Center for
Mathematical Economics
Working Papers

223

January 1994

Population and Environmental Quality

Mark B. Cronshaw and Till Requate

Center for Mathematical Economics (IMW)
Bielefeld University
Universitätsstraße 25
D-33615 Bielefeld · Germany

e-mail: imw@uni-bielefeld.de
http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 223

Population and Environmental Quality

by

Mark B. Cronshaw and Till Requate

January 1994

University of Bielefeld

33501 Bielefeld, Germany

Institute of Mathematical Economics
— IMW —
University of Bielefeld
Postfach 10 01 31
D—33501 Bielefeld
Germany

Telefon: (0521) 1064898
Telefax: (0521) 1062997
Telex: 932362 unibi
e—mail: REQUATE @ NW42.WIWI.UNI—BIELEFELD.DE

Printed at University of Bielefeld

ISSN 0931–6558

Population and Environmental Quality

MARK B. CRONSHAW †

Department of Economics,
and E & B Program Institute of Behavioral Science
University of Colorado, Boulder
CO 80309-0256
USA

and

Till Requate ‡

Institute of Mathematical Economics
University of Bielefeld
P.O. Box 100131
33501 Bielefeld
Germany

January 1994

† CRONSHAW thanks the Institute of Mathematical Economics, Bielefeld University, and the Graduiertenkolleg for their hospitality and for funding this research. Email: CRONSHAW@SPOT.COLORADO.EDU.

‡ REQUATE thanks the Department of Economics, University of Colorado at Boulder, for their hospitality and the state government of Nordrhein–Westfalen (von Bennigsen Foerder Preis) for financial support. Email: REQUATE@NW42.WIWI.UNI-BIELEFELD.DE.

Abstract

This paper considers first best allocations in an economy where a consumption good is produced using labor. Production results in pollution, which is a public bad. Pollution abatement can be achieved either by restricting production output or by using labor. We consider how the first best allocation varies with population size. Consumers are unambiguously worse off when the population is larger. However, surprisingly, there is no single optimal policy on how pollution and labor should vary with population size. For standard models of preferences and technology it might be desirable either to increase or to reduce emissions and/or labor, depending on parameters. Despite such ambiguity in the first best level of emissions, the Pigouvian tax which implements the first best is a non-decreasing function of the population size. We conclude that, since the comparative statics of the first best are so ambiguous, sensible debate on environmental policy cannot proceed without a careful determination of actual preferences and technology.

1 Introduction

In typical neoclassical growth theory with an exogenous population growth rate and without technical progress, an economy can approach a steady state growth in which per capita consumption is constant, and aggregate consumption grows without bound. This is in stark contrast with a Malthusian view of an economy in which the presence of some fixed factor(s) would not allow sustainable growth in aggregate consumption (see Malthus (1798), book I).

In this paper we consider a variation on the Malthusian theme, in which even though there are no fixed factors, consumers are worse off in larger populations due to the presence of pollution. We model pollution as a public bad which is generated by the productive sector of the economy.

Previous authors have alluded to the issue of population and pollution, but work in this area is far from complete. Keeler, Spence and Zeckhauser (1971) state that no steady state equilibrium exists to an optimal pollution control problem with exogenous population growth. Gruver (1976) states that "the relationships between population growth and pollution are important and hotly debated", but considers an optimal pollution control model with a fixed population size to avoid complications. Tahvonen and Kuuluvainen (1993) consider a growth model with pollution but with a single representative consumer. To the best of our knowledge the only growth model which considers both, population growth and pollution, has been recently published by Gradus and Smulders (1993). In contrast to the Keeler–Spence–Zeckhauser model where pollution is a *flow* input of production, in their paper pollution results from the *stock* of production capital, but the negative externality can be mitigated by abatement capital. In apparent contradiction to the statement of Keeler et al., Gradus and Smulders focus on steady states. The existence of steady states in output and consumption in their model, however, depends heavily on the special functional form of pollution as a function of capital and abatement.

In general, the role of a growing population in a world where production generates pollution does not seem to be well understood yet. Hence, as a first step and for the sake of simplicity we do not consider a growth model here. Rather, we present a comparative statics analysis of an economy in which population is treated as a parameter. This simplification still leaves interesting results. Although it might be desirable to discuss such ideas in the context of a growth model, this would require making some assumptions about technological progress and the dynamics of pollution decay processes which would be quite speculative.

We consider first best allocations, as a benchmark of what might be achieved. Of course the public bad aspect of pollution may make it difficult to obtain the first best! In our economy identical consumers can supply labor to produce a consumption good. Production results in pollution. Pollution abatement can be achieved by supplying additional labor.

