INSTITUTE OF MATHEMATICAL ECONOMICS
WORKING PAPERS

No. 213

Star–Shapedness of the Kernel for Homogeneous Games and Application to Weighted Majority Games

by
Peter Sudhölter

September 1992

University of Bielefeld
4800 Bielefeld, Germany

Abstract

Homogeneous games and weighted majority games were introduced by von Neumann–Morgenstern (1944) in the constant–sum case. Peleg (1966, 1968) studied the kernel and nucleolus for these classes of games. The general theory of homogeneous, not necessarily constant–sum, games was developed by Ostmann (1987a), Rosenmüller (1982, 1984, 1987), and Sudhölter (1989). Peleg–Rosenmüller (1992) used it to discuss several solution concepts for homogeneous games without steps. A reduction theorem for the nucleolus and kernel of homogeneous games with steps was proved by Rosenmüller–Sudhölter (1992) and Peleg–Rosenmüller–Sudhölter (1992) respectively. Based on these results, this paper shows that the kernel of each homogeneous game is star–shaped and that the kernel of a weighted majority game arises from the one of a certain homogeneous game in a canonical way. The weighted majority game occurs as a reduced game of this homogeneous extension. Moreover, the kernels of partition games turn out to be singletons.

0. Introduction

Two classes of simple games, the weighted majority games and the subset of homogeneous games, are considered in this paper. A simple game is a cooperative multi–person game in which each coalition either wins, i.e. obtains a fixed positive payoff, or loses, i.e. obtains no payoff. If it is possible to separate winning coalitions from losing ones by assigning non–negative weights to the players such that the aggregated weight of each winning coalition exceeds or is equal to a positive level, whereas the weight of each losing coalition is less than the level, the game is a weighted majority game. The vector which consists of both, the level and the weights, is a representation of the game. If, in addition, there is a representation such that each winning coalition contains a "smallest" winning coalition, i.e. a minimal winning one, with a weight exactly hitting the level, then the game is homogeneous. For the explicit definitions Section 1 is referred to.

The terms "simple", "weighted majority", and "homogeneous" were introduced by von Neumann–Morgenstern (1944). They, however, were dealing with constant–sum games only. Both, simple and weighted majority games, appear in many applications of game theory (see, e.g., Shapley (1962)). Concerning the structure of homogeneous games Isbell (1956, 1958, 1959) and Peleg (1968) in the constant–sum case, Ostmann (1987a), Rosenmüller (1982, 1984, 1987), and Sudhölter (1989) in the general case, should also be mentioned.

Sections 1–4 are organized as follows. Section 1 presents the notation, partially adopted from Peleg–Rosenmüller–Sudhölter (1992). Moreover, necessary foundations and results concerning weighted majority games and, in particular, homogeneous games are summarized.

In Section 2 the main result of this paper, Theorem 2.3, is stated and proved. It turns out that the kernel of a homogeneous game is star–shaped and that the center of this set coincides with the normalized vector of weights of the minimal representation. Peleg (1966) proved the same assertion for a subclass of the homogeneous constant–sum games – the partition games –, but his proof cannot be generalized to the class of homogeneous games or even of those with the constant–sum property in an obvious way. Basically, it is shown that the maximum surplus of one player over another is attained by a minimal winning coalition as in the just mentioned paper; but the approaches are

totally different and the characterization of homogeneous games via "incidence" vectors (see Sudhölter (1989)) plays an important role in this paper. Additionally, the kernel of a partition game, indeed, is a singleton as proved in Section 4.

Section 3 shows that the kernel of an arbitrary weighted majority games is a "canonical" image of the kernel of a homogeneous game called homogeneous extension. Therefore it is sufficient to restrict the attention to homogeneous games in the class of weighted majority games with respect to the kernel.

Finally, at the end of Section 4, some illustrating examples are presented.

1. Notation

During this paper let $\mathbb{N} = \{1,2,3,...\}$ denote the universe of players. Finite subsets of $\mathbb{N}$ are called **coalitions**, **intervals** are subsets of $\mathbb{N} \cup \{0\}$ of the form

$$[a,b] = \{i \in \mathbb{N} \cup \{0\} \mid a \leq i \leq b\},$$

where $a, b \in \mathbb{N} \cup \{0\}$.

The **grand coalition** is an interval $\Omega = \Omega_n = [1, n]$.
If

$$v: \mathcal{P}(\Omega) \to \{0,1\}, \ v(\emptyset) = 0,$$

where

$$\mathcal{P}(\Omega) = \{S \mid S \subseteq \Omega\},$$

is a mapping (the **characteristic function**) then $(\Omega, \mathcal{P}(\Omega), v)$ is a **simple n–person game**. Since the nature of Ω and $\mathcal{P}(\Omega)$ is determined by the characteristic function, v is called simple game as well. A coalition $S \subseteq \Omega$ is often identified with the indicator function 1_S, considered as n–vector. A coalition S is **winning**, if $v(S) = 1$, and **losing**, otherwise. The set of winning coalitions is abbreviated by W_v.

In a **monotone** simple game all subcoalitions of losing coalitions are losing as well. If each proper subcoalition of a winning coalition is a losing one, then this winning coalition is called a **minimal** winning (**min–win**) coalition. It should be noted that a monotone simple game is completely determined by the set of min–win coalitions, denoted by W_v^m. To simplify matters we exclude the "degenerate" monotone simple games having no winning coalitions at all.

Let v be a simple game. The relation $\precsim \subseteq \Omega^2$, defined by $i \precsim j$, if $v(S \cup \{i\}) \leq v(S \cup \{j\})$, for all $S \subseteq \Omega \setminus \{i,j\}$ is called **desirability relation** of v (see Maschler–Peleg (1966)). Note that $\precsim$ is a relation w.r.t. players which can be generalized to coalitions (see e.g. Einy (1985)).

If $i \sim j$ (i.e. $i \precsim j$ and $j \precsim i$), then i and j are interchangeable or of the same type. A monotone simple n–person game v is an ordered game if its desirability relation is complete. An ordered game is a **directed** game if additionally

$$n \precsim n-1 \precsim ... \precsim 1$$

is valid. Concerning this notation it is referred to Ostmann (1987b), Ostmann (1989), and Krohn–Sudhölter (1990).

If a simple game is ordered, it is always assumed that it is directed since this can be enforced by just renaming the players.

A **weighted majority game** (with n players) is a simple n–person game having a **representation** $(\lambda; m)$, i.e. a level $\lambda \in \mathbb{R}_{>0}$ and a vector of weights – a measure – $m \in \mathbb{R}^n_{\geq 0}$ such that

$$v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda \\ 0, & \text{if } m(S) < \lambda \end{cases}.$$

Here, we use $m(S) = \sum_{i \in S} m_i \ (S \subseteq \Omega)$ and call m(S) the **weight** of coalition S. Clearly $i \precsim j$, if $m_i \leq m_j \ (i, j \in \Omega)$, is valid in this case and thus v is directed by monotonicity and the above assumption. That is to say, there exists a representation $(\lambda;m)$ of v such that $i < j$ implies $m_i \geq m_j \ (i, j \in \Omega)$.

A measure $m \in \mathbb{R}^n_{\geq 0}$ is said to be **homogeneous** w.r.t. $\lambda \in \mathbb{R}_{>0}$ – written m hom λ – if, for any $T \subseteq \Omega$ with $m(T) \geq \lambda$, there is $S \subseteq T$ with $m(S) = \lambda$.

A weighted majority game is **homogeneous** if it has a **homogeneous representation**. That is a representation $(\lambda;m)$ with m hom λ and $m(\Omega) \geq \lambda$. We write

$$\ell(S) = \max S$$

for $S \subseteq \Omega$, sometimes calling this the **length** of S. Note that in a directed game v, $\ell(S)$ is a "weakest" player of coalition S w.r.t. the desirability relation.

Let v be a directed game. There is a unique min–win coalition with minimal length. This coalition is an interval of the form $[1,t]$ and the **lexicograghically maximal** (**lex–max**) min–win coalition of v. Player $i \in \Omega$ is a **null player** if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \Omega$, and a **winning player**, if $v(\{i\}) = 1$. Moreover i is a **veto player** if each winning coalition contains this player. Clearly winning players are interchangeable as null players and veto players are. Types of players establish a decomposition of Ω.

