (side-payment) game with the characteristic function v.

Every vector $x \in \mathbb{R}^n$ can also be conceived as an additive set-function $x : \mathcal{P}(\Omega) \to \mathbb{R}$, if we define $x(S) := \sum_{i \in S} x_i$ for all $S \in \mathcal{P}(\Omega)$.

Now the following solution concepts are fundamental in game theory:

For every game $\Gamma = (\Omega, \mathcal{P}(\Omega), v)$ we have

$$\mathcal{J}^*(\Gamma) = \{x \in \mathbb{R}^n \mid x(\Omega) = v(\Omega)\}$$

the set of pre-imputations of Γ,

$$\mathcal{J}(\Gamma) = \{x \in \mathcal{J}^*(\Gamma) \mid x_i \geq v(\{i\}) \quad (i \in \Omega)\}$$

the set of imputations of Γ,

$$\mathcal{C}(\Gamma) = \{x \in \mathcal{J}^*(\Gamma) \mid x(S) \geq v(S) \quad (S \in \mathcal{P}(\Omega))\}.$$

the core of Γ.

As we want to study a special class of games, namely the market games, we also have to introduce some objects from the theory of markets.

A quadrupel $\mathfrak{m} = (\Omega, \mathbb{R}^{m+} \times \mathbb{R}, (\tilde{u}_i)_{i\in\Omega}, ((a^i,0))_{i\in\Omega})$ is called a TU-market (market with transferable utility) where $\mathbb{R}^{m+} \times \mathbb{R}$ is the commodity-space with one monetary component, $(\tilde{u}_i)_{i\in\Omega}$ are the real valued utility functions on $\mathbb{R}^{m+} \times \mathbb{R}$, and $(a^i,0)_{i\in\Omega}$ are the initial allocations of all players with $a^i \in \mathbb{R}^{m+}$ for all $i \in \Omega$.

From each market we can construct a corresponding market game $\Gamma^{\mathfrak{m}} = (\Omega, \mathcal{P}(\Omega), v^{\mathfrak{m}})$: In the usual way we define $v^{\mathfrak{m}}(S)$ to be the maximal total utility achievable for coalition S.

$$v^{\mathfrak{m}}(S) := \max \{ \sum_{i\in S} \tilde{u}_i(x^i,\xi^i) \mid (x^i,\xi^i) \in \mathbb{R}^{m+} \times \mathbb{R} \ (i \in S),$$

$$\sum_{i\in S} (x^i,\xi^i) = \sum_{i\in S} (a^i,0)\} \quad (S \in \mathcal{P}(\Omega)).$$

In order to enlarge the market by taking multiplicities of each player we define (cf. Debreu-Scarf [1963], Rosenmüller [1971], [1982]:

For every integer $k \in \mathbb{N}$ the quadrupel

$$\mathfrak{m}^k = (\Omega^k, \mathbb{R}^{m+} \times \mathbb{R}, (\tilde{u}_i)_{i\in\Omega^k}, ((a^i,0))_{i\in\Omega^k})$$

is called the <u>k-fold replication of $\mathfrak{m}$</u>, with

$$\Omega^k := \{1,\ldots,nk\} = \bigcup_{l=1}^{k} \Omega_l^k ,$$

$$\Omega_l^k := \{i \in \Omega^k \mid i \equiv l \ (\mathrm{mod}\ n)\} \quad (l \in \Omega) ,$$

$$\tilde{u}_i := \tilde{u}_l \qquad (l \in \Omega,\ i \in \Omega_l^k) ,$$

$$a^i := a^l \qquad (l \in \Omega,\ i \in \Omega_l^k) .$$

Every vector $x \in \mathbb{R}^n$ can also be replicated in the analogous way defining $x^k \in \mathbb{R}^{kn}$ by

$$x_i^k := x_l \quad (i \in \{1,\ldots,nk\} ,\ i \equiv l \ (\mathrm{mod}\ n)).$$

If $\Gamma^{\mathfrak{m}} = (\Omega, \mathcal{P}(\Omega), v^{\mathfrak{m}})$ is a market game, we call $\Gamma^{\mathfrak{m}k} := (\Omega^k, \mathcal{P}(\Omega^k), v^{\mathfrak{m}k})$ the <u>k-fold replication of $\Gamma^{\mathfrak{m}}$</u>.

3. The kernel

We now want to consider the solution concept kernel which is due to Davis and Maschler [1965].

Let $\Gamma^{m^k} = (\Omega^k, \mathcal{P}(\Omega^k), v^{m^k})$ be the k-fold replication of a market game Γ^{m} (i.e. $k \in \mathbb{N}$). Then we define excess, maximum surplus and kernel as follows:

•) $e(x,S,v^{m^k}) := v^{m^k}(S) - x(S) \qquad (x \in \mathbb{R}^{kn},\ S \in \mathcal{P}(\Omega^k))$

the <u>excess of S in x (w.r.t. v^{m^k})</u> ;

•) $s_{ij}^k(x) := s_{ij}^k(x, v^{m^k}) := \max\limits_{S \in \underline{\underline{T}}_{ij}^k} e(x,S,v^{m^k}) \quad (x \in \mathbb{R}^{kn},\ i,j \in \Omega^k,\ i \neq j)$

the <u>maximum surplus of x (w.r.t. v^{m^k})</u>

where $\underline{\underline{T}}_{ij}^k$ is defined as $\{S \in \Omega^k \mid i \in S,\ j \notin S\}$;

•) $\mathcal{K}(\Gamma^{m^k}) := \{x \in \mathcal{J}(\Gamma^{m^k}) \mid (s_{ij}^k(x) - s_{ji}^k(x))\,(x_j - v(\{j\})) \leq 0 \quad (i,j \in \Omega^k,\ i \neq j)$

the <u>kernel (of the grand coalition) of Γ^{m^k}</u> .

It is already known for the class of 0-monotonic games, that the kernel and pre-kernel coincide (cf. Maschler, Peleg, Shapley [1979], or Rosenmüller [1981] III. 9). The type of market games we study here is superadditive and therefore 0-monotonic. So for computing the kernel of these market games it is sufficient to work with the <u>pre-kernel</u> $\mathcal{K}^*(\Gamma^{m^k})$ <u>(of the grand coalition) of Γ^{m^k}</u> :

•) $\mathcal{K}^*(\Gamma^{m^k}) := \{x \in \mathcal{J}^*(\Gamma^{m^k}) \mid s_{ij}^k(x) = s_{ji}^k(x) \quad (i,j \in \Omega^k,\ i \neq j)\}$

where $\mathcal{J}^*(\Gamma^{m^k})$ is the set of pre-imputations.