A first best allocation involves trade-offs between the benefits of consumption and the disutility of supplying labor and of experiencing pollution. Since pollution is a public bad, in some sense its cost rises when the population is larger. In the next section we show that consumers are worse off when the population grows. In section 3 we consider how first best consumption varies with the population size. Although per capita consumption may fall or rise in general, we give conditions on preferences and technology which guarantee that consumption per capita decreases. In section 4 we consider how the first best levels of pollution and per capita labor depend on the population size. We find that the dependence is very sensitive to the nature of preferences and technology. Such ambiguity makes it difficult for policy makers to decide on appropriate environmental targets! In Section 5 we consider a Pigouvian tax. We show that the optimal emission tax rises as population grows under mild conditions on preferences. Section 6 concludes. An appendix contains more general results than in the main text, as well as the solution of several examples.

2 A Model

We consider first best allocations in an economy with n identical agents who have convex preferences which depend on a consumption good, labor and pollution. Preferences are represented by a quasi-concave utility function $U : (q, l, E) \mapsto U(q, l, E)$ which is assumed to be strictly increasing in the consumption good q, and strictly decreasing in labor l, and pollution E. We also refer to pollution as emissions. For simplicity, and to obtain more crisp results, we only consider symmetric allocations in which each agent gets the same amount of consumption good and supplies the same amount of labor. We model pollution as a public bad, so that each agent also suffers from the same quantity of pollution.

The consumption good can be manufactured using labor as an input. This process results in emissions of a by-product which is a pollutant. Labor can be used to abate emissions. With the symmetry restriction, technically efficient production can be represented by the transformation function $T(nq, nl, E) = 0$. We assume

that the transformation function is homothetic, so that efficient production can also be represented by $T(q, l, E/n) = 0$.

Let V_n be the utility of each consumer in the symmetric Pareto efficient allocation[1]. That is,

$$V_n = \max_{q,l,E\geq 0} \quad U(q, l, E) \qquad \text{s.t.} \quad T(q, l, E/n) = 0$$

Let q_n, l_n be the first best levels of per capita consumption and labor, and E_n be the aggregate emissions. Note that the first best utility level would be independent of the population size if there were no pollution. However, we will show that with pollution consumers are worse off when the population is larger.

For the remainder of the paper we assume that the technology can be represented by a continuous production function f, such that

$$q \leq f(l, E/n) \tag{1}$$

which we assume to be strictly increasing in both labor and emissions. It is convenient to define per capita emissions $e = E/n$. By monotonicity, at a Pareto efficient allocation we have $q = f(l, e)$.

Our first result, which is quite general but not very surprising, is that consumers are worse off if the population is larger.

Theorem 1 *The first best utility level, V_n, is non-increasing in the population size, n. If $f_e(l, 0) = \infty$ for all l then V_n is strictly decreasing.*

Proof: Consider $m < n$. Since f is increasing in emissions we get $f(l_n, E_n/m) \geq f(l_n, E_n/n) \geq q_n$. So q_n, l_n, E_n is feasible with m people. Since V_m is optimal given m, it follows that $V_m \geq U(q_n, l_n, E_n) = V_n$.

If $f_e(l, 0) = \infty$ then emissions E_n are strictly positive. In this case $f(l_n, E_n/m) > f(l_n, E_n/n) \geq q_n$. So there is some $q > q_n$ such that q, l_n, E_n is feasible with m people. Since V_m is optimal $V_m \geq U(q, l_n, E_n) > U(q_n, l_n, E_n) = V_n$. □

In the next two sections we examine the properties of the first best allocation. In particular, we are interested in what direction per capita consumption and labor, and aggregate emissions change in response to popuation growth. One would suspect that per capita consumption decreases in n, and indeed, we will show that this is the case under some regularity conditions on preferences and technology. It would also be interesting to know what policy implications can be drawn with respect to environmental and labor policy, e.g., in the face of a greater population are

[1] There is a symmetric Pareto efficient allocation because preferences are convex.

tougher environmental standards required, and/or should more effort be expended in abatement activities ? In section 4 we show by examples that the answers to such questions depend critically on the exact preferences and technology. Perhaps surprisingly, it can even be the case that environmental standards should be weakened when the population is larger.