There is another decomposition of Ω in the case of a homogeneous n–person game v into sets of players of equal character. Let $(\lambda;m)$ be a homogeneous representation of v. There are three characters, called "sum", "step", and again "null player". The definition of a null player was given above and remains unchanged. So the two others have to be defined. Fix a non null player $i \in \Omega$ and consider the minimal length of min–win coalitions containing i, say

$$\ell^{(i)} := \min \{\ell(S) \mid i \in S \in W_v^m\}.$$

The domain of i is

$$C^{(i)} := [\ell^{(i)}+1, n].$$

Player i is a sum, if

$$m_i \geq m(C^{(i)})$$

and otherwise i is a step.

A sum can be replaced in at least one min–win coalition by a coalition of smaller players, the weight being exactly the sum of the weights of these smaller players by the homogeneity of $(\lambda;m)$. On the other hand, "steps rule their followers", i.e., whenever a smaller player – a player with a larger index – is a member of a min–win coalition, any preceding step also is a member.

Note that a winning player may be sum or step, whereas a veto player is a step. A homogeneous game may have no null players or sums (e.g. the unanimous game of the grand coalition) but steps are always present. The smallest non null player is always a step. To simplify matters, we say that v is a homogeneous game without steps if this player is the only step.

Now the necessary definitions concerning the "(pre–)kernel" are recalled. Let v be a directed n–person game.

Definition 1.1: $X^* = \{x \in \mathbb{R}^n \mid x(\Omega) = 1\}$ is the set of pre–imputations. For different players $i, j \in \Omega$ we write

$$T_{ij} = \{S \subseteq \Omega \mid i \in S \not\ni j\}.$$

Let

$$e(S,x,v) := e(S,x) := v(S) - x(S)$$

denote the excess of $S \subseteq \Omega$ at $x \in \mathbb{R}^n$ w.r.t. v.

The maximal excess of $x \in \mathbb{R}^n$ w.r.t. v is

$$\mu(x) := \mu(x,v) := \max_{S \subseteq \Omega} e(S,x);$$

and

$$s_{ij}(x) := s_{ij}(x,v) := \max_{S \in T_{ij}} e(S,x)$$

is the maximum surplus of i over j.

The corresponding systems of coalitions reaching maximal excess or maximum surp are given by

$$\mathscr{D}(x) = \mathscr{D}(x,v) = \{S \subseteq \Omega \mid e(S,x) = \mu(x)\}$$

and

$$\mathscr{D}_{ij}(x) = \mathscr{D}_{ij}(x,v) = \{S \in T_{ij} \mid e(S,x) = s_{ij}(x)\}.$$

The pre–kernel of v is given by

$$\mathscr{P\!K}(v) = \{x \in X^* \mid s_{ij}(x) = s_{ji}(x) \ (i,j \in \Omega, i \neq k)\}.$$

The kernel is the set

$$\mathscr{K}(v) = \{x \in X^* \mid x_j \geq v(\{j\}) \text{ and } (s_{ij}(x) \leq s_{ji}(x) \text{ or } x_j = v(\{j\})), i,j \in \Omega, i \neq j\}.$$

The kernel was introduced by Davis–Maschler (1965), see also Maschler–Peleg–Shap (1979), Maschler–Peleg (1966, 1967), and Peleg (1966).

It is obvious that a game which arises by dropping some or all null players inherits directedness, the weighted majority property, and the homogeneity respectively. Mo over the (pre–)kernel of the new game arises from the original one by dropping corresponding zero components of each element. Therefore only directed games with null players are considered from now on.

By Corollary 1.7, Theorems 2.1 and 2.2 of Peleg–Rosenmüller–Sudhölter (1992) we assume that veto players and winning players are absent. Then the pre–kernel a kernel coincide. The reduction theorem (Theorem 4.5) of the same paper allows to strict the attention to homogeneous games without non interchangeable steps in the homogeneous case. Therefore we assume from now on that each considered homogeneous game is a one without steps of different type, without null players, without winning players, and without veto-players. To the end of this section some important assertions and definitions concerning homogeneous games are recalled.

Remark 1.2:

1. A homogeneous game v has a unique minimal representation – i.e., an integer valued $(\bar{\lambda};\bar{m})$ representing v such that $\bar{m}(\Omega)$ is minimal among all integer representations of v – which is automatically homogeneous itself (see Ostmann (1987a) and Rosenmüller (1982)). Moreover $\bar{m}_i = \bar{m}_j$, iff i and j are interchangeable and $\bar{m}_k \geq \bar{m}_{k+1}$ $(i,j,k \in \Omega,\ k < n)$.

2. Let $(\lambda;m)$ be a homogeneous representation of the homogeneous game v and $S \in W_v^m$.

   If the length of S is minimal among all min-win coalitions, then S is an interval $[1,\ell(S)]$ and thus the lex-max min-win coalition.

   The set

   $$\{i \in \Omega \mid S \not\ni i < \ell(S)\}$$

   is the set of dropouts of S. If S is not the lex-max min-win coalition, then S possesses dropouts. In this case the last dropout is denoted by

   $$r(S) = \max\{i \in \Omega \mid i \text{ dropout of } S\}.$$

   Clearly there exists a unique $t \in [r(S)+1, \ell(S)]$ such that

   $$\varphi(S) := S \cup \{r(S)\} \setminus [t, \ell(S)]$$

   is a min-win coalition. That means, φ inserts the last dropout and cuts off a tail of S to generate a min-win coalition. The aggregated weight of this tail coincides with the weight of player r(S) by homogeneity. If α is the number of dropouts of S, then $\varphi^{\alpha}(S)$ – i.e. the α iterate of φ applied to S – coincides with the lex-max min-win coalition.

   To define the "inverse" map, let $j < \ell(S)$ such that

   $$[j,\ell(S)] \subseteq S \text{ and } S \setminus \{j\} \cup [\ell(S)+1,n] \in W_v.$$

   Then j is expendable in S, i.e. replaceable by a "tail" $[\ell(S)+1,t]$. To be more precise, t is defined to be the player such that

   $$\rho_j(S) := S \setminus \{j\} \cup [\ell(S)+1,t]$$

   is a min-win coalition. Again the aggregated weight of the tail coincides with the weight of j.

   Clearly

   $$\rho_{r(S)}(\varphi(S)) = S, \text{ if } S \text{ is not lex-max,}$$

   and

   $$\varphi(\rho_j(S)) = S, \text{ if } j \text{ is expendable in } S.$$

3. Let k be a player of the homogeneous game v such that all persons 1,...,k are sums. Then there exists a sequence of min-win coalitions $S_1,\ldots,S_{k+1} \in W_v^m$ such that the following conditions are satisfied:

   (i) S_1 is the lex-max min-win coalition.

   (ii) $S^j := \{S_i \mid i \in [1,j],\ j \in S_i\} \neq \emptyset$ for each $j \in [1,k]$.

   (iii) $S_{j+1} = \rho_j(S_{i_0})$, where i_0 is minimal w.r.t. $S_{i_0} \in S^j$ and $\ell(S_{i_0}) = \min\{\ell(S) \mid S \in S^j\}$, for each $j \in [1,k]$.

   This theorem follows directly from Theorem 2.3 and Definition 2.4 in Sudhölter (1989).

   Moreover, let $j \in [1,k+1]$, $r_0 = \ell(S_j)$, and $r_1 > \ldots > r_\alpha = 0$ be defined by

   $$\{r_i \mid i \in [1,\alpha-1]\}$$

   is the set of dropouts of coalition S_j. Then

   $$\ell(\varphi^{\beta}(S_j)) = \min\{\ell(S) \mid r_{\beta+1} \notin S \in W_v^m,\ \ell(S) > r_\beta\}$$
   $$= \min\{\ell(S) \mid r_\beta \in S \in W_v^m\}$$

   for each $\beta \in [1,\alpha-1]$ (for a proof it is referred to the same paper).

2. Star-Shapedness

Recall that each directed game is assumed to be a game without null players, without winning players, without veto players, and – in the homogeneous case – without steps of different type.