In order to get a further simplification in computing the kernel we now use a result stated by Maschler and Peleg [1966]. In their paper it is shown that every payoff which lies in the kernel gives the same amount to players of the same type. So in particular for all k-fold replications of market games this means an "equal-treatment"-property of the kernel:

For all $x \in \mathcal{K}(\Gamma^{m^k})$ we have

$$x_i = x_j \qquad (l \in \Omega,\ i,\ j \in \Omega_l^k).$$

Shorter we write for this

$$\mathcal{K}(\Gamma^{m^k}) \subseteq \mathcal{J}^{et}(\Gamma^{m^k}) \tag{3.1}$$

where $\mathcal{J}^{et}(\Gamma^{m^k}) := \{x \in \mathcal{J}(\Gamma^{m^k}) \mid x_i = x_j$ if there exists $l \in \Omega$ such that $i \in \Omega_l^k$ and $j \in \Omega_l^k\}$

is called the set of <u>equal-treatment-imputations of Γ^{m^k}</u>.

For the maximum surplus we have also an invariancy under permutation of players of the same type. Clearly, if we have an imputation with constant payoff for every class of players of the same type, the excess of an excess-maximizing coalition remains unchanged by replacing one player by another of the same type. So it is easily seen that for all $x \in \mathbb{R}^{kn}$ with $x_i = x_j$ for all $l \in \Omega$, $i,j \in \Omega_l^k$ the following equations hold true:

$$s_{ih}^k(x) = s_{lg}^k(x) \qquad (l, g \in \Omega,\ i \in \Omega_l^k,\ h \in \Omega_g^k)\ .$$

This means the maximum surplus of two players in Ω^k is equal to the maximum surplus of their "typical" representatives in Ω.

Now using these properties of the kernel and maximum surplus and looking at the definition of the pre-kernel, it turns out that we get the following form of the kernel:

$$\mathcal{K}(\Gamma^{m^k}) = \{x \in \mathcal{J}(\Gamma^{m^k}) \mid s_{ij}(x) = s_{ji}(x) \quad (i,j \in \Omega,\ i \neq j),$$
$$x_i = x_j \quad (l \in \Omega^k,\ i,j \in \Omega_l^k)\ .$$

The advantage of this representation is that for every k-fold replication of the game we only have to compute a constant number of $n \cdot (n-1)$ functions s_{ij}^{k} instead of $kn \cdot (kn-1)$.

4. The inclusion $\mathcal{C} \subset \mathcal{K}$ for replications of market games

In this paper we want to investigate the set inclusion between kernel and core for market games.

Let $\mathfrak{m} = (\Omega, \mathbb{R}^{m+} \times \mathbb{R}, (\tilde{u}_i)_{i\in\Omega}, ((a^i,0))_{i\in\Omega})$ be a market with transferable utility and $\Gamma^{\mathfrak{m}} = (\Omega, \mathcal{P}(\Omega), v^{\mathfrak{m}})$ the induced market game as defined before.

The theorem of Debreu-Scarf states that the core of the market "converges" to the Walras-equilibrium (cf. Debreu-Scarf [1963], Hildenbrand [1974]). With the help of market games this result can be formulated in the theory of games. It means that the set of all payoffs in the core converges under replication to the set of equilibrium payoffs.

If we were able to show that the core includes the kernel of a market game under replication, this theorem of Debreu-Scarf would also be true for the kernel. But in this paragraph it turns out that the core lies in the kernel for each k-fold replication of market games, if $k \geq 2$.

Now the following lemma states the equal-treatment property of the core of replicated market games, that is players of equal types get equal payoffs.

4.1. Lemma

Let $\Gamma^{\mathfrak{m}} = (\Omega, \mathcal{P}(\Omega), v^{\mathfrak{m}})$ be the market game of a TU-market with continuous and concave utility functions. Further let $k \in \mathbb{N}$ and $k \geq 2$.

If $\qquad x \in \mathcal{C}(\Gamma^{\mathfrak{m}^k})$,

then $\qquad x_i = x_1 \qquad (1 \in \Omega,\ i \in \Omega_1^k)$.

Proof: The given properties of the utility functions imply (cf. Rosenmüller [1971], II, §1, Satz 1.2):

$$v^{\mathfrak{m}^k}(\Omega^k) = k \cdot v^{\mathfrak{m}}(\Omega) \quad . \tag{4.1}$$

Now let $(S^{(r)})_{r=1,\ldots,k} \subseteq \mathcal{P}(\Omega^k)$ be a partition of Ω^k with $|S^{(r)} \cap \Omega_1^k| = 1$ for all $r = 1,\ldots,k$. This means all coalitions of this partition of Ω^k have the same typical composition of players as Ω.

Then for every $x \in \mathcal{C}(\Gamma^{\mathfrak{m}^k})$ we have

$$x(S^{(r)}) \geq v^{\mathfrak{m}^k}(S^{(r)}) = v^{\mathfrak{m}^k}(\Omega) \qquad (r=1,\ldots,k)$$

and

$$x(\Omega^k) = \sum_{r=1}^{k} x(S^{(r)}) \geq k \cdot v^{\mathfrak{m}^k}(\Omega) = v^{\mathfrak{m}^k}(\Omega^k) = x(\Omega^k).$$

Therefore it is

$$x(S^{(r)}) = v^{\mathfrak{m}^k}(\Omega) \qquad (r=1,\ldots,k).$$

So for every coalition $S \in \mathcal{P}(\Omega^k)$ that has the same composition of types as Ω, the equality

$$x(S) = v^{\mathfrak{m}^k}(\Omega) \tag{4.2}$$

holds true, because it is always possible to find a partition $(S^{(r)})_{r=1,\ldots,k}$ of Ω^k with $S = S^{(1)}$.

Then, on account of (4.2), we have in particular

$$\begin{aligned} x(\Omega) &= v^{\mathfrak{m}^k}(\Omega) \\ &= x(\Omega - 1 + i) \\ &= x(\Omega) - x_1 + x_i \qquad (1 \in \Omega,\ i \in \Omega_1^k) \end{aligned}$$

which implies

$$x_1 = x_i \qquad (1 \in \Omega,\ i \in \Omega_1^k)\ . \ \square$$

Now we want to extend our considerations of replicated market games to the solution concept kernel.