3 First Best Consumption

The first order conditions for the social optimum are most conveniently expressed by:

$$-n\frac{U_E(q_n, l_n, E_n)}{U_q(q_n, l_n, E_n)} = f_e(l_n, E_n/n) \qquad (2)$$

$$-\frac{U_l(q_n, l_n, E_n)}{U_q(q_n, l_n, E_n)} = f_l(l_n, E_n/n) \qquad (3)$$

$$q_n = f(l_n, E_n/n) \qquad (4)$$

We make the following assumption:

Assumption 1 *f is twice differentiable, monotonic, concave and satisfies $f_{le} \geq 0$.*

Note that we allow for constant and decreasing returns to scale. When there are constant returns to scale the deterioration in first best utility as population grows is not due to some fixed factor[2]

Theorem 2 *If f satisfies Assumption 1 and U is concave and additively separable (i.e., $U_{ql} = U_{lE} = U_{qE} = 0$), then first best consumption q_n is decreasing in n.*

Recall that monotonic transformations of utility represent the same preferences. So, for example, Cobb-Douglas and CES utility functions are covered by the theorem.

Proof: The result is a consequence of a more general result (Theorem 6) given in Appendix 2. There we weaken additive separability into a condition requiring that the cross effects are not too large. On the other hand, we have to impose a joint condition on preferences *and* the production function which automatically holds for additive utility functions, and production functions satisfying Assumption 1.

If labor does not affect the consumers' utility and is supplied inelastically we can show that under fairly general conditions first best consumption decreases. For this purpose we make

[2]One might think of the environment as a fixed factor. However, in example 4 below the first best emissions increase without bound as the population increases.

Assumption 2 *The consumers' preferences are represented by a twice differentiable, quasi-concave utility function $U : (q, E) \mapsto U(q, E)$, satisfying $U_q > 0$, $U_E < 0$, and*

$$U_{qq}U_E - U_{qE}U_q > 0 \qquad (5)$$

$$U_{EE}U_q - U_{qE}U_E < 0 \qquad (6)$$

Note that the conditions (5) and (6) imply that the utility function is quasi-concave, but not all quasi-concave utility functions satisfy these conditions. With this assumption both, consumption and abatement, would be normal goods if the consumer were a price-taker for these goods.

Theorem 3 *If labor supply is inelastic and Assumption 2 holds, and f is a strictly concave function of pollution only, then the first best per capita consumption is decreasing in the population size.*

Proof: See Appendix 2.

If Assumption 2 is not satisfied, it can happen that first best per capita consumption *increases* with population size. This occurs for example 4 of table 2 in the next section. In that example, both emissions and consumption increase with a larger population.

4 First Best Environmental and Labor Policy

We now consider how the first best levels of labor and emissions vary with population size. The choice of these levels depends on standard economic trade-offs. If population increases but the aggregate level of emissions and per capita labor are held fixed, then per capita consumption falls (see (1)). Consumption could be increased by increasing emissions. It would be desirable to do so if the extra utility from consumption compensated for the additional pollution disutility. If, on the other hand, the disutility from pollution were high, it might be desirable to maintain the emission level, but for the consumers to work more to increase production and/or reduce pollution. Table 1 presents a taxonomy of various pollution and labor policies in response to an increase in population. We use the notation E' and l' to denote the signs of changes in the first best levels of emissions and labor when the population becomes larger.

It is not possible to tell in general whether emissions should be reduced or increased as population increases. It is equally ambiguous as to whether people

	$E' \leq 0$	$E' > 0$
$l' \leq 0$	Make love, not pollution (1,3)	Relax, and don't worry (2b)
$l' > 0$	Conservation (2a)	Bountiful (4)

(Numbers in parentheses refer to examples listed in table 2.)

Table 1: Environmental and Labor Policies

should work harder or relax more. Table 2 presents examples of utility and production functions to illustrate this. Apart from example 4, all other examples employ quite standard utility and production functions, widely accepted as typical in the literature. Examples 1 through 2b have Cobb–Douglas preferences in consumption, leisure and abatement. In example 3, utility is additively separable into a concave function of consumption and into convex disutility in labor, and emissions, respectively. Production functions are Cobb–Douglas, CES, or Leontief. The examples are solved in Appendix 1.