It is the aim of this section to show that the kernel of a homogeneous game is star-shaped. This assertion will be a consequence of Theorem 2.3, in which it turns out that the maximum surplus of i over j $(i,j \in \Omega)$ is attained by at least one minimal winning coalition. Peleg (1966) showed the same assertion for certain pairs (i,j) in the constant-sum case. However, his approach cannot be generalized to arbitrary homogeneous games and the "theory of incidence vectors" (see Sudhölter (1989)) is strongly used in this paper. At first two lemmata are needed.

Lemma 2.1: If v is a directed game and $x \in \mathcal{PrK}(v)$, then $\mathcal{D}(x) \subseteq W_v$.

Proof: By Lemma 1.4 and 1.5 of Peleg–Rosenmüller–Sudhölter (1992), see also Peleg (1989), we know that $x \geq 0$ and $x_1 \geq \ldots \geq x_n$ holds true. Assume, on the contrary, that $\mathcal{D}(x)$ contains a losing coalition, say T. For each winning coalition S we have

$$e(S,x) \geq 1 - x(\Omega) = 0$$

and

$$e(T,x) = -x(T) \leq 0.$$

Thus $\quad e(T,x) = 0$

and $x_i = 0$ for all $i \in T$ by the assumption. Moreover the excess of S must vanish thus $x(S) = 1$ for each $S \in W_v$.
Consequently each player i with $x_i > 0$ must be a member of each winning coalition, thus i is a veto player which is excluded. q.e.d.

Lemma 2.2: Let i and j be different players of the directed game v and $x \in \mathcal{PrK}(v)$. Then

(a) $s_{ij}(x) \geq \mu(x) - x_i$
and

(b) $s_{ij}(x)$ is attained by a min–win coalition or by a coalition of the form $S \cup \{i\}$, where $S \in W_v^m$ is a coalition with maximal excess and $\ell(S) < i$; formally written

$$\mathcal{D}_{ij}(x) \cap (W_v^m \cup \{S \cup \{i\} \mid S \in \mathcal{D}(x) \cap W_v^m, \ell(S) < i\}) \neq \emptyset$$

holds true.

Proof: (a) Take any minimal winning coalition T with maximal excess. If $j \notin T$, then

$$e(T \cup \{i\},x) \geq \mu(x) - x_i,$$

thus assertion (a) is true. If $j \in T$, take any $k \in \Omega \setminus T - \Omega \setminus T \neq \emptyset$ since v is assumed to have no veto–players – and observe that there is a coalition $S \in \mathcal{D}(x)$ with $k \in S$, $j \notin S$. We can assume w.l.o.g. that S is a minimal winning coalition (otherwise take any minimal winning subcoalition of S). Then, again,

$$e(S \cup \{i\},x) \geq \mu(x) - x_i$$

holds true.

(b) Take any $T \in \mathcal{D}_{ij}(x)$ and define

$$\bar{T} = T \cap [1,i-1].$$

If $\bar{T} \in W_v$, choose any minimal winning subcoalition S of $\bar{T}$ and observe that

$$e(S,x) - x_i = e(S \cup \{i\},x) \geq e(\bar{T} \cup \{i\},x) \geq e(T,x) \geq \mu(x) - x_i,$$

thus $S \in \mathcal{D}(x) \cap W_v^m$, $\ell(S) < i$, $j \notin S$ and the proof is completed.
If $\bar{T} \notin W_v$, then there is a min–win coalition $\bar{T} \cup \{i\} \subseteq S \subseteq T$ and $e(S,x) \geq e(T,x)$, thus the proof is finished. q.e.d.

Now the main result can be formulated and proved.

Theorem 2.3: Let v be a homogeneous n–person game, $x \in \mathcal{K}(v)$, and i,j different players such that min {i,j} is a sum. Then $s_{ij}(x)$ is attained by a min–win coalition.

The prerequisite "min {i,j} is a sum" means that if the first step does not coincide with n – but is interchangeable with n by the absence of steps of different type – then it is excluded that both i and j are interchangeable with n.

Proof: Assume, on the contrary, $s_{ij}(x)$ is not attained by a min–win coalition, i.e.

$$\mathscr{D}_{ij}(x) \cap W_v^m = \emptyset. \tag{1}$$

Therefore

$$s_{ij}(x) = \mu(x) - x_i \tag{2}$$

by Lemma 2.2. Let t be the index of the last non vanishing component of x, i.e.

$$x_1 \geq \ldots \geq x_t > 0 = x_{t+1} = \ldots = x_n. \tag{3}$$

For each coalition $S \in W_v$ there is a unique $t(s) \in S$ such that

$$S \cap [1,t(S)] \tag{4}$$

is a min–win coalition – i.e. this min–win coalition arises from the winning coalition by dropping "superfluous" small players.

For each $T \in W_v^m$ let $\alpha(T) \in \mathbb{N} \cup \{0\}$ be minimal such that $\varphi^{\alpha(T)}(T)$ has no dropout $k > \max\{j,t\}$.

Define

$$\tilde{S} = \varphi^{\alpha(S \cap [1,t(S)])}(S \cap [1,t(S)])$$

for each $S \in W_v$. We conclude

$$\begin{aligned} e(S,x) &= 1 - x(S) \leq 1 - x(S \cap [1,t(S)]) && \text{(by (3))} \\ &= 1 - x(\tilde{S}) && \text{(by (3))} \\ &= e(\tilde{S},x) \end{aligned} \tag{5}$$

and thus

$$\mathscr{M} := \{S \in \mathscr{D}(x) \cap W_v^m \mid \alpha(S) = 0\} \neq \emptyset \text{ (by (5) and Lemma 2.1).} \tag{6}$$

Two subsets of $\mathscr{M}$ are defined as follows:

$$\mathscr{M}^- = \{S \in \mathscr{M} \mid [\min\{i,j\}, n] \cap S = \emptyset\}$$

$$\mathscr{M}^+ = \{S \in \mathscr{M} \mid [\min\{i,j\}, \max\{j,t\}] \subseteq S\}.$$

1st STEP: $\mathscr{M} = \mathscr{M}^- \cup \mathscr{M}^+$ and $\mathscr{M}^- \neq \emptyset \neq \mathscr{M}^+$.

Indeed, as soon as the equality is shown, it is easy to deduce the second part of the assertion. Take $S \in \mathscr{M}$. If $S \in \mathscr{M}^+$, then there is a player $k \notin S$ – since S cannot be the grand coalition by the absence of veto players. In this case the "balancedness property of x" applied to (k, min {i,j}), i.e., $s_{k\,\min\{i,j\}} = s_{\min\{i,j\}\,k}$, guarantees the existence of $T \in \mathscr{D}(x)$ with $k \in T$, $\min\{i,j\} \notin T$. By definition $\tilde{T} \in \mathscr{M}$ thus $\tilde{T} \in \mathscr{M}^-$ holds true. In each case $\mathscr{M}^-$ is nonempty. But $\mathscr{M}^+ \neq \emptyset$ is valid as well, which can be seen by distinguishing two cases: If $t < \min\{i,j\}$, take $S \in \mathscr{M}$ with $\ell(S)$ is maximal. Clearly $\ell(S) \geq t$ holds true. The case $S \in \mathscr{M}^-$ cannot occur since then $T := \rho_{\ell(S)}(S)$ exists and $e(T,x) \geq e(S,x) = \mu(x)$, thus $\ell(T) > \ell(S)$, $T \in \mathscr{M}$, a contradiction.

If $t \geq \min\{i,j\}$, take $S \in \mathscr{M}^-$ and observe that there is $T \in \mathscr{D}(x)$, $\min\{i,j\} \in T$, $k \notin T$ for each $k \in S$. Clearly $\tilde{T} \in \mathscr{M}^+$.

To show that $\mathscr{M} = \mathscr{M}^+ \cup \mathscr{M}^-$ it suffices to verify that there is no coalition $S \in \mathscr{M}$ such that

$$S \cap [\min\{i,j\}, n] \neq \emptyset \neq [\min\{i,j\}, \max\{j,t\}] \setminus S \tag{7}$$

is satisfied.