4.2 Theorem

Let $\mathfrak{m}$ be a t.u.-market with continuous and concave utility functions. Then the following inclusion holds true for all $k \in \mathbb{N}$ with $k \geq 2$:

$$\mathcal{C}(\Gamma^{\mathfrak{m}^k}) \subseteq \mathcal{K}(\Gamma^{\mathfrak{m}^k}) .$$

Proof: Let $x \in \mathcal{C}(\Gamma^{\mathfrak{m}^k})$, then we have

$$v^{\mathfrak{m}^k}(S) - x(S) = e\,(x,S,v^{\mathfrak{m}^k}) \leq 0 \qquad (S \in \mathcal{P}(\Omega^k)).$$

Of course we also have

$$s_{ij}(x,v^{\mathfrak{m}^k}) = \max_{S \in \underline{\underline{T}}^k_{ij}} e(x,S,v^{\mathfrak{m}^k}) \leq 0 \quad (i,j \in \Omega^k,\ i \neq j). \qquad (4.3)$$

Now, for every pair $(i,j) \in \Omega^k \times \Omega^k$ with $i \neq j$ we are able to construct a set $S_0^{ij} \in \underline{\underline{T}}^k_{ij}$ which contains player i but not player j and which is composed with the same types of players as Ω:

Without l.o.g. assume $i \in \Omega^k_1$ and $j \in \Omega^k_{l_0}$ for a certain $l_0 \in \Omega$. We define

$$S_0^{ij} := \begin{cases} (\Omega \setminus \{1,l_0\}) \cup \{i,l_0+n\} & \text{if } j \in \Omega \text{ and } l_0 \neq 1 \\ (\Omega \setminus \{1\}) \cup \{i\} & \text{else .} \end{cases}$$

Then we have $S_0^{ij} \in \mathcal{P}(\Omega^k)$, for the player l_0+n exists in Ω^k because of $k \geq 2$.

As $i \in S_0^{ij}$ and $j \notin S_0^{ij}$ by construction, we have $S_0^{ij} \in \underline{\underline{T}}^k_{ij}$. Also on account of the above definition S_0^{ij} has the same composition of types, as Ω, that is

$$|\ \Omega^k_l \cap S_0^{ij}\ | = 1 = |\ \Omega^k_l \cap \Omega\ | \qquad (l \in \Omega) .$$

Hence, using the given properties of the utility function, we get the following equalities:

i) $\quad x(S_o^{ij}) = \sum_{l \in \Omega} x_l \cdot | \Omega_l^k \cap S_o^{ij} | = \sum_{l \in \Omega} x_l = x(\Omega)$

because of lemma 4.1 ("equal treatment");

ii) $\quad v^{\mathfrak{m}^k}(S_o^{ij}) = v^{\mathfrak{m}^k}(\Omega) = \frac{1}{k} \cdot v^{\mathfrak{m}^k}(\Omega^k)$

because of equation (4.1).

Both equalities together with the pareto optimality of the core imply finally:

$$
\begin{aligned}
s_{ij}(x, v^{\mathfrak{m}^k}) &\geq e(x, S_o^{ij}, v^{\mathfrak{m}^k}) \\
&= v^{\mathfrak{m}^k}(S_o^{ij}) - x(S_o^{ij}) \\
&= \frac{1}{k} \cdot v^{\mathfrak{m}^k}(\Omega^k) - x(\Omega) \\
&= \frac{1}{k} \cdot (v^{\mathfrak{m}^k}(\Omega^k) - x(\Omega^k)) \\
&= 0 \qquad\qquad (i,j \in \Omega,\ i \neq j).
\end{aligned}
$$

Using (5.3) we have

$$s_{ij}(x, v^{\mathfrak{m}^k}) = 0 \qquad\qquad (i,j \in \Omega,\ i \neq j)\ .$$

This implies immediately

$$x \in \mathcal{K}(\Gamma^{\mathfrak{m}^k})\ .$$

5. LP-games

In the preceeding chapters we studied the behaviour of market games under replication. We now want to turn to the LP-games, that is a special class of market games (cf. Owen [1974], Billera, Raanan [1981], Rosenmüller [1982]). For these games replications are easily practicable and it is possible to use theorems from linear programming and duality for instance to compute the set of equilibrium payoffs (cf. Owen [1975], Schmidt [1980]).

Now let every player of $\Omega = \{1,\dots,n\}$ be equipped with m resources that can be used for the production of p goods (i.e. $m, p \in \mathbb{N}$). The distribution of these "raw materials" to playes and coalitions respectively is given by the following additive set-function.

Let $\qquad b : \mathcal{P}(\Omega) \to \mathbb{R}^{m+}$

be an additive mapping with

$$b(i) := b^i \qquad (i \in \Omega),$$

$$b(S) := \sum_{i \in S} b^i \qquad (S \in \quad (\Omega)),$$

and $\qquad b(\Omega) > 0$.

A game $\Gamma^{\mathcal{L}} = (\Omega, \mathcal{P}(\Omega), v^{\mathcal{L}})$ with

$$v^{\mathcal{L}}(S) := \max \{\langle c,x\rangle \mid x \in \mathbb{R}^{p+}, Ax \leq b(S)\} \qquad (S \in \mathcal{P}(\Omega))$$

is called LP-game, where $\mathcal{L} = (A,b(\cdot),c)$ characterizes the above linear program.

In a paper of Owen [1975] it is shown that LP-games are totally balanced and therefore they can be represented as market games. In order to study replications of LP-games, we replicate linear programs $\mathcal{L} = (A,b(\cdot),c)$ canonically by replicating the vector of resources $b(\cdot)$, as it is done in Rosenmüller [1982].

Let $\mathcal{L}^k := (A, b^k(\cdot), c)$ be the

k-fold replication of $\mathcal{L}$ with

$$b^k : \mathcal{P}(\Omega^k) \to \mathbb{R}^{m+}$$

and $\quad b^k(i) := b(l) \qquad (l \in \Omega,\ i \in \Omega_l^k)$.

Then $\quad \Gamma^{\mathcal{L}^k} = (\Omega^k, \mathcal{P}(\Omega^k), v^{\mathcal{L}^k})$ with

$$v^{\mathcal{L}^k}(S) := \max \{ \langle c,x \rangle \mid x \in \mathbb{R}^{p+},\ Ax \leq b^k(S) \} \qquad (S \in \mathcal{P}(\Omega^k))$$

is called k-fold replication of $\Gamma^{\mathcal{L}}$.