Example	$U(q,l,E)$	$f(l,e)$	Parameter restrictions
1	$q^\alpha(\bar{l}-l)^\beta(\overline{E}-E)^\gamma$	$l^\delta e^\theta$	$0<\alpha,\beta,\gamma,\delta,\theta<1$
2a	$q^\alpha(\bar{l}-l)^\beta(\overline{E}-E)^\gamma$	$(l^\rho+e^\rho)^{1/\rho}$	$0<\alpha,\beta,\gamma<1, 0<\rho<1$
2b	$q^\alpha(\bar{l}-l)^\beta(\overline{E}-E)^\gamma$	$(l^\rho+e^\rho)^{1/\rho}$	$0<\alpha,\beta,\gamma<1, \rho<0$
3	$\frac{a}{\alpha}q^\alpha-\frac{b}{\beta}l^\beta-\frac{c}{\gamma}E^\gamma$	$A\cdot l^\delta e^\zeta$	$0<\alpha,\delta,\zeta<1, \beta,\gamma>1, A>0$
4	$(q-1)/(E-1)^2$	$\min\{l,e\}$	$q,E>1$

$\bar{l}>0$ is the per capita labor endowment.

$\overline{E}>0$ is a pollution level at which consumers die.

Table 2: Examples

For example 2a, when population increases it is optimal to restrict emissions and for consumers to work more. We refer to this policy as the "Conservation" policy. Conversely, in example 2b when the population is larger it is optimal for consumers to enjoy more leisure but at the same time to cope with more pollution (to support production). We refer to this as the "Relax, and don't worry (about pollution)" policy[3]. One could also call it the "traditional growth economists'" policy, since standard growth theory models ignore pollution and predict that labor will be reduced by virtue of technical progress.

[3]This is a bit loose since the *optimal* pollution level may be fall short of the *real* or *laissez faire* pollution levels.

Note that the only difference between examples 2a and 2b is that the elasticity of substitution between labor and emissions is greater than 1 in the former, and less than 1 in the latter.

In example 3 the optimal policy is to work and pollute less. This results in lower per capita consumption. We refer to this as the "Make love, not pollution" policy[4]. The counterpart to this policy is obtained in example 4 where it is optimal to increase both labor and emissions as population grows. We refer to this as the "bountiful" policy. Note, however, the preferences represented by the utility function in example 4 have a special feature. Consumption is extremely inferior against abatement. These preferences are such that consumption would be a Giffen good in a standard model of budget constrained choice, with prices for consumption and emissions. So here, even though consumption becomes socially more expensive through population pressure, the first best level of consumption goes up as population rises. In the previous section we have shown that this cannot happen if consumption and abatement are normal goods, and if there is no disutility from working (Theorem 3), or if preferences are additively separable (Theorem 2).

Finally, example 1 with Cobb Douglas preferences and technology represents a kind of knife-edge case where labor per capita and aggregate pollution are constant, independent of population size. Such a scenario would be convenient for policy makers since constant emissions over time are much more easily implemented than those which require continual adjustment of standards, effluent taxes or tradeable permits.

If the reader studies the proof of Theorem 6 in Appendix 2, she or he will see from the formulas for l_n' and E_n', in particular from the numerators given by (29) and (30), that there is not much hope to find simple conditions for labor per capita and aggregate pollution being monotonic in n.

5 Implementation by Taxes[5]

Assume now that there is a competitive output market for the only consumption good, with output price p, and a competitive labor market with wage w. The

[4]Assuming that more love does not increase population endogenously!

[5]We are grateful to Edward Morey for suggesting this section.

government sets a tax τ per unit of emissions and redistributes tax revenues lump sum to the consumers. Consumers also receive the profits of the firm, denoted by π. Then the household maximizes $U(q, l, E)$ over q, l s.t. $pq \leq wl + \pi + \tau\frac{E}{n}$, leading to $U_q/U_l = p/w$. The firms maximize $\pi(l, E) = pf(l, E/n) - wl - \tau E/n$, giving $f_l = w/p$ and $f_e = \tau/p$.

From the first order conditions of the social optimum, in particular from (2) it follows immediately that in order to implement the social optimum the regulator has to set the tax in the following way.

$$\tau_n = f_e(l_n, E_n/n) = -n\frac{U_E(q_n, l_n, E_n)}{U_q(q_n, l_n, E_n)} \tag{7}$$

In the following we show that under mild restrictions on preferences and technology the emission tax is increasing as population grows.

Theorem 4 *If U is concave and additively separable, and the technology satisfies Assumption 1, then $\partial\tau/\partial n > 0$.*

This result is a corollary of the more general Theorem 7 in Appendix 2, which employs weaker but more technical assumptions on the preferences.

Analogously to Theorem 3 we make a statement about the model with inelastic labor supply:

Theorem 5 *If labor supply is inelastic and Assumption 2 holds, and f is a strictly concave function of pollution only, then $\partial\tau/\partial n > 0$. If f has constant returns to scale, i.e. $f'' = 0$, the tax does not depend on n.*

Proof: see Appendix 2.