Assume, on the contrary, there is a coalition $S \in \mathscr{M}$ with property (7). Four cases are distinguished:

(a) $i < j$, $t < j$:

If $i \in S$ then $j \in S$ by (1). Since $t < j$ there is $k \in [i,j]$ with $k \notin S$ by (6). Then $S \cup \{k\} \setminus \{j\}$ is winning (since $j \lesssim k$) and contains a min–win coalition T containing i (see (4)).

Now

$$x(T) \leq x(S) + x_k \leq x(S) + x_i \text{ (by (3))}$$

hold true, a contradiction to (1).

If $i \notin S$ and $j \in S$, then the observation that $S \cup \{i\} \setminus \{j\}$ is winning yields a contradiction in the same way as before without using $t < j$.

If $i \notin S$ and $j \notin S$ then there is $k \in S \cap [i+1,n]$ (by (7)). Again a contradiction is obtained by considering $S \cup \{i\} \setminus \{k\}$ without using $t < j$.

(b) $i < j \le t$:

We can assume w.l.o.g. $i \in S$, thus $j \in S$, since all other cases can be treated in the same way as in (a). Again there is $k \in [i+1,t] \setminus S$ by (7).

Since $s_{jk}(x) = s_{kj}(x)$ it follows that $\mathscr{D}_{kj}(x) \subseteq \mathscr{D}(x)$, thus $\mathscr{D}_{kj}(x) \cap \mathscr{M} \neq \emptyset$ (by (5), (6) and $x_k > 0$).

Take $T \in \mathscr{D}_{kj}(x) \cap \mathscr{M}$ and observe that in case $i \notin T$ a contradiction is obtained analogously to the last subcase of (a). The case $i \in T$ cannot occur since then

$$e(T,x) = \mu(x) > \mu(x) - x_i = s_{ij}(x) \quad \text{(by (2) and } x_i > 0\text{)}.$$

(c) $t < j < i$:

Property (7) directly implies $j \notin S$, $[j+1, \ell(S)] \subseteq S$, $\ell(S) \ge j+1$, thus $\ell(S) < i$ (by (1)). Take $T \in \mathscr{D}(x) \cap W^m_v$ with $\ell(T) < i$ such that $\ell(T)$ is maximal with these properties. Since all steps – they are interchangeable – either occur as a block or do not occur at all in a fixed min–win coalition (by "steps rule their followers"), the last player $\ell(T)$ of T must be a sum – clearly $\ell(T) < n$ by definition. Therefore $\ell(T)$ is expendable in T and a min–win coalition

$$R = \rho_{\ell(T)}(T)$$

is obtained. It is obvious that

$$\mu(x) = e(R,x) \qquad \text{(by } \ell(T) > t\text{)},$$

thus either $i \in R$ – a contradiction to (1) – or $\ell(R) < i$ – a contradiction to the maximality of $\ell(T)$.

(d) $j < i$ and $j \le t$:

If $j \notin S$ then there is $k > j$ with $k \in S$ (by (7)) and w.l.o.g. $k \le t+1$, since $[t+1, \ell(S)] \subseteq S$ whenever $\ell(S) > t$ by (6). If $\ell(S) > i$ (implying $t \ge i$), then the consideration of a min–win coalition contained in $S \cup \{i\} \setminus \{\ell(S)\}$ again yields a contradiction. If $t \le \ell(S) < i$, then we can proceed in the same way as in the last subcase of (c) by choosing any $T \in \mathscr{D}(x) \cap W^m_v$ with $\ell(T) < i$, $j \notin T$ such that $\ell(T)$ is maximal. If $\ell(S) < \min\{i,t\}$, assume that $\ell(S)$ is maximal with these properties and take any $T \in \mathscr{M}$ with $\ell(S) \notin T$, $\ell(S) + 1 \in T$. There exists $k \in T$, $k > \ell(S)$, such

that

$$R = T \cup \{\ell(S)\} \setminus [k,n] \in W^m_v \qquad \text{(see (4))}.$$

Thus

$$x_{\ell(S)} \ge x(T \cap [k,n]) \qquad \text{(since } T \in \mathscr{M}\text{)}. \tag{8}$$

But – by homogeneity –

$$Q := S \setminus \{\ell(S)\} \cup (T \cap [k,n]) \in W^m_v$$

holds true. Consequently (8) is, indeed, an equality and $Q \in \mathscr{M}$, $\ell(Q) > \ell(S)$. Clearly Q satisfies (7) and thus $\ell(Q) \ge \min\{i,t\}$ is impossible as seen above. But $\ell(Q) < \min\{i,t\}$ contradicts the maximality of $\ell(S)$.

If $j \in S$ then there is $k \in [j+1,t] \setminus S$, thus there is $\tilde{S} \in \mathscr{M}$, $k \in \tilde{S}$, $j \notin \tilde{S}$ which is impossible by the first part of (d).

From now on let i,j be chosen in such a way that $i + j$ is minimal with the desired properties. Moreover write $k := \min\{i,j\}$.

2nd STEP: Let $S \in \mathscr{M}^+$ with $r := r(S)$ maximal. Then

$$\ell(\varphi(S)) = r \text{ and } \varphi(S) \in \mathscr{M}^-. \tag{9}$$

Recall that r(S) denotes the last dropout of S which exists because S cannot be the lex–max min–win coalition. Since $r < k$ is valid, i.e. $r+j < i+j$, there is a min–win coalition $T \in \mathscr{D}_{rj}(x)$ by the minimality of $i+j$. By the balancedness property of x, namely $\mu(x) = s_{jr}(x) = s_{rj}(x)$, coalition T has maximal excess, thus $T \in \mathscr{M}^-$ by the first step.
If $\ell(T) > r$, then there is $R \in \mathscr{D}(x)$ with $\ell(T) \notin R$, $k \in R$, since $T \in \mathscr{D}(x) \cap \mathscr{D}_{\ell(T)k}$.
Again by the minimality of $i + j$ and $\ell(T) + k < i+j$ we can assume w.l.o.g. that $R \in W^m_v$ and $R \in \mathscr{M}^+$ is valid. Now, the existence of R contradicts the maximality of r.
Therefore $\ell(T) = r$. We conclude

$$x_r \ge x([\ell(\varphi(S)) + 1, \ell(S)]) \qquad \text{(by } S \in \mathscr{M}\text{)}$$

and

$$x_r \le x([\ell(\varphi(S)) + 1, \ell(S)]) \qquad \text{(by } T \in \mathscr{M} \text{ and homogeneity)},$$

thus the assertion (9).

3rd STEP: Now the proof can be completed. Let S be the coalition of the 2nd Step and again $r = r(S)$. Moreover, let $r_0 := \ell(S_{k+1})$ and

$$k = r_1 > \ldots > r_\alpha = 0$$

be defined via $\{r_1,\ldots,r_{\alpha-1}\}$ is the set of dropouts of S_{k+1} – for the definition of S_{k+1} it is referred to the third part of Remark 1.2. By construction and this remark we have

$$r_{\beta+1} \notin \varphi^\beta(S_{k+1}) \ni r_\beta \qquad (10)$$

and

$$\ell(\varphi^\beta(S_{k+1})) = \min\{\ell(S) \mid r_{\beta+1} \notin S \in W^m_v, \ \ell(S) > r_{\beta+1}\}$$
$$= \min\{\ell(S) \mid r_\beta \in S \in W^m_v\} \qquad (11)$$

for all $\beta \in [0,\alpha-1]$.

Let f be defined by $r_{f+1} < r \leq r_f$ and let

$$T := \varphi^{f-1}(S_{k+1}).$$

Three cases are distinguished:

(a) $f + 1 = \alpha$:

Then $r \leq \ell(\varphi(T)) = \ell(S_1)$ (12)

holds true, where S_1 is the lex–max min–win coalition.

But $\ell(S_1) = \min\{\ell(S) \mid S \in W^m_v\} \leq \ell(\varphi(S)) = r$ (by (9)), thus (12) is an equality. Consequently $\ell(S)$ is minimal such that $k \in S$, thus k is expendable in S by Sudhölter (1989), a contradiction to the 1st Step. Therefore we assume $f + 1 < \alpha$ from now on.