This definition of replications of LP-games coincides with the replication of LP-games as market games: Every type of player of an LP-game is characterized by his vector of resouces. The construction of the corresponding market $\mathcal{M}^{\mathcal{L}}$ does not change this characterization, because just the vectors of resources in the game become the initial allocations in the market. So under k-fold replication we have the following equation:

$$\Gamma^{(\mathcal{M}^{\mathcal{L}})^k} = \Gamma^{\mathcal{M}(\mathcal{L}^k)} = \Gamma^{\mathcal{L}^k} .$$

All statements of market games with continuous and concave utility functions are also true for LP-games, because these can be represented by market games with concave and piecewise linear utility functions (cf. Schmidt [1980]).

6. Two examples of the pulsation of the kernel

A theorem of Peleg states that for all n-person games with at most four playes the core includes the kernel. This statement is not true for $n \geq 5$. A counter-example for that case is published in the same paper (cf. Peleg [1966]).

We now want to show that even under replication the kernel of LP-games is not necessarily a subset of the core, and that the "convergence"-theorem of Debreu and Scarf does not hold in general for the kernel.

At first we introduce some suitable and shorter notations which are in a certain sense generalizations of the case of usual replications. Let the following sets and vectors be given:

$\Omega = \{1,2\}$	the set of <u>types</u> of players;
$K = \{k_1,k_2\} \in \mathbb{N}_0^2$	the vector which gives the number of players of type 1 and type 2; in the following we shall use $k_1=3$, $k_2=2$ (i.e. $K=(3,2)$);
$\Omega_1^K = \{1,3,\dots,2k_1-1\} = \{1,3,5\}$	the set of players of type 1;
$\Omega_2^K = \{2,4,\dots,2k_2\} = \{2,4\}$	the set of players of type 2;
$\Omega^K = \Omega_1^K \cup \Omega_2^K$	the set of all players;
$M^K = \{0,1,\dots,k_1\} \times \{0,1,\dots,k_2\}$	the set of all possible vectors which represent the typewise composition of coalitions in $\mathcal{P}(\Omega^K)$.

Let $(r,s) \in M^K$, then we call $S \in \mathcal{P}(\Omega^K)$ to be <u>of type (r,s)</u>, if

$$| S \cap \Omega_1^K | = r \quad \text{and} \quad | S \cap \Omega_2^K | = s \; .$$

In the following we want to consider games $\Gamma^K = (\Omega^K, \mathcal{P}(\Omega^K), v^K)$ with characteristic functions v^K that depend only on the number of players of each type. This means, more exactly:

For all coalitions $S,T \in \mathcal{P}(\Omega^K)$ with
$|S \cap \Omega_1^K| = |T \cap \Omega_1^K|$ and $|S \cap \Omega_2^K| = |T \cap \Omega_2^K|$ it is

$$v^K(S) = v^K(T) \quad . \tag{6.1}$$

So the next expression is well defined:

$$v^K(r,s) := v^K(S) \qquad ((r,s) \in M^K,\ S \in \mathcal{P}(\Omega^K) \text{ of type } (r,s)).$$

We know that all imputations of the kernel give the same amount to players of the same type (cp. chapter 3). So analogous to (3.1) we can write here

$$\mathcal{K}(\Gamma^K) \subseteq \mathfrak{J}^{et}(\Gamma^K) \tag{6.2}$$

where $\mathfrak{J}^{et}(\Gamma^K) := \{x \in \mathfrak{J}(\Gamma^K) \mid x_i = x_j$ if there exists $l \in \Omega$ such that $i \in \Omega_l^K$ and $j \in \Omega_l^K\}$

are also called <u>equal treatment imputations of Γ^K</u> .

If $x = (x_1,x_2) \in \mathbb{R}^2$ we define

$$x^K = (x_1,x_2)^K := (x_i)_{i \in \Omega^K} \quad \text{where}$$

$$x_i = \begin{cases} x_1 & \text{if} \quad i \in \Omega_1^K \\ x_2 & \text{if} \quad i \in \Omega_2^K \ . \end{cases}$$

Let $x \in \mathfrak{J}^{et}(\Omega^K)$ and $(r,s) \in M^K$, then because of $1 \in \Omega_1^K$ and $2 \in \Omega_2^K$ we have for each coalition $S \in \mathcal{P}(\Omega^K)$ of type (r,s)

$$x(S) = rx_1 + sx_2 = (x_1,x_2)^K(S) \ .$$

Therefore the following excess is well defined on the set of equal treatment imputations $\mathfrak{J}^{et}(\Gamma^K)$:

$$e(x,r,s,v^K) := e(x,S,v^K) \qquad (x \in \mathfrak{J}^{et}(\Gamma^K),\ (r,s) \in M^K,\ S \in \mathcal{P}(\Omega^K) \text{ of type } (r,s)).$$

If S is a coalition of type (r,s), we get

$$\begin{aligned} e(x,r,s,v^K) &= v^K(S) - x(S) \\ &= v^K(r,s) - rx_1 - sx_2 \qquad (x \in \mathfrak{J}^{et}(\Gamma^K)) . \end{aligned}$$

Sometimes we write shorter $e(x,r,s)$ for this excess, if there is no doubt about the game Γ^K .

The intersection of core and equal treatment imputations can now be written as

$$\mathcal{C}(\Gamma^K) \cap \mathfrak{J}^{et}(\Gamma^K) = \{x \in \mathfrak{J}^{et}(\Gamma^K) \mid e(x,r,s,\Gamma^K) \leq 0 \ ((r,s) \in M^K)\} .$$

The maximum surplus can also be represented in dependence of types of coalitions:

Let

$$T^K_{12} := \{(r,s) \in M^K \mid r \geq 1 \ , \ s \leq k_2 - 1\}$$

be the set of type-vectors of coalitions which contain at least one player of type 1 and in which at least one player from Ω^K of type 2 is missing.