So, whereas we cannot tell in general whether emission quotas should be increased or decreased as population grows, at least we can say that the under fairly general conditions the social price for pollution, reflected by the optimal emission tax, goes up if there are more people. This immediately implies that, if the number of firms rises proportional to the population, each single firm emits less. Whether the aggregate emission level, which is implemented by the optimal taxes, goes up or down, of course, remains ambiguous.

6 Conclusions

We have shown that consumers are worse off as population grows in the first best outcome of an economy with a pollution externality. This result is quite Malthusian

although there is no resource limit in the model presented here. The deterioration is not due to resource scarcity. The environment is in some sense a scarce resource. However, in example 4 emissions increase without bound. To the extent that environmental externalities are important, our result is at odds with the predictions of neo-classical growth theory, which suggests that balanced growth is possible with no deterioration in consumer well-being.

We would like to offer advice to policy-makers concerning desirable pollution levels depending on population size. However, we have shown by example that optimal policy depends critically on the nature of preferences and technology. So no general advice can be offered. It may be desirable to allow pollution levels to increase as population grows. Conversely, it may be better to reduce pollution levels, and to promote more labor intensive and cleaner production.

We have only considered first best outcomes. There is a rich literature on decentralized methods of attempting to achieve these in the static setting considered here. There remains substantial work to be done on decentralization if one recognizes the dynamic aspects of population growth and environmental processes. A dynamic model could also include accumulation of capital, which might compensate for increased population pressure. Verification of this conjecture would be an interesting next step.

Appendix 1 - Examples

This appendix gives the solutions of the examples presented in the paper.

Example 1 : Since utility is increasing in q, the production constraint will bind at the first best allocation. Letting λ be the multiplier on this constraint the first order conditions are

$$\frac{\alpha u}{q_n} = \lambda$$

$$\frac{\beta u}{\bar{l} - l_n} = \frac{\lambda \delta q_n}{l_n}$$

$$\frac{\gamma u}{\overline{E} - E_n} = \frac{\lambda \theta q_n}{E_n}$$

$$q_n = l_n^{\delta}\left(\frac{E_n}{n}\right)^{\theta} .$$

These are sufficient given our convexity assumptions. The labor and emissions which solve the first order conditions are

$$l_n = \frac{\alpha\delta}{\alpha\delta + \beta}\ \bar{l}$$

$$E_n = \frac{\alpha\theta}{\alpha\theta + \gamma}\ \overline{E} .$$

The first best labor per capita and aggregate emissions are independent of the population size. Consumers are worse off in this example solely because per capita consumption, $f(l_n, E_n/n)$, falls with n.

Example 2 : The first order conditions for this example, which are sufficient, are

$$\frac{\alpha u}{q_n} = \lambda \tag{8}$$

$$\frac{\beta u}{\bar{l} - l_n} = \lambda l_n^{\rho-1}\left[l_n^{\rho} + \left(\frac{E_n}{n}\right)^{\rho}\right]^{\frac{1}{\rho}-1} \tag{9}$$

$$\frac{\gamma u}{\overline{E} - E_n} = \lambda n^{-\rho} E_n^{\rho-1}\left[l_n^{\rho} + \left(\frac{E_n}{n}\right)^{\rho}\right]^{\frac{1}{\rho}-1} \tag{10}$$

$$q_n = \left[l_n^{\rho} + \left(\frac{E_n}{n}\right)^{\rho}\right]^{1/\rho} . \tag{11}$$

Although there are no closed form solutions for q_n, l_n, and e_n, one can give fairly complete characterization, as follows. Dividing (10) by (9) gives

$$\frac{\gamma(\bar{l} - l_n)}{\beta(\overline{E} - E_n)} = n^{-\rho}\left(\frac{E_n}{l_n}\right)^{\rho-1} . \tag{12}$$

Dividing (9) by (8), and substituting for q_n using (11) gives

$$\frac{\beta}{\alpha}\frac{l_n^{\rho} + (E_n/n)^{\rho}}{\bar{l} - l_n} = l_n^{\rho-1} .$$

Multiplying by $\alpha l_n^{1-\rho}/\beta$ gives

$$\frac{l_n + E_n n^{-\rho}(E_n/l_n)^{\rho-1}}{\bar{l} - l_n} = \frac{\alpha}{\beta} .$$

Now use (12) to substitute for $n^{-\rho}(E_n/l_n)^{\rho-1}$ to give

$$\frac{l_n}{\bar{l} - l_n} + \frac{\gamma}{\beta}\frac{E_n}{\overline{E} - E_n} = \frac{\alpha}{\beta} \tag{13}$$

This equation shows that at the optimal solution l_n and E_n are inversely related[6]. That is, if per capita labor, l_n, increases with n then aggregate pollution, E_n, decreases, and vice versa.