(b) $r_{f+1} \notin S$:

Then $r_{f+1} \in \varphi(S)$ and $\ell(\varphi(T)) \geq r_f \geq r$.
By the minimality of $\ell(\varphi(T))$ – see (11) – and (9) we obtain $r_f = r$. Now – by homogeneity – $\ell(T)$ has to coincide with $\ell(S)$, thus $f = 1$, but k must be expendable in T by (11), thus in S, a contradiction.

(c) $r_{f+1} \in S$:

If $r_f = r$, we obtain a contradiction analogously to (b). Therefore $r_f > r$ is assumed. Choose $R \in \mathscr{M}$ with $r_{f+1} \notin R$, $r \in R$. The existence of R is guaranteed by the minimality of $i + j$ and the balancedness

$$\mu(x) = s_{r_{f+1}r}(x) = s_{rr_{f+1}}(x).$$

R cannot be a member of $\mathscr{M}^-$, since otherwise – by $\ell(R) \geq \ell(T) > r$ (see (11)) – there is a coalition containing k and not $\ell(R)$ with maximal excess, which can be chosen to be min–win by $k + \ell(R) < i+j$. This contradicts the maximality of r.

Therefore $R \in \mathscr{M}^+$ holds true. Let $(\lambda;m)$ be the minimal representation of v (see Remark 1.2).

Now we have

$$m([r+1, \ell(S)]) = m_r \leq m_b;\ b \leq r \text{ (by (9) and Remark 1.2)}$$

and

$$m_{r+1} \geq m_k \quad \text{(by Remark 1.2)}.$$

Since k is not expendable in S, we conclude

$$m([\ell(S) + 1,n]) < m_{r+1},$$

thus $\ell(\varphi(R)) \leq r$ is valid. Now, r_{f+1} cannot be a dropout of $\varphi(R)$ (see (11)). Since r_f is expendable in $\varphi(T)$ but not in R, the inequality

$$r \geq \ell(\varphi(R)) > \ell(\varphi^2(T))$$

is satisfied. Thus

$$m([\ell(\varphi(T)) + 1, \ell(R)]) = m([\ell(\varphi^2(T)) + 1, \ell(\varphi(R))]) \geq m_r$$

but

$$\begin{aligned} m([\ell(\varphi(T)) + 1, \ell(R)]) &\leq m([\ell(\varphi(T)) + 1, n]) \\ &\leq m([r+1,n]) - m_{r+1} \\ &= m([r+1, \ell(S)]) + m([\ell(S) + 1,n]) - m_{r+1} \\ &< m_r + m_k - m_{r+1} \\ &\leq m_r, \end{aligned}$$

a contradiction.

q.e.d.

This section is concluded by formulating and proving the explicit results concerning the star–shapedness of the (pre–)kernels of homogeneous games.

Corollary 2.4: If the assumptions of Theorem 2.3 and $x_n > 0$ are satisfied, then

$$s_{ij}(x) > \mu(x) - x_i$$

and

$$\mathscr{D}_{ij}(x) \subsetneq W^m_v .$$

Proof: Again

$$x_1 \geq ... \geq x_t > 0 = x_{t+1} = ... = x_n .$$

Thus $t = n$. The second assertion is directly implied by the first one and Lemma 2.2. In order to verify the first assertion, a part of the proof of Theorem 2.3 has to be repeated: Start again with the 1st Step – only parts (b) and (d) have to be taken into consideration – and observe that all constructed contradictions to (1) are also contradictions to (2), if $t = n$. The 2nd Step and 3rd Step can be left unchanged. q.e.d.

Proposition 2.5:

(a) The (pre–)kernel of a homogeneous game v is star–shaped with center $m/m(\Omega)$, where $(\lambda;m)$ is the minimal representation of v.

(b) The normalized vector of minimal weights $m/m(\Omega)$ is an extreme point of the convex hull of $\mathscr{PK}(v)$.

Proof: Let $y = m/m(\Omega)$ and τ be the index of the first step. Then

$$y_\tau = y_{\tau+1} = ... = y_n$$

is valid (see Remark 1.2). Clearly y is a member of the kernel of v (see e.g., Peleg–Rosenmüller (1992)). Let $x \in \mathscr{PK}(v)$.

ad (a): It suffices to show the following:
If $i, j \in \Omega$, $i \neq j$, and $\min\{i,j\} < \tau$, then
$s_{ij}(cx+(1-c)y) = c \cdot s_{ij}(x) + (1-c)\, s_{ij}(y)$ for all $c \in \mathbb{R}$,
$0 \leq c \leq 1$.

Lemma 2.1 implies $\mu(y) = e(S,y)$ for all $S \in W^m_v$.

Theorem 2.3 directly shows

$$s_{ij}(y) = \mu(y), \quad \mathscr{D}_{ij}(y) \subsetneq W^m_v.$$

Take any min–win coalition S attaining $s_{ij}(x)$.
Then $s_{ij}(cx + (1-c) \cdot y) = e(S, cx + (1-c)\, y) = c\, s_{ij}(x) + (1-c)\, s_{ij}(y)$.

ad (b): It suffices to show:
If $x_n \geq y_n$ then $x = y$.

Let $x_n \geq y_n$. Then $x_i \geq y_i$ for all $i \geq \tau$.
Assume $x_i \geq y_i$ for some $\tau > k \geq 1$ and all $i > k$. Choose a min–win coalition $S \in \mathscr{D}_{nk}(x)$ – S exists because k is a sum (see Theorem 2.3).

Then the coalition $S \cup \{k\} \setminus \{n\}$ contains a min–win coalition T with

$$T = (S \cup \{k\} \setminus \{n\}) \cap [1, \ell(T)]$$

by (4). Clearly player k is a member of T. Therefore the balancedness of x implies that

$$\begin{aligned} x_k &\geq x(S \cap [k+1,n]) - x(T \cap [k+1,n]) \\ &\geq y(S \cap [k+1,n]) - y(T \cap [k+1,n]) \\ &= y_k \qquad \text{(by the homogeneity of } (\lambda;m)). \end{aligned}$$

An inductive argument finishes the proof. q.e.d.

Corollary 2.6: Let v be a homogeneous game minimally represented by $(\lambda;m)$. If x is a member of the kernel of v other than $m/m(\Omega)$ with a non vanishing last component, then there exists $y \in \mathscr{K}(v)$, $y_n = 0$ such that

(i) x is a convex combination of y and $m/m(\Omega)$

and

(ii) all convex combinations of y and $m/m(\Omega)$ are elements of the kernel

hold true.

The proof is a direct consequence of Corollary 2.4 and Proposition 2.5 (b) and therefore skipped here.

Up to the end of this section it will be shown how different steps can be avoided: the steps can be collected to one player and all elements of the kernel can be treated in the same way resulting in the kernel of the new game. To be more precise, let v be a homogeneous n–person game and τ be the first step, i.e. $n \sim \ldots \sim \tau$ and $\tau-1 \precsim \tau$. Let w be the simple τ–person game, defined via

$$w(S) = \begin{cases} v(S), & \text{if } \tau \notin S \\ v(S \cup [\tau,n]), & \text{if } \tau \in S \end{cases}.$$

Let $(\lambda;m)$ be the minimal representation of v. Then $m_{\tau-1} = m([\tau,n])$ and $(\frac{\lambda}{m_{\tau-1}}; \frac{m_1}{m_{\tau-1}}, \ldots, \frac{m_{\tau-1}}{m_{\tau-1}}, 1)$ is the minimal representation of w, i.e., w is a homogeneous game without steps (see Ostmann (1987a) and Sudhölter (1989)).