Analogous let

$$T^K_{21} := \{(r,s) \in M^K \mid r \leq k_1 - 1 \ , \ s \geq 1\} .$$

Then it is easy to verify that for every pair $(i,j) \in \Omega^K_1 \times \Omega^K_2$ with $i \neq j$ the next equations hold true (we write $s_{ij}(x)$ instead of $s_{ij}(x,\Gamma^K)$) :

$$s_{ij}(x) = s_{12}(x) = \max_{(r,s)\in T^K_{12}} e(x,r,s,v^K) \qquad (x \in \mathfrak{J}^{et}(\Gamma^K)) \tag{6.3}$$

and

$$s_{ji}(x) = s_{21}(x) = \max_{(r,s)\in T^K_{21}} e(x,r,s,v^K) \qquad (x \in \mathfrak{J}^{et}(\Gamma^K)) . \tag{6.4}$$

This simplifies the computation of the kernel and we are able to formulate the following lemma:

6.1 Lemma:

Given the game $\Gamma^K = (\Omega^K, \mathcal{P}(\Omega^K), v^K)$ which characteristic function v^K fulfills (6.1) and is 0-monotonic. Then the following equations hold true:

$$\mathcal{K}(\Gamma^K) = \{x \in \mathcal{J}^{et}(\Gamma^K) \mid s_{12}(x) = s_{21}(x)\}$$
$$= \{x \in \mathcal{J}^{et}(\Gamma^K) \mid \max_{(r,s)\in T^K_{12}} e(x,r,s,v^K) = \max_{(r,s)\in T^K_{21}} e(x,r,s,v^K)\} .$$

Proof: The assertion follows immediately from the equations (6.3) and (6.4) for the maximum surplus and from the representation of the kernel for 0-monotonic games (cp. chapter 3).

As $$s_{ij}(x) = s_{ji}(x) \qquad (x \in \mathcal{J}^{et}(\Gamma^K))$$

holds true for all $i,j \in \Omega^K_1$ or $i,j \in \Omega^K_2$, that is for all pairs of players of the same type, we have

$$\mathcal{K}(\Gamma^K) = \{x \in \mathcal{J}^{et}(\Gamma^K) \mid s_{ij}(x) = s_{ji}(x) \quad (i,j \in \Omega^K)\}$$
$$= \{x \in \mathcal{J}^{et}(\Gamma^K) \mid s_{12}(x) = s_{21}(x)\} \quad . \square$$

So for computing the maximum surplus and the kernel we don't have to maximize the excess over sets of coalitions but only over sets of types of coalitions.

In our examples we shall consider LP-games with two resources where each player of type 1 has a_1 units of resource 1 and nothing of resource 2 while players of type 2 possess a_2 units of resource 2 and nothing of the first resource. So let $\mathcal{L}^K = (A, b(\cdot), c)$ be a linear program with

i) $A := \binom{1}{1} \in \mathbb{R}^2$,

ii)
$$b(i) := \begin{cases} \begin{pmatrix} a_1 \\ 0 \end{pmatrix} & \text{if } i \in \Omega_1^K \\ \begin{pmatrix} 0 \\ a_2 \end{pmatrix} & \text{if } i \in \Omega_2^K \end{cases} \qquad (6.5)$$

where $a_1, a_2 \in \mathbb{R}^+ \setminus \{0\}$,

iii) $c := 1$

As the additive function b depends only on the type of coalitions, we can write for every $S \in \mathcal{P}(\Omega^K)$ of type $(r,s) \in M^K$:

$$b(S) = \begin{pmatrix} ra_1 \\ \\ sa_2 \end{pmatrix} .$$

So we get an LP-game $\Gamma^{\mathcal{L}^K} = (\Omega^K, \mathcal{P}(\Omega^K), v^{\mathcal{L}^K})$ which characteristic function also depends only on the type of coalitions:

For every $S \in \mathcal{P}(\Omega^K)$ of type $(r,s) \in M^K$ we get

$$\begin{aligned} v^{\mathcal{L}^K}(S) &= \max \{x \in \mathbb{R}^+ \mid x \leq ra_1,\ x \leq sa_2\} \\ &= \min \{ra_1, sa_2\} \\ &= v^{\mathcal{L}^K}(r,s) . \end{aligned}$$

Changing the types of players we can assume without l.o.g.

$$k_1 a_1 \leq k_2 a_2 \qquad (6.6)$$

i.e. $\quad v^{\mathcal{L}^K}(\Omega^K) = v^{\mathcal{L}^K}(k_1, k_2) = k_1 a_1$.

If we write $\tilde{x}^1 := (a_1,0)^K = (a_1,0,a_1,0,a_1)$

and $\tilde{x}^2 := (0, \frac{k_1 a_1}{k_2})^K = (0, \frac{k_1 a_1}{k_2}, 0, \frac{k_1 a_1}{k_2}, 0)$,

we get the following equation for the equal-treatment imputations by using (6.6):

$$\mathcal{J}^{et}(\Gamma^{\mathscr{L}^K}) = \{(x_1,x_2)^K \in \mathbb{R}_+^{k_1+k_2} \mid k_1x_1+k_2x_2 = v^{\mathscr{L}^K}(\Omega^K) = k_1a_1\}$$

$$= \{(x_1,x_2)^K \in \mathbb{R}_+^{k_1+k_2} \mid x_2 = \frac{k_1}{k_2}(a_1-x_1)\}$$

$$= \text{conv } \{(a_1,0)^K, (0, \frac{k_1}{k_2} a_1)^K\}$$

$$= \text{conv } \{\tilde{x}^1,\tilde{x}^2\} \quad . \qquad (6.7)$$

So $\mathcal{J}^{et}(\Gamma^{\mathscr{L}^K})$ is a straight line between the endpoints $\tilde{x}^1$ and $\tilde{x}^2$. The excess is a linear function and therefore completely determined by its values at $\tilde{x}^1$ and $\tilde{x}^2$. This will be usefull for the computation of the kernel.

The k-fold replication $(\Gamma^{\mathscr{L}^K})^k = ((\Omega^K)^k, \mathcal{P}((\Omega^K)^k), (v^{\mathscr{L}^K})^k)$ of the LP-game $\Gamma^{\mathscr{L}^K} = (\Omega^K, \mathcal{P}(\Omega^K), v^{\mathscr{L}^K})$ will be written as $\Gamma^{\mathscr{L}^{k\cdot K}} = (\Omega^{k\cdot K}, \mathcal{P}(\Omega^{k\cdot K}), v^{\mathscr{L}^{k\cdot K}})$, because up to reordering $\Omega^{k\cdot K}$ and $(\Omega^K)^k$ have the same typical composition

$$k \cdot K = (kk_1,kk_2) = (3k,2k)$$

(this means $3k$ players of type 1 and $2k$ players of type 2).

