

INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 297

An Axiomatization of the Core

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March 1998



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Abstract

We prove that the core on the set of all transferable utility games with players contained in a universe of at least five members can be axiomatized by nonemptiness for two-person flat games, covariance under strategic equivalence, anonymity, individual rationality, the converse reduced game property, the weak reduced game property, and the reduced game property from below (RGPB). Here, a solution satisfies RGPB, if for every member of the solution of the game the following condition is satisfied: Every feasible payoff vector belongs to the solution, whenever its restriction to some coalition is a member of the solution of the reduced game and its restriction to the complement coalition coincides with the corresponding restriction of the initial vector. Moreover, individual rationality can be replaced by boundedness. Finally we prove that these properties also characterize the core on certain subsets of games, e.g., on the set of totally balanced games, on the set of balanced games, and on the set of superadditive games.

Key words: TU-game, core, kernel.

0 Introduction

On balanced cooperative transferable utility games and on some subclasses the core can be axiomatized (see, e.g., Peleg (1986,1989)). However, in the well-known axiomatizations either nonemptiness or the property of "coincidence with the core on two-person games" are employed. The characterization of the core presented in this paper does neither refer to balanced games nor does it use one of the axioms just mentioned. That may be regarded as an advantage over the axiomatizations that are known from literature. Except nonemptiness (which is relaxed) the axioms employed in the present results have been used to characterize the core or the prenucleolus.

The paper is organized as follows: In Section 1 the notation and some definitions are presented and some relevant well-known results are recalled. Two axioms which were not frequently used up to now are nonemptiness for two-person flat games (NETPG) and the reduced game property from below (RGPB). For the detailed description see Definitions 1.5 and 1.6. However, the first property is weaker than nonemptiness and the second property is, alike the reduced game property, a set-valued generalization of the reduced game property for single-valued solutions as introduced by Sobolev (1975).

In Section 2 it is shown that on the set of games with player set contained in a universe of at least five members the core is the unique solution that satisfies nonemptiness for two-person flat games, covariance under strategic equivalence, anonymity, the (weak) reduced game property and its converse, RGPB, and individual rationality (IR). Especially the last property can be weakened. Boundedness is a property which is able to replace IR.

In Section 3 it is shown that Theorem 2.1 is also valid for every subset of games that contains every totally balanced game and does not contain nonbalanced two-person games. The considered set of games is, thus, "closed under weak reduction" with respect to members of the core (meaning that every two-person reduced game with respect to a member of the core belongs to the considered set of games). Among others the subset of all superadditive (balanced) games has the required properties. On such sets of games AN can be dropped as a condition.

In Section 4 it is shown that the axioms that occur in Theorems 2.1 and 3.1 are logically independent. Moreover, it turns out that boundedness can be weakened and it can even be replaced by AN in the second theorem, if the converse reduced game property is formulated with respect to feasible payoffs instead of preimputations.

Some results of Section 4 are proved in Section 5.

1 Notation and Definitions

A cooperative game with transferable utility – a game – is a pair (N, v) , where N is a finite nonvoid set and

$$v : 2^N \rightarrow \mathbb{R}, v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \subseteq N\}$ is the set of coalitions of (N, v) .

If (N, v) is a game, then N is the **grand coalition** or the set of **players** and v is called **coalitional function** of (N, v) . If $\emptyset \neq S \subseteq N$, then (S, v) denotes the **subgame** of (N, v) w.r.t. the coalition S . (The coalitional function of the subgame w.r.t. S is the restriction of v to subsets of S .)

The set of **feasible payoff vectors** of G is denoted by

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

whereas

$$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

is the set of **preimputations** of (N, v) (also called set of **Pareto optimal feasible payoffs** of (N, v)). Here

$$x(S) = \sum_{i \in S} x_i \quad (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S , i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S.$$

For disjoint coalitions $S, T \subseteq N$ and $x \in \mathbb{R}^N$ let $(x_S, x_T) = x_{S \cup T}$.

A **solution** σ on a set Γ of games is a mapping that associates with every game $(N, v) \in \Gamma$ a set $\sigma(N, v) \subseteq X^*(N, v)$.

If $\bar{\Gamma}$ is a subset of Γ , then the canonical restriction of a solution σ on Γ is a solution on $\bar{\Gamma}$. We say that σ is a solution on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution on every set of games. Typically we shall assume that a solution σ is defined on a subset of Γ_U . Here Γ_U denotes the set of **all games** with player set contained in U . The **universe** U of **players** is assumed to be a set.