Remark 2.7: It is well know (see, e.g. Maschler–Peleg (1966)) that the pre–kernel of a game is a finite union of polytopes. Let v be a homogeneous n–person game and $\tilde{m}$ be the normalized vector of a minimal weight. If

$$\mathcal{PK}(v) = \bigcup_{i=1}^{r} P^i$$

for some polytopes P^i $(i \in [1,r])$ and some $r \in \mathbb{N}$, then P^i can be enlarged to a polytope $\tilde{P}_i$ containing $\tilde{m}$ as an extreme point and no other extreme point with a positive last component such that $P^i \subseteq \tilde{P}^i \subseteq \mathcal{PK}(v)$. To see this take any extreme point x from P^i other than $\tilde{m}$ with a positive last component and observe that

$$(1+\delta) \cdot x - \delta\tilde{m} =: x^{1+\delta},$$

where δ is defined via $x_n^{1+\delta} = 0$, is a member of the pre–kernel by Corollary 2.6. By the same result the straight line connecting $x^{1+\delta}$ with $\tilde{m}$ belongs to the pre–kernel. If y is in P^i then the triangle $CH(\{y,x,\tilde{m}\})$ (CH denotes the convex hull) is contained in the pre–kernel by Proposition 2.5. Therefore again Corollary 2.6 directly implies

$$CH(\{y, x^{1+\delta}, \tilde{m}\}) \subseteq \mathcal{PK}(v)$$

holds true (see Figure 1). Indeed, $CH\{z, \bar{z}\} \subseteq \mathcal{PK}(v)$, since $\bar{z}_n \geq 0$ (for each $z \in CH\{x,y\}$).

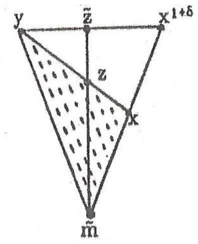

Figure 1

Lemma 2.8: $\mathcal{K}(v) = \{x \in \mathbb{R}^n \mid x_\tau = \ldots = x_n \text{ and } (x_1,\ldots,x_{\tau-1}, x[\tau,n]) \in \mathcal{K}(w)\}$.

Proof:

1. If $x \in \mathcal{K}(v)$ and $x_n = 0$, then $(x_1,\ldots,x_{\tau-1},0) \in \mathcal{K}(w)$.
2. If $x \in \mathcal{K}(w)$ and $x_\tau = 0$, then $(x_1,\ldots,x_{\tau-1}, \underbrace{0,\ldots,0}_{n-\tau+1}) \in \mathcal{K}(v)$.

A proof of these assertions – even without Theorem 2.3 – is straight forward and therefore skipped. The proof is completed with the help of the preceding corollary and proposition. Let $(\lambda;m)$ be the minimal representation of v and $x = \frac{m}{m(\Omega)}$. Then $y \in \mathbb{R}^\tau$, defined by

$$y_i = \begin{cases} x_i, & i < \tau \\ x_{\tau-1} = x([\tau,n]), & i = \tau \end{cases}$$

is a center of $\mathcal{K}(w)$. If CH(M) denotes the convex hull of $M \subseteq \mathbb{R}^m$, then

$$\mathcal{K}(v) = \bigcup_{\substack{a \in \mathcal{K}(v) \\ a_n = 0}} CH\{a,x\}$$

and

$$\mathcal{K}(w) = \bigcup_{\substack{b \in \mathcal{K}(v) \\ b_\tau = 0}} CH\{b,y\}$$

by Corollary 2.6 and Proposition 2.5 and the proof is finished by construction. q.e.d.

3. The Weighted Majority Case

Each weighted majority game is assumed to have no null players, no winning players, and no veto players (see Section 1).

It is the aim of this section to show that the pre-kernel of each weighted majority game coincides with the "relevant" subset of the kernel of a homogeneous game with a larger number of players. This homogeneous game, a "homogeneous extension" of the original one, is not uniquely determined and can be chosen superadditively. Here are the details.

Definition 3.1: Let v be a weighted majority n-person game. A homogeneous k-person game $\tilde{v}$ with minimal representation $(\tilde{\lambda};\tilde{m})$ is a **homogeneous extension** of v, if the following conditions are satisfied:

(i) $k > n$;
(ii) $\tilde{m}_i = 1$ for $i \in [n+1,k]$;
(iii) $\tilde{m}_i$ is even for $i \in [1,n]$;
(iv) $\lambda := \tilde{\lambda} - m([n+1,k-1])$ is even;
(v) $(\lambda;m)$, where $m = (\tilde{m}_1,\ldots,\tilde{m}_n)$, is a representation of v.

It should be noted that a weighted majority game is the reduced game (in the sense of Davis-Maschler (1965)) of each of its homogeneous extensions w.r.t. Ω and all pre-imputations with vanishing components for players $i \notin \Omega$.

Lemma 3.2: Each weighted majority game v possesses a superadditive homogeneous extension without steps.

Proof: Let $(\lambda;m)$ be an arbitrary integer representation of the n-person game v satisfying

$$m_1 \geq \ldots \geq m_n \text{ and } \lambda = \min\{m(S) \mid S \in W_v^m\}.$$

Let w.l.o.g. λ, m_i be even numbers for $i \in \Omega$ – otherwise $(\lambda;m)$ can be replaced by $(2\lambda;2m)$. Then the vector

$$(\tilde{\lambda};\tilde{m}) := (m(\Omega);\, m_1,\ldots,m_n,\underbrace{1,\ldots,1}_{1+m(\Omega)-\lambda \text{ times}})$$

is a minimal homogeneous representation of a superadditive $(n+1+m(\Omega)-\lambda)$-person game $\tilde{v}$ without steps. This can be verified directly using, e.g., the procedure of testing for homogeneity described by Sudhölter (1989). Clearly – by Definition 3.1 – $\tilde{v}$ is a homogeneous extension of v. q.e.d.

Theorem 3.3: Let $\tilde{v}$ be a homogeneous k-person extension of the weighted majority v. Then

$$\mathcal{PK}(v) = \{x \in \mathbb{R}^n \mid (x,\underbrace{0,\ldots,0}_{k-n \text{ times}}) \in \mathcal{PK}(\tilde{v})\}$$

is valid.

Proof: Let $(\tilde{\lambda};\tilde{m})$ and $(\lambda;m)$ be constructed according to Definition 3.1.

Note that $S \in W_v$ implies $m(S) \geq \lambda$ and thus $\tilde{m}(S \cup [n+1,k-1]) \geq \tilde{\lambda}$. Conversely, if $S \in W_{\tilde{v}}$, then $\tilde{m}(S \cap [1,n]) \geq \tilde{\lambda} - (k-n)$, $\tilde{m}(S \cup [1,n])$ is even, and $\tilde{\lambda} - (k-n)$ is odd. Consequently we have $m(S \cup [1,n]) \geq \lambda$.
This motivates the definition of two mappings

$$\sigma: \quad \mathcal{P}([1,n]) \to \mathcal{P}([1,k]),\ S \mapsto S \cup [n+1,k-1]$$

and

$$\nu: \quad \mathcal{P}([1,k]) \to \mathcal{P}([1,n]),\ S \mapsto S \cap [1,n].$$

These maps have the following properties

(a) $\nu \circ \sigma = \mathrm{id}$;
(b) $\sigma(S) \in W_{\tilde{v}}$ iff $S \in W_v$;
(c) $\nu(S) \in W_v$ iff $S \in W_{\tilde{v}}$.

Let $x \in \mathcal{PK}(v)$ and $\tilde{x} = (x,\underbrace{0,\ldots,0}_{k-n \text{ times}})$. It is clear that $s_{ij}(x) = s_{ij}(\tilde{x})$ for $i,j \in [1,n]$ and $i \neq j$. In view of the fact that all players of $[n+1,k]$ are interchangeable it suffices to show that $s_{ik}(\tilde{x}) = s_{ki}(\tilde{x})$ for all $i \in [1,n]$. Indeed, by a balancedness argument, there is a coalition $S \subseteq [1,n]$ with $i \in S$, $e(S,x) = \mu(x)$ and a coalition $T \subseteq [1,n]$ with $i \notin T$, $e(T,x) = \mu(x)$. Therefore the coalitions $\sigma(S)$ and $\sigma(T) \setminus \{n+1\} \cup \{k\}$ show that $s_{ik}(\tilde{x}) = s_{ki}(\tilde{x}) = \mu(\tilde{x})$ holds true.

Conversely, if $\tilde{x} \in \mathcal{PK}(\tilde{v})$ with $\tilde{x}_k = 0$ define $x = (\tilde{x}_1,\ldots,\tilde{x}_n)$. It is well–known (see, e.g., Peleg (1989)) that the pre–kernel has the reduced game property in the sense of Sobolev (1975), thus the proof can be finished using this property. Besides, the last assertion can also be verified directly using the surjective mapping ν. q.e.d.