6.2 Example

Let $a_1 = 1$ and $a_2 = 2$, that is players of type 1 have one unit of resource 1 and players of type 2 have two units of resource 2. Then the k-fold replications of the linear program $\mathcal{L}^K$ is given as in (6.5) by

$$b(i) := \begin{cases} \binom{1}{0} = b(1) & \text{if} \quad i \in \Omega_1^{k\cdot K} \\ \\ \binom{0}{1} = b(2) & \text{if} \quad i \in \Omega_2^{k\cdot K} \end{cases}$$

So for the corresponding LP-games $v^{\mathcal{L}^{k\cdot K}}$ we get for the types of coalitions $(r,s) \in M^{k\cdot K}$:

$$v^{\mathcal{L}^{k\cdot K}}(r,s) = \min\{r,2s\} .$$

As we know that LP-games are 0-monotonic and in our examples their characteristic functions depend only on types of coalitions, we can use Lemma 6.1 for computing the kernel.

For shorter notation we write again

$$s_{ij}^k(x) = s_{ij}(x, v^{\mathcal{L}^{k\cdot K}}) = \max_{(r,s)\in T_{ij}^{k\cdot K}} e(x,r,s,v^{\mathcal{L}^{k\cdot K}})$$

for all $k \in \mathbb{N}$; $x \in \mathcal{J}^{et}(\Gamma^{\mathcal{L}^{k\cdot K}})$; $i,j \in \{1,2\}$.

For every imputation $x \in \mathcal{J}^{et}(\Gamma^{\mathcal{L}^{k\cdot K}})$ the values of the functions s_{12}^k and s_{21}^k can be taken from the following list (here $\lceil y \rceil$ means the smallest integer greater or equal to $y \in \mathbb{R}$) :

	k=1	k=2	k=3
	$0 \quad -x_2$	$s_{21}(x)$	$s_{21}^2(x)$
$s_{12}^k(x) = \max$		$3-3x_1-2x_2$	$6-6x_1-3x_2$
	$1-x_1-x_2$	$4-4x_1-2x_2$	$7-7x_1-4x_2$
	$2-2x_1-x_2$	$5-5x_1-3x_2$	$8-8x_1-4x_2$
$s_{12}^k(x) = \max$	$0-x_1$	$6-6x_1-3x_2$	$9-9x_1-5x_2$
		$s_{12}(x)$	$s_{12}^2(x)$

(6.8)

Here types of coalitions which are irrelevant for the maximizing of the excess are not taken into consideration (for example (0,2) or (3.1)).

In general we have for $k \geq 2$:

$$s_{21}^k(x) = \max \begin{cases} s_{21}^{k-1} \\ (3k-3)\,(1-x_1) - \lceil\frac{3k-3}{2}\rceil\, x_2 \\ (3k-2)\,(1-x_1) - \lceil\frac{3k-2}{2}\rceil\, x_2 \\ (3k-1)\,(1-x_1) - \lceil\frac{3k-1}{2}\rceil\, x_2 \end{cases}$$

$$s_{12}^k(x) = \max \begin{cases} (3k-3)\,(1-x_1) - \lceil\frac{3k-3}{2}\rceil\, x_2 \\ (3k-2)\,(1-x_1) - \lceil\frac{3k-2}{2}\rceil\, x_2 \\ (3k-1)\,(1-x_1) - \lceil\frac{3k-1}{2}\rceil\, x_2 \\ 3k\,(1-x_1) - \lceil\frac{3k}{2}\rceil\, x_2 \\ s_{12}^{k-1}(x) \end{cases} \qquad (6.9)$$

The set of equal treatment imputations $\mathcal{J}^{et}(\Gamma^{k\cdot K})$ is given for every $k \in \mathbb{N}$ by (6.7)

$$\mathcal{J}^{et}(\Gamma^{k\cdot K}) = \{(x_1,x_2)^{k\cdot K} \in \mathbb{R}_+^{5k} \mid 3x_1+2x_2 = 3\}$$

$$= \text{conv}\,\{(1,0)^{k\cdot K}, (0, \tfrac{3}{2})^{k\cdot K}\} .$$

We now shall compute the maximum surplus at these endpoints

$$\tilde{x}^1 := (\tilde{x}^1)^k = (1,0)^{k\cdot K} \quad ,$$

$$\tilde{x}^2 := (\tilde{x}^2)^k = (0, \tfrac{3}{2})^{k\cdot K} \quad .$$

Obviously we have for $\tilde{x}^1$

$$s_{21}^k \; (\tilde{x}^1) = s_{12}^k \; (\tilde{x}^1) = 0 \qquad\qquad (k \in \mathbb{N}) \; .$$

For $\tilde{x}^2$ we have to examine two cases by means of (6.8) and (6.9):

1. Let k be odd. Then we have

$$s_{21}^k \; (\tilde{x}^2) = 3k-1 - \frac{3k-1}{2} \cdot \frac{3}{2} = \frac{1}{4} \; (3k-1)$$

and $s_{12}^k \; (\tilde{x}^2) = s_{21}^k \; (\tilde{x}^2)$, because of simple computation for $k = 1$ and in case of $k \geq 3$ because of

$$s_{12}^{k-1} \; (\tilde{x}^2) = s_{21}^k \; (\tilde{x}^2) \quad \text{and}$$

$$3k - \lceil \tfrac{3k}{2} \rceil \cdot \frac{3}{2} = 3k - \frac{3k+1}{2} \cdot \frac{3}{2} = \frac{1}{4} \; (3k-1) - \frac{1}{2} < s_{21}^k \; (\tilde{x}^2) \; .$$

2. Let k be even. In this case we get

$$s_{21}^k \; (\tilde{x}^2) = 3k-2 - \frac{3k-2}{2} \cdot \frac{3}{2} = \frac{1}{4} \; (3k-2)$$

and

$$s_{12}^k \; (\tilde{x}^2) = 3k - \frac{3k}{2} \cdot \frac{3}{2} = \frac{1}{4} \cdot 3k > s_{21}^k \; (\tilde{x}^2) \; .$$

Both cases together with linearity of the excess imply the following equation for the kernel:

$$\mathcal{K}(\Gamma^{k\cdot K}) = \begin{cases} \{(1,0)^{k\cdot K}\} & \text{if} \quad k \equiv 0 \pmod 2 \\ \\ \operatorname{conv} \{(1,0)^{k\cdot K}, \; (0,\tfrac{3}{2})^{k\cdot K}\} & \text{if} \quad k \equiv 1 \pmod 2 \end{cases} \tag{6.10}$$

Now let us compute the intersection of core and the set of equal treatment imputations for every k-fold replication of the given LP-game $\Gamma^{\mathfrak{X}^K}$.