Some convenient and well-known properties of a solution σ on a set Γ of games are as follows.

- (1) σ is **anonymous** (satisfies **AN**), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \rightarrow N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j}$ ($x \in \mathbb{R}^N$, $j \in N'$, $T \subseteq N'$)). In this case (N, v) and $(N', \tau v)$ are **isomorphic games**.

- (2) σ is **covariant under strategic equivalence** (satisfies **COV**), if for $(N, v), (N, w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}^N$

$$\sigma(N, w) = \alpha \sigma(N, v) + \beta$$

holds. The games v and w are called **strategically equivalent**.

- (3) σ satisfies **nonemptiness** (**NE**), if $\sigma(N, v) \neq \emptyset$ for $(N, v) \in \Gamma$.

- (4) σ is **Pareto optimal** (satisfies **PO**), if $\sigma(N, v) \subseteq X(N, v)$ for $(N, v) \in \Gamma$.

(5) σ satisfies **individual ratiounality**, if $x_i \geq v(\{i\})$ for every $i \in N$ holds true for $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$.

Some more notation will be needed. Let (N, v) be a game and $x \in \mathbb{R}^N$. The **excess** of a coalition $S \subseteq N$ at x is the real number

$$e(S, x, v) = v(S) - x(S).$$

For **different** players $i, j \in N$ let

$$s_{ij}(x, v) = \max\{e(S, x, v) \mid i \in S \subseteq N \setminus \{j\}\}$$

denote the **maximum surplus of i over j at x** .

The **core** of (N, v) is the set

$$\mathcal{C}(N, v) = \{x \in X^*(N, v) \mid e(S, x, v) \leq 0 \forall S \subseteq N\}$$

of feasible payoff vectors which generate nonpositive excesses. The **prekernel** of (N, v) is the set

$$\mathcal{PK}(N, v) = \{x \in X(N, v) \mid s_{ij}(x, v) = s_{ji}(x, v) \forall i, j \in N \text{ with } i \neq j\}$$

of preimputations that **balance** the maximum surplus of the pairs of players.

The **pre-nucleolus** of (N, v) , abbreviated by $\mathcal{PN}(N, v)$, is the set of preimputations that lexicographically minimize the nonincreasingly ordered vector of excesses of the coalitions. The pre-nucleolus of a game is a singleton.

On the set Γ_U the prekernel as well as the pre-nucleolus satisfy all properties mentioned so far except individual rationality. The core satisfies all axioms except NE. On the subset of **balanced** games the core satisfies NE.

For these notations and assertions see Davis and Maschler (1965), Schmeidler (1969), Bondareva (1963), and Shapley (1967).

Axiomatizations of the pre-nucleolus and the prekernel on Γ_U are due to Sobolev (1975) and Peleg (1986). On the subsets of balanced or totally balanced games (a game is **totally balanced**, if each of its subgames is balanced) the core can be axiomatized (see Peleg (1986, 1989)). In order to precisely formulate some of these characterizations we recall some definitions.

Definition 1.1 Let (N, v) be a game, let $\emptyset \neq S \subseteq N$, and $x \in X^*(N, v)$. The **reduced game w.r.t. S and x** is the game $(S, v^{S,x})$ defined by

$$v^{S,x}(T) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(N) - x(N \setminus S), & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\}, & \text{otherwise} \end{cases}$$

Definition 1.2 Let σ be a solution on a set Γ of games. Then σ satisfies the

- (1) **reduced game property (RGP)**, if the following condition holds: If $(N, v) \in \Gamma$, $\emptyset \neq S \subseteq N$, and $x \in \sigma(N, v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.
- (2) **weak reduced game property (WRGP)**, if the following condition holds: If $(N, v) \in \Gamma$, $\emptyset \neq S \subseteq N$, $|S| \leq 2$, and $x \in \sigma(N, v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.
- (3) **converse reduced game property (CRGP)**, if the following condition holds: If $(N, v) \in \Gamma$, $x \in X(N, v)$, and for every $S' \subseteq N$ with two members $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N, v)$.

Note that Definition 1.2(2) is due to Peleg (1989) and that RGP implies WRGP. Furthermore, note that the prekernel and the core satisfy CRGP and RGP, if the set Γ of games is rich enough. Now two results are recalled.