Remark 3.4: Let $\tilde{v}$ be a homogeneous extension of the n–person weighted majority game v. Then

$$\mathcal{PK}(\tilde{v}) = \bigcup_{i=1}^{r} \tilde{P}^i$$

holds true for some $r \in \mathbb{N}$ and some polytopes $\tilde{P}^i$ containing the normalized minimal vector of weights $\tilde{m}$ and no other extreme point with a non vanishing last component. Let P^i be the polytope which arises from $\tilde{P}^i$ by taking the convex hull of all extreme points other than $\tilde{m}$. Then the pre–kernel of v is the projection of the union of the P^i ($i \in [1,r]$) to $\mathbb{R}^n$.

Example 3.5: Kopelowitz (1967) gave examples of weighted majority games with disconnected kernels. Here is one 6–person game v, given by the representation (10;5,4,3,2,2,2) or $(\lambda;m) := (20;10,8,6,4,4,4)$. Kopelowitz computed the kernel of this game and came up with

$$\mathcal{K}(v) = \{(2,1,1,1,1,1)/7,\ (1,1,0,0,0,0)/2\}.$$

Then $\tilde{v}$, minimally represented by

$$(\tilde{\lambda};\tilde{m}) = (32;10,8,6,4,4,4,1,1,1,1,1,1,1,1,1,1,1,1,1)$$

is a homogeneous extension of v. Here $\tilde{\lambda}$ is chosen minimally such that m hom $\tilde{\lambda}$ is valid. This homogeneous extension is not the one suggested in Lemma 3.2 – having level $m([1,6]) = 36$ and therefore four additional players of weight 1.

For the sake of completeness the kernel of $\tilde{v}$ is described explicitly. Define two vectors

$$x^1 = (2,1,1,1,1,1,\underbrace{0,\ldots,0}_{13\text{ times}})/7,$$

$$x^2 = (1,1,\underbrace{0,\ldots,0}_{17\text{ times}})/2.$$

Then $\mathcal{K}(\tilde{v}) = CH\{x^1, \tilde{m}/49\} \cup CH\{x^2, \tilde{m}/49\}$ (see Figure 2).

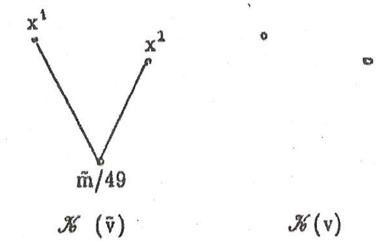

Figure 2

4. The Kernels of Partition Games and Examples

Peleg (1966) showed that the kernels of certain homogeneous constant–sum games called partition games are star–shaped. Partition games were introduced by Isbell (1956, 1958). He observed that a monotone simple constant–sum game (without null players) has at least as many min–win coalitions as players. And, up to one famous exception, the partition games are exactly those with this minimal number. The exception is the projective 7–person game, introduced by Richardson (1956). This game has a very symmetric kernel with equal treatment of the players in the center – the center being no extreme point of the convex hull.

In this paper it turns out that the pre–kernel of a partition game is not only star–shaped – by homogeneity – but a singleton. We start recalling the definition of partition games. Let $n \geq 4$ in this section.

Definition 4.1: The game v is an n–person partition game if there is a number $r \in \mathbb{N} \setminus \{1\}$ and a vector $t \in \mathbb{N}^r$ with $t_1 = 1$, $t_2 \geq 2 \leq t_r$, and $t([1,r]) = n$ – i.e., $[1,n] = \bigcup_{i=1}^{r} T_i$, where $T_{i+1} = [1+t([1,i]), t([1,i+1])]$, $i \in \{0\} \cup [1,r-1]$ – such that

$$S \in W_v^m, \text{ iff } \tilde{S} \in \{\tilde{S}^j \mid j \in [1,r]\},$$

where $\tilde{S} \in (\mathbb{N} \cup \{0\})^r$ is defined by $\tilde{S}_i = | S \cap T_i |$ $(i \in [1,r])$

$$\text{and } \tilde{S}^j_i = \begin{cases} t_i, \text{ if } j-i \equiv 0 \bmod 2 \text{ and } i \leq j \\ 0, \text{ if } (j-1 \equiv 1 \bmod 2 \text{ and } i < j) \text{ or } i > j+1 . \\ 1, \text{ otherwise} \end{cases}$$

The following table shows the minimal representations of all 7–person partition games.

(6;5,1,1,1,1,1,1)
(9;5,4,4,1,1,1,1)
(10;7,3,3,3,1,1,1)
(11;7,4,4,3,1,1,1)
(9;7,2,2,2,2,1,1)
(12;7,5,5,2,2,1,1)
(11;8,3,3,3,2,1,1)
(13;8,5,5,3,2,1,1)

Theorem 4.2: If v is a partition game, then the (pre–)kernel of v is a singleton.

Proof: Let $x \in \mathcal{K}(v)$, r, t_i, T_i, $\tilde{S}^i$ be defined according to Definition 4.1. In view of Proposition 2.5, it suffices to show that $x = m/m(\Omega)$, where $(\lambda;m)$ is the minimal representation of v. Again, by the star–shapedness we assume $x_n > 0$.

Define $\mathscr{D} = \{\tilde{S} \mid S \in \mathscr{D}(x) \cap W_v^m\}$.

Claim: $\mathscr{D} = \{\tilde{S}^j \mid j \in [1,r]\} =: \mathscr{M}$.

As soon as this last equality is shown, the proof is finished, since $m/m(\Omega)$ is uniquely determined by $y(S) = \text{const.}$ for $S \in W_v^m$ and $y(\Omega) = 1$ – note that all players of T_i are interchangeable and thus obtain equal weights according to x.

Since $x_n > 0$ there is $\tilde{S} \in \mathscr{D}$ with $\tilde{S}_r > 0$, but $\tilde{S}^{r-1}$ and $\tilde{S}^r$ are the only elements of $\{\tilde{S} \in \mathscr{M} \mid \tilde{S}_r > 0\}$. If $\tilde{S}^{r-1} \in \mathscr{D}$, then $s_{ij}(x) = \mu(x)$ for $i \in T_{r-1}$, $j \in T_r$ since $\tilde{S}^{r-1}_{r-1} > 0$ and $\tilde{S}^{r-1}_r = 1 < t_r$ hold true. By the balancedness of x we have $s_{ji}(x) = \mu(x)$ implying the existence of $\tilde{S} \in \mathscr{D}$ with $\tilde{S}_r > 0$, $\tilde{S}_{r-1} < t_{r-1}$ and thus $\tilde{S} = \tilde{S}^r$. Therefore, in each case, $\tilde{S}^r \in \mathscr{D}$ is valid.

Now, $s_{ij}(x) = \mu(x)$ shows that there is $\tilde{S} \in \mathscr{D}$ with $\tilde{S}_{r-1} > 0$, $\tilde{S}_r < T_r$, thus $\tilde{S} \in \{\tilde{S}^{r-1}, \tilde{S}^{r-2}\}$. Assume $\tilde{S} = \tilde{S}^{r-2}$ (i.e. $r \geq 3$), thus $\tilde{S}_{r-2} > 0$and $\tilde{S}_r = 0$. As a consequence we obtain

$$s_{ij}(x) = \mu(x) \text{ for } i \in T_{r-2}, j \in T_r$$

and – by balancedness – $s_{ij}(x) = \mu(x)$. We conclude that there is $\tilde{S} \in \mathscr{D}$ with $\tilde{S}_r > 0$, $\tilde{S}_{r-2} < t_{r-2}$, thus $\tilde{S} = \tilde{S}^{r-1}$. Up to now we proved $\tilde{S}^{r-1}, \tilde{S}^r \in \mathscr{D}$.