For all $x \in \mathcal{J}^{et}(\Gamma^{\mathfrak{X}^{k\cdot K}})$ we have because of pareto-optimality:

$$x_2 = \frac{3}{2}(1-x_1) \quad .$$

The type (2,1) of coalitions lies in $M^{k\cdot K}$ for all $k \in \mathbb{N}$. This type gives the excess

$$\begin{aligned} e(x,2,1,v^{\mathfrak{X}^{k\cdot K}}) &= v^{\mathfrak{X}^{k\cdot K}}(2,1) - 2x_1 - x_2 \\ &= 2 - 2x_1 - \frac{3}{2}(1-x_1) \\ &= \frac{1}{2}(1-x_1) \qquad\qquad (x \in \mathcal{J}^{et}(\Gamma^{\mathfrak{X}^{k\cdot K}})). \end{aligned}$$

So, if $x_1 < 1$, the excess $e(x,2,1,v^{\mathfrak{X}^{k\cdot K}})$ will be positive, that is

$$x \notin \mathcal{C}(\Gamma^{\mathfrak{X}^{k\cdot K}}) \qquad (x \in \mathcal{J}^{et}(\Gamma^{\mathfrak{X}^{k\cdot K}}) \setminus \{(1,0)^{k\cdot K}\}) \; . \tag{6.11}$$

On the other hand the payoff $(1,0)^{k\cdot K}$ is in the core, because its excess is always zero. Hence we get

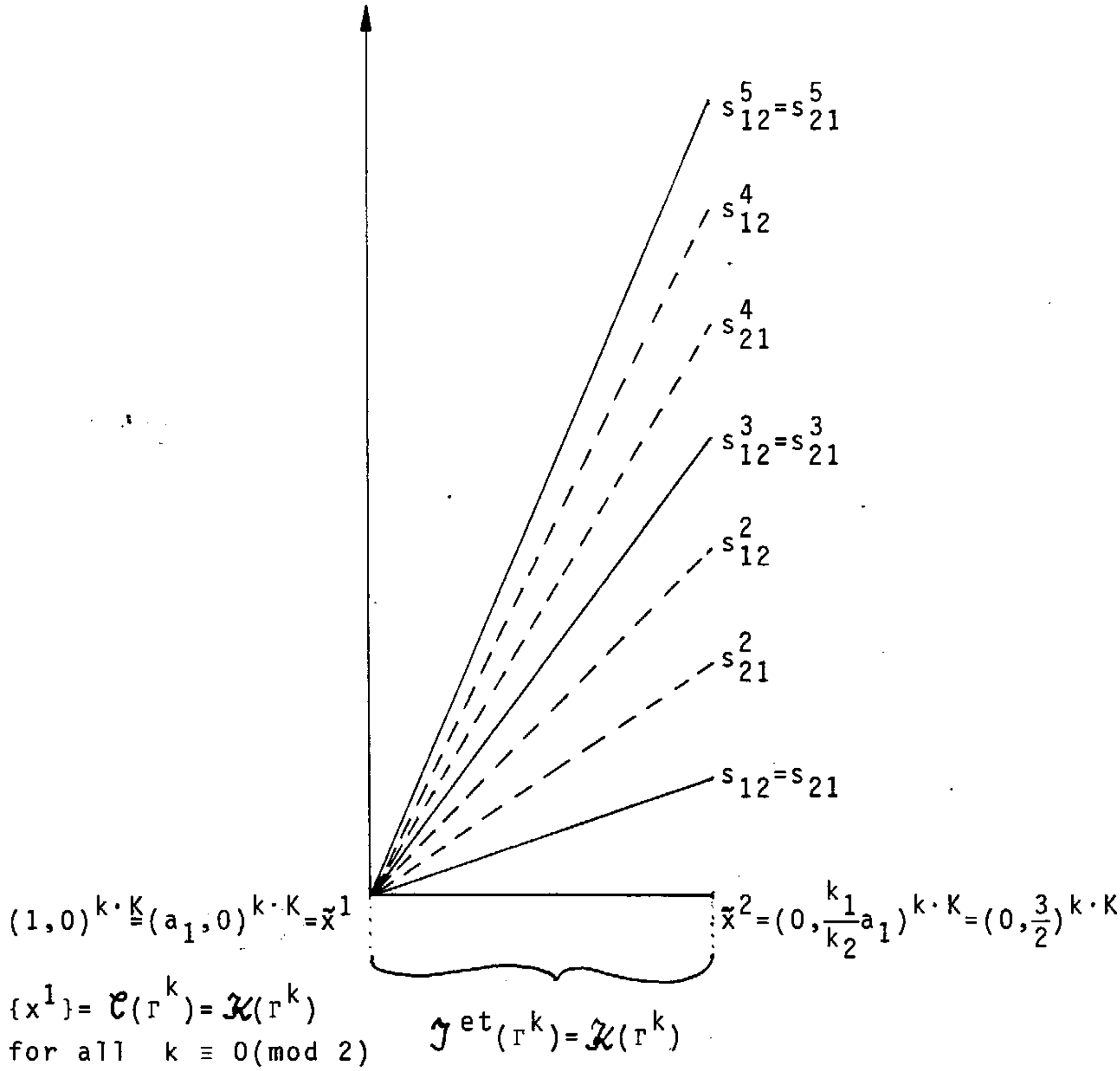
$$\mathcal{C}(\Gamma^{\mathfrak{X}^{k\cdot K}}) \cap \mathcal{J}^{et}(\Gamma^{\mathfrak{X}^{k\cdot K}}) = \{(1,0)^{k\cdot K}\} \qquad (k \in \mathbb{N}) \; . \tag{6.12}$$

As in the kernel equal treatment holds true for equal types of players, the statements (6.10), (6.11), and (6.12) imply

$$\begin{aligned} \mathcal{K}(\Gamma^{\mathfrak{X}^{k\cdot K}}) &\subseteq \mathcal{C}(\Gamma^{\mathfrak{X}^{k\cdot K}}) \qquad \text{if} \quad k \equiv 0 \pmod 2 \\ \mathcal{K}(\Gamma^{\mathfrak{X}^{k\cdot K}}) &\not\subseteq \mathcal{C}(\Gamma^{\mathfrak{X}^{k\cdot K}}) \qquad \text{if} \quad k \equiv 1 \pmod 2 \; . \end{aligned}$$

Finally for this example we consider the equilibrium payoff μ^K, which can be computed by using the solution of the dual program of μ^K (cf. Owen [1975], Schmidt [1980]): Then we have $\mu^K = (1,0)^K$.