Now it is a matter of determining which of labor or emissions increases with the population size. To do so, rearrange (12) to give

$$\frac{\beta E_n^{\rho-1}(\overline{E} - E_n)}{\gamma l_n^{\rho-1}(\bar{l} - l_n)} = n^{\rho} .$$

If $0 < \rho \leq 1$ then the right hand side increases with n. This means that E_n must fall with n, since $E_n^{\rho-1}(\overline{E} - E_n)$ is a decreasing function of E_n for values less than $\overline{E}$, and E_n and l_n are inversely related. That is, the first best level of pollution falls and the per capita labor rises as the population increases if ρ is between 0 and 1.

If $\rho < 0$, then the right hand side of the last equation decreases with n, and so the first best level of pollution *rises* with n and the per capita labor supply falls.

Example 3: The first order conditions of social optimum imply

$$E_n = K \cdot n^{\epsilon} \tag{14}$$

$$l_n = \left[\frac{a}{b}\delta A^{\alpha}\right]^{\frac{1}{\beta-\alpha\delta}} \cdot \left[\frac{E}{n}\right]^{\frac{\alpha\zeta}{\beta-\alpha\delta}} \tag{15}$$

where K is a constant depending on $a, b, c, A, \alpha, \beta, \gamma, \delta, \zeta$ (we omit the formula), and the exponent of n is given by

$$\epsilon = \frac{1}{1 - \frac{\gamma(\beta-\alpha\delta)}{\alpha\beta\zeta}} .$$

[6]Note that $l/(\bar{l} - l)$ is increasing in l between 0 and $\bar{l}$.

ε is negative, if $\frac{\gamma(\beta-\alpha\delta)}{\alpha\beta\zeta} > 1 \Leftrightarrow \alpha\beta\zeta < \gamma(\beta-\alpha\delta) \Leftrightarrow \alpha[\beta\zeta+\gamma\delta] < \gamma\beta$. But this holds since $\alpha[\beta\zeta+\gamma\delta] < \alpha\cdot\max\{\beta,\gamma\}(\zeta+\delta) < \max\{\beta,\gamma\} < \gamma\beta$ since $\beta,\gamma > 1$, and $\alpha,\delta,\zeta < 1$, and $\delta+\zeta \le 1$. Since E_n is decreasing, and since $\beta > \alpha\delta$, it must be the case that l_n and hence also q_n are decreasing.

Example 4: In this example, the utility function does not depend on labor. The problem of solving for the symmetric Pareto efficient allocation is

$$\max_{q,E\geq 1} \frac{q-1}{(E-1)^2} \quad \text{s.t.} \quad q \le \min\{l, E/n\} \quad .$$

Providing that $n \ge 2$ the optimum is $E_n = 2n-1$ and $l_n = q_n = 2 - 1/n$. Note that emissions increase without bound as the population increases.

Appendix 2 – Generalizations and Proofs of the Comparative Statics Results

We will first present more general versions of Theorems 2 and 4, and then jointly prove both of them. For this purpose we make the following assumption which generalizes Assumption 2 to the case of three commodities.

Assumption 3 *The consumers' utility function U is twice differentiable, monotonic, quasi-concave and satisfies:*

$$A := U_{qq}U_l - U_{ql}U_q > 0\,, \quad (16)$$
$$B := U_{qq}U_E - U_{qE}U_q > 0\,, \quad (17)$$
$$C := U_{EE}U_q - U_{qE}U_E < 0\,, \quad (18)$$
$$D := U_{ll}U_q - U_{ql}U_l < 0\,, \quad (19)$$
$$G := U_{ql}U_E - U_{lE}U_q \ge 0\,, \quad (20)$$
$$H := U_{qE}U_l - U_{lE}U_q \ge 0\,, \quad (21)$$
$$A\cdot G + B\cdot D < 0\,, \quad (22)$$
$$A\cdot C + B\cdot H < 0\,, \quad (23)$$

The inequalities (16) – (23) are a subset of sufficient conditions for consumption, leisure and abatement of pollution to be *normal* goods if the consumer were a price-taker for these goods.