Assume $\tilde{S}^r, \tilde{S}^{r-1}, \dots, \tilde{S}^{\alpha+1} \in \mathscr{D}$ for some $\alpha \leq r-2$. If $\alpha = 0$, the proof is complete. Therefore assume $\alpha \geq 1$.
Again, since $\tilde{S}^{\alpha+1}_{\alpha+2} > 0$, $\tilde{S}^{\alpha+1}_{\alpha} = 0$ there is $\tilde{S} \in \mathscr{D}$ with $\tilde{S}_\alpha > 0$, $\tilde{S}_{\alpha+2} < t_{\alpha+2}$. Observe that $\tilde{S} = \tilde{S}^\alpha$, if $\alpha = 1$. Therefore assume $\alpha > 1$. Then clearly $\tilde{S} \in \{\tilde{S}^\alpha, \tilde{S}^{\alpha+1}\}$ is valid. Assume $\tilde{S} = \tilde{S}^{\alpha-1}$, thus $\tilde{S}_{\alpha-1} > 0$, $\tilde{S}_{\alpha+1} = 0$ and – by balancedness – there is $T \in \mathscr{D}$ with $\tilde{T}_{\alpha+1} > 0$, $\tilde{T}_{\alpha-1} < t_{\alpha-1}$. Consequently, $\tilde{T} = \tilde{S}^\alpha$ holds true. q.e.d.

Finally, some examples are presented showing the following assertions.

(i) The kernel of a homogeneous constant-sum game need not be a singleton or even convex.

(ii) An element of the kernel of a homogeneous game – even in the constant sum case – need not satisfy the condition that the maximum surplus of player i or j coincides with the maximal excess, even for non interchangeable player i and j.

(iii) The least core of a homogeneous game need not be contained in the kernel of the game.

(iv) An element of the kernel of a weighted majority game need not satisfy that the maximum surplus of player i or player j is attained by a min–win coalition for non interchangeable players i and j.

Note that a possible example showing (iii) has to be a non constant-sum game, because otherwise the least core is a singleton consisting of the nucleolus as Peleg (1968) has shown. Nevertheless, this assertion may be surprising because pre-kernel, nucleolus and least core behave in the same way w.r.t. homogeneous games with steps of different type, which was shown by Rosenmüller–Sudhölter (1992) and Peleg–Rosenmüller–Sudhölter (1992).

Examples 4.3:

(a) Let v be the homogeneous 11–person constant-sum game minimally represented by

$$(\lambda;m) = (16;10,6,4,2,2,2,1,1,1,1,1).$$

Define

$$x^1 = (2,1,1,1,1,1,0,0,0,0,0) / 7$$
$$x^2 = (1,1,0,0,0,0,0,0,0,0,0) / 2$$
$$x^3 = (6,3,3,2,2,2,0,0,0,0,0) / 18.$$

It is easy to verify that

$$\mathcal{K}(v) = \text{CH}\,\{x^1, m/31\} \cup \text{CH}\,\{x^2, x^3, m/31\}.$$

There are only two "types" of min–win coalitions in T_{32}, namely one consisting of players 1,3, and one additional player in [4,6] and the other consisting of players 1,3 and two additional players in [7,11]. The excess w.r.t. x^3 is 7/18 and 1/2 respectively. The maximal excess w.r.t. x^3 is attained by, e.g. $\{1,4\} \cup [7,10]$, and is 10/18.

This example, thus, shows assertions (i) and (ii).

(b) Let v be the 5–person weighted majority game, represented by $(\lambda;m) = (8;4,3,2,2,1)$. Then $x := m/12 \in \mathcal{K}(v)$ – indeed, m/12 is the nucleolus of the game. Moreover $s_{52}(x)$ cannot be attained by a min–win coalition because there is no min–win coalition containing player 5 but not player 2. As a consequence we have assertion (iv) and, additionally, $s_{52}(x) < \mu(x)$.

(c) Let v be the homogeneous 7–person game, represented by

$$(\lambda;m) = (14;6,5,3,3,2,1,1).$$

Define $x^1 = (2,2,1,1,0,0,0) / 6$, $\quad x^2 = (2,1,1,1,1,0,0) / 6$.

Then the least core is the convex hull of x^1, x^2 and m/21. Besides, the nucleolus of v can be computed as $\frac{7}{11} \cdot \frac{m}{21} + \frac{2}{11} x^1 + \frac{2}{11} x^2$.

It can easily be verified that $x^1, x^2 \notin \mathcal{K}(v)$. With the help of a computer it was checked that this is the only 7–person example, showing assertion (iii).

References

Davis, M. and Maschler, M. (1965): The Kernel of a Cooperative Game. Naval Res. Logist. Quarterly 12, pp.223–259

Einy, E. (1985): The Desirability Relation of Simple Games. Math. Social Sci. 10, pp.155–168

Isbell, J.R. (1956): A Class of Majority Games. Quarterly Journal Math. 7, pp.183–187

Isbell, J.R. (1958): A Class of Simple Games. Duke Math. Journal 25, pp.423–439

Isbell, J.R. (1959): On the Enumeration of Majority Games. Math. Tables Aids Comput. 13, pp.21–28

Kopelowitz, A. (1967): Computation of the Kernel of Simple Games and the Nucleolus of N–Person Games. Research Memorandum No. 31, Research Program in Game Theory and Math. Ec., Dept. of Math., The Hebrew University, Jerusalem, Israel

Krohn, I. and Sudhölter, P. (1990): Directed and Weighted Majority Games. Working Paper 195, Inst. of Math. Ec., University of Bielefeld

Maschler, M. and Peleg, B. (1966): A Characterization, Existence Proof and Dimension Bounds for the Kernel of a Game. Pacific J. Math. 18, pp.289–328

Maschler, M. and Peleg. B. (1967): The Structure of the Kernel of a Cooperative Game. SIAM J. Applied Math. 15(3), pp.569–604

Maschler, M., Peleg, B. and Shapley, L.S. (1979): Geometric Properties of the Kernel, Nucleolus and Related Solution Concepts, Math. of Operations Research 4, pp.303–338

Ostmann, A. (1987a): On the Minimal Representation of Homogeneous Games. Int. Journal of Game Theory 16, pp.69–81

Ostmann, A. (1987b): Life–Length of a Process with Elements of Decreasing Importance. Working Paper 156, Inst. of Math. Ec., University of Bielefeld

Ostmann, A. (1989): Simple Games: On Order and Symmetrie. Working Paper 169, Inst. of Math. Ec., University of Bielefeld

Peleg, B. (1966): On the Kernel of Constant–Sum Simple Games with Homogeneous Weights. Ill. Journal Math. 10, pp.39–48

Peleg, B. (1968): On Weights of Constant–Sum Majority Games. SIAM J. of Appl. Math. 16, pp.527–532

Peleg, B. (1989): Introduction to the Theory of Cooperative Games. Research Memoranda No. 81–88, Center for Res. in Math. Ec. and Game Theory, The Hebrew University, Jerusalem, Israel

Peleg, B. and Rosenmüller, J. (1992): The Least–core, Nucleolus, and Kernel of Homogeneous Weighted Majority Games. To appear in Games and Economic Behaviour

Peleg, B., Rosenmüller, J. and Sudhölter, P. (1992): The Kernel of Homogeneous Games with Steps. Working Paper 209, Inst. of Math. Ec., University of Bielefeld

Richardson, M. (1956): On Finite Projective Games. Proceedings of the American Math. Soc. 7, pp. 458–465

Rosenmüller, J. (1982): On Homogeneous Weights for Simple Games. Working Paper 115, Inst. of Math. Ec., University of Bielefeld

Rosenmüller, J. (1984): Weighted Majority and the Matrix of Homogeneity. Zeitschrift für Operations Research 12, pp.123–141

Rosenmüller, J. (1987): Homogeneous Games: Recursive Structure and Computation. Math. of Operations Research 12, pp.309–330

Rosenmüller, J. and Sudhölter, P. (1992): The Nucleolus of Homogeneous Games with Steps. To appear in Discrete Applied Mathematics

Shapley, L.S. (1962): Simple Games: An Outline of the Descriptive Theory. Behavioral Sci. 7, pp.59–66

Sobolev, A.I. (1975): A Characterization of Optimality Principles in Cooperative Games by Functional Equations. N. Vorobjev, Ed., Math. Meth. Social Sci. 6 (Academy of the Lithuanian SSR, Vilnius), pp.94–151

Sudhölter, P. (1989): Homogeneous Games as Anti Step Functions. Int. Journal of Game Theory 18, pp.433–469

von Neumann, J. and Morgenstern, O. (1944): Theory of Games and Economic Behavior. Princeton University Press, New Jersey