Here we see that the kernel does not "converge" to the equilibrium payoff in the sense of Debreu-Scarf's theorem, because for every odd $k \in \mathbb{N}$ the kernel $\mathcal{K}(\Gamma^{\mathcal{X}^{k \cdot K}})$ is the whole straight line of $\mathcal{J}^{et}(\Gamma^{\mathcal{X}^{k \cdot K}})$ with the replicated equilibrium payoff $\mu^{k \cdot K}$ as one endpoint.



In this picture we can see the function s^k_{12} and s^k_{21} on the set of equal treatment imputations for some $k \in \mathbb{N}$.

- - - - means different values of s^k_{12} and s^k_{21},

——— means equal values of s^k_{12} and s^k_{21}.

6.3 Example

In this example the parameters of the linear program $\mathcal{L}^K$ are given as in 6.2 except of the resources:

Let $a_1 = \frac{7}{6}$ and $a_2 = 2$.

The computation of the kernel is analogous to example 6.2, but here we get four cases:

$$\mathcal{K}(\Gamma^{\mathcal{L}^{k\cdot K}}) = \begin{cases} \operatorname{conv}\{(\frac{7}{6},0)^{k\cdot K}, (0,\frac{7}{4})^{k\cdot K}\} & k \equiv 1 \pmod 4 \\ \operatorname{conv}\{(\frac{7}{6},0)^{k\cdot K}, (\frac{1}{6},\frac{3}{2})^{k\cdot K}\} & k \equiv 2 \pmod 4 \\ \operatorname{conv}\{(\frac{7}{6},0)^{k\cdot K}, (\frac{1}{2},1)^{k\cdot K}\} & k \equiv 3 \pmod 4 \\ \{(\frac{7}{6},0)^{k\cdot K}\} & k \equiv 0 \pmod 4 \end{cases}$$

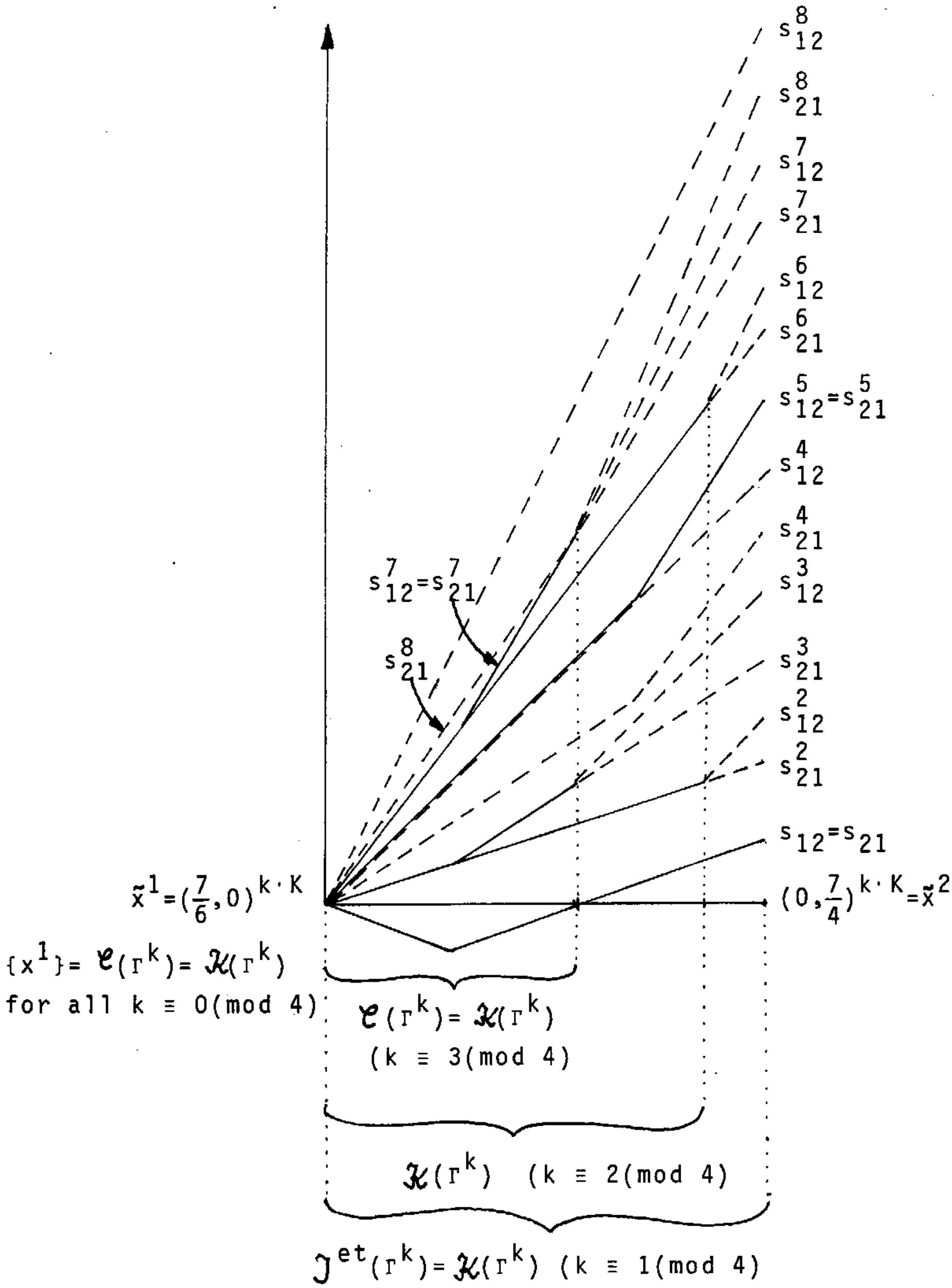
Furthermore the following equation holds true for the intersection of core and equal treatment imputations:

$$\mathcal{K}(\Gamma^{\mathcal{L}^{k\cdot K}}) \cap \mathcal{J}^{et}(\Gamma^{k\,K}) = \begin{cases} \operatorname{conv}\{(\frac{7}{6},0)^{k\cdot K}, (\frac{1}{2},1)^{k\cdot K}\} & k = 1 \\ \{(\frac{7}{6},0)^{k\cdot K}\} & k \geq 2\,. \end{cases}$$

The equilibrium payoff is

$$\mu^K = (\tfrac{7}{6},0)^K\,.$$

We see, that the kernel pulsates again and so he does not converge to the equilibrium payoff with which he coincides only in the case $k \equiv 0 \pmod 4$.



Here the analogous picture to example 6.2.

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