Theorem 6 *If U satisfies Assumption 3 and f satisfies Assumption 1 and if each feasible triple (q,l,E) satisfies the following joint condition:*

$$(U_{ll}U_q - U_{ql}U_l)f_E + (U_{qE}U_l - U_{lE}U_q)f_l < 0 \quad , \quad (24)$$

then q_n is decreasing in n.

If U is additively separable, Assumption 3 will be satisfied, and (24) will reduce to $U_{ll}U_q f_E < 0$ which obviously holds. Thus Theorem 6 implies Theorem 2.

For the optimal tax we get:

Theorem 7 *If preferences satisfy Assumption 3 and technology satisfies Assumption 1, then $\partial\tau/\partial n > 0$.*

Note that we can dispense with the joint condition (24). Theorem 7 also implies immediately Theorem 4.

Proof of Theorems 6 and 7: Denote by q_n', l_n', and E_n' the derivatives with respect to n of q_n, l_n, and E_n, respectively. Differentiating (2), (3) and (4) with respect to n we get:

$$q_n' = f_l l_n' + f_e(nE_n' - E)/n^2 \quad (25)$$
$$-U_E/U_q + n(Bq_n' + Gl_n' - CE_n')/U_q^2 = f_{el}l_n' + f_{ee}(E_n'n - E)/n^2 \quad (26)$$
$$(Bq_n' - Dl_n' + HE_n')/U_q^2 = f_{ll}l_n' + f_{le}(E_n'n - E)/n^2 \quad (27)$$

Using Cramer's rule and rearranging we get:

$$q_n' = J/N\,, \quad l_n' = K/N\,, \quad E_n' = L/N\,,$$

where

$$\begin{aligned} J = {} & nf_eE(CD - GH) + nU_qU_E(Df_e + nHf_l) \\ & + U_q^2E(H[f_{le}f_e - f_{ee}f_l] + nC[f_{ll}f_e - f_{le}f_l]) \\ & + nU_q^3U_E(f_{ll}f_e - f_{le}f_l) \end{aligned} \quad (28)$$
$$\begin{aligned} K = {} & nE(AC + BH) + nU_qU_E(Af_e + nH) \\ & + U_q^2E(Hf_{ee} - nCf_{le}) - nU_q^3U_Ef_{le} \end{aligned} \quad (29)$$
$$\begin{aligned} L = {} & nf_eE(AG + BD) + U_q^4E(f_{le}^2 - f_{ll}f_{ee}) \\ & + U_q^2E\{nB[f_{ll}f_e - f_{le}f_l] + A[f_{ee}f_l - f_{le}f_e] - Df_{ee} - nGf_{le}\} \\ & + n^2U_qU_e(D - Af_l + U_q^2f_{ll}) \end{aligned} \quad (30)$$
$$N = n\{f_en(AG + BD) + n^2(GH - CD) + n^2f_l(AC + BH)$$

$$+U_q^2[A(f_{ee}f_l - f_{le}f_e) - f_{ee}D$$
$$+nB(f_{ll}f_e - f_{le}f_l) - nf_{le}(G+H) - n^2Cf_{ll}]$$
$$+U_q^4[f_{le}^2 - f_{ee}f_{ll}]\} \tag{31}$$

First we show that N is negative. To see this observe first that (22) and (23) imply that $GH - CD$ in the second term of (31) is negative. Moreover, Assumption 1 implies

$$f_{ee}f_l - f_{le}f_e \le 0\,, \tag{32}$$
$$f_{ll}f_e - f_{le}f_l \le 0\,. \tag{33}$$
$$f_{ee}f_{ll} - f_{le}^2 \ge 0 \tag{34}$$

Using (16) to (23), and $U_q > 0$, $U_l < 0$, and $U_E < 0$, it can be easily checked by inspection that all the other terms of (31) are negative.

The terms of the numerator J all are positive by (16) - (23), and (32) - (34), apart from the second one. Now $Df_e + nHf_l < 0$ is equivalent to $Df_e/n + Hf_l = Df_E + Hf_l < 0$, which is condition (24). This proves Theorem 2.

Since the optimal tax has to satisfy $\tau_n = f_e$, we get

$$\tau_n' = f_{le}l_n' + f_{ee}\left[\frac{E_n'n - E}{n^2}\right]. \tag{35}$$

Substituting from (29), (30), and (31) into (35) and rearranging we obtain:

$$\tau_n' = M/N \tag{36}$$

where

$$M = n\cdot\{[AC + BH]\cdot[f_{le}f_e - f_{ee}f_l] + f_{ee}H[CD - GH]$$
$$+U_qU_E[A(f_{le}f_e - f_{ee}f_l)] + Df_{ee} + nHfle$$
$$+U_q^2[CE + U_qU_e][f_{ee}f_{ll} - f_{le}^2]\} \tag{37}$$

By inspection one can check again that each term is negative by (16) to (23), and (32) to (34). Since M and N are negative, τ_n must be increasing. This proves Theorem 4. □

Proof of Theorems 3 and 5:

Since labor is supplied inelastically, we abuse notation by writing $f(e)$ for $f(\bar{l}, e)$, where $\bar{l}$ is the per capita labor endowment. The first order conditions for Pareto–optimality yield

$$-\frac{U_q}{U_E} = \frac{n}{f'(E_n/n)} \tag{38}$$
$$q_n = f(E_n/n) \tag{39}$$

Differentiating both equations with respect to n yields

$$-\frac{(U_{qq}q_n' + U_{qE}E_n')U_E - (U_{qE}q_n' + U_{EE}E_n')U_E}{[U_E]^2} = \frac{1}{f'} - \frac{f''}{[f']^2}\left[E_n' - \frac{E_n}{n}\right]$$
$$q_n' = \frac{f'}{n^2}[E_n'n - E_n]$$

By rearranging we obtain

$$-\left[\frac{U_{qq}U_E - U_{qE}U_q}{[U_E]^2}\right]q_n'$$
$$-\left[\frac{U_{qE}U_E - U_{EE}U_q}{[U_E]^2} - \frac{f''}{[f'(e_n)]^2}\right]E_n' = \frac{1}{f'} - \frac{f''}{[f']^2}\frac{E_n}{n} \tag{40}$$
$$E_n' = \frac{q'n}{f'} + \frac{E_n}{n} \tag{41}$$

Eliminating E_n' and solving for q' yields

$$q' = -\frac{\frac{[U_E]^2}{f'} + W}{V + \frac{n}{f'}\left[W - \frac{f''}{[f']^2}\right]} \tag{42}$$

where

$$V = U_{qq}U_E - U_{qE}U_q\,,$$
$$W = U_{qE}U_E - U_{EE}U_q\,.$$

V and W are positive by assumption. Since $f' > 0$, $f'' \le 0$, it follows that $q' < 0$. It is clear from (41) that the sign of E_n' is ambiguous. This proves Theorem 3.

To prove Theorem 5, we differentiate $\tau(n) = f_e$, which gives

$$\tau_n' = f''\cdot\left[\frac{E_n'n - E_n}{n^2}\right] \tag{43}$$

Now substituting from (42) into (41) and solving for E_n' yields

$$E_n' = \frac{-\frac{U_E}{U_q} + \left[\frac{f''}{n} - \frac{Vf'}{U_q^2}\right]\frac{E_n}{n}}{\frac{f''}{n} - [Vf' + nW]\frac{1}{U_q^2}} \tag{44}$$

Substituting this expression into (43), yields

$$\tau_n' = \frac{f''}{nU_q}\left[\frac{E_nW}{U_q} - U_E\right]\frac{1}{Q} \tag{45}$$

where

$$Q = \frac{f''}{n} - \frac{1}{U_q^2}[Vf' + nW] \tag{46}$$

Now since $V > 0$ and $W > 0$ by Assumption 2, Q is negative if $f'' < 0$. Since the term in brackets from (45) is positive, the tax is increasing if $f'' < 0$. If $f'' = 0$, obviously $\tau_n' = 0$. This proves Theorem 5. □

References

[1] R. Gradus and S. Smulders, (1993), "The trade-off between environmental care and long-term growth — pollution in three prototype growth models", *Journal of Economics* **58**, 25–51

[2] G. Gruver, (1976), "Optimal investment in pollution control capital in a neo-classical growth context", *Journal of Environmental Economics and Management* **3**, 165–177

[3] E. Keeler, M. Spence, and R. Zeckhauser, (1972), "The optimal control of pollution", *Journal of Economic Theory* **4**, 19-34

[4] O. Tahvonen and J. Kuuluvainen, (1993), "Economic growth, pollution, and renewable resources", *Journal of Environmental Economics and Management* **24**, 101–118

[5] T. Malthus, (1798), "An essay on the principle of population — or a View of its past and present effects on human happiness", London, Reeves and Turner, 7th ed. (1. ed. 1798), reprinted (1971) by Augustus M. Kelly Publishers, Reprints of Economic Classics, New York, New York