Theorem 1.3 (Sobolev (1975)) *If the universe U of players is infinite, then the prenucleolus is the unique solution on Γ_U that satisfies single-valuedness, COV, AN, and RGP.*

A solution on a set Γ of games satisfies **superadditivity (SUPA)**, if $x^1 + x^2 \in \sigma(N, v^1 + v^2)$, whenever $(N, v^1), (N, v^2), (N, v^1 + v^2) \in \Gamma$, $x^1 \in \sigma(N, v^1)$ and $x^2 \in \sigma(N, v^2)$.

Theorem 1.4 (Peleg (1989)) *If the universe U contains at least four players, then the core is the unique solution on the set of totally balanced games in Γ_U that satisfies NE, SUPA, WRGP, CRGP, and individual rationality.*

Let Γ_U^b and Γ_U^{tb} respectively denote the set of all balanced and totally balanced games in Γ_U .

We shall present an axiomatization of the core (see Theorem 2.1) which can be compared with the preceding results. We do not want to employ NE, because our attention is **not** restricted to balanced games. Therefore we demand less than NE.

Definition 1.5 *A solution on a set Γ of games satisfies **nonemptiness for two-person flat games (NETPFG)**, if for every flat two-person game $(N, v) \in \Gamma$, i.e. $|N| = 2$, $v(S) = 0$ for $S \subseteq N$,*

$$\sigma(N, v) \neq \emptyset.$$

Note that we do not impose from a solution which satisfies NETPFG that it yields $(0, 0) \in \mathbb{R}^N$ for every flat two-person game; we only require nonemptiness. An idea of this property is as follows. Suppose two players bargain about how to share the worth of the grand coalition **with respect to** (w.r.t.) their two-person game. If this worth is positive or negative, it may happen that they do not reach any agreement. If the game is flat they should be indifferent between leaving the game, thus obtaining zero (0), and sharing the worth (0) of the grand coalition equally. NETPFG only requests that the two players do not leave the flat game without an agreement.

Definition 1.6 Let σ be a solution on a set Γ of games.

- (1) The solution σ satisfies the **reduced game property from below**, if the following condition is satisfied: If $(N, v) \in \Gamma$, $x \in \sigma(N, v)$ and $S \subseteq N$ such that $(S, v^{S,x}) \in \Gamma$, then $(y, x_{N \setminus S}) \in \sigma(N, v)$ holds true for every $y \in \sigma(S, v^{S,x})$.
- (2) σ satisfies the **strong reduced game property (SRGP)**, if σ satisfies RGP and the reduced game property from below.

For interpretations of the notion of the reduced game, the reduced game property, and the converse reduced game property see, e.g., Maschler (1992). In some sense RGP is a reduced game property from above. Indeed, if a solution satisfies RGP, then the restriction of any member of the solution of a game belongs to the solution of the corresponding reduced game. RGPB reflects, in some sense, the opposite direction. Every member of the solution of a reduced game yields an element of the solution of the game, whenever it is combined with the corresponding restriction of the initial element of the solution. More precisely, on Γ_U the reduced game properties can be described as follows. A solution σ satisfies RGP or RGPB respectively, if for every game $(N, v) \in \Gamma_U$, every $x \in \sigma(N, v)$, and every nonempty coalition $S \subseteq N$

$$\{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in \sigma(N, v)\} \subseteq \sigma(S, v^{S,x})$$

or

$$\{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in \sigma(N, v)\} \supseteq \sigma(S, v^{S,x})$$

holds true respectively.

Note that RGP and the reduced game property from below are equivalent for single-valued solutions on Γ_U . Moreover, it should be remarked that the core satisfies RGPB on every set of games.

A solution on a set Γ of games is said to satisfy the **strong reduced game property (SRGP)**, if it satisfies RGP and RGPB. It satisfies the **semi strong reduced game property (SSRGP)**, if it satisfies WRGP and RGPB.

Now an axiomatization of the core on Γ_U can be formulated.

Theorem 1.7 If $|U| \geq 5$, then the core is the unique solution on Γ_U that satisfies NETPFG, AN, COV, SRGP, CRGP, and individual rationality.

This Theorem is a direct consequence of Theorem 2.1, in which SRGP is replaced by SSRGP and individual rationality is replaced by BOUND. A solution σ on a set Γ satisfies **boundedness (BOUND)**, if $\sigma(N, v)$ is bounded (from below) for every game $(N, v) \in \Gamma_U$. Of course individual rationality implies BOUND. Moreover, Theorem 1.7 constitutes an axiomatization as shown in Section 4.

2 A Characterization of the Core

Our main result is the following theorem.

Theorem 2.1 *If the universe U of players contains at least 5 elements, then the core is the unique solution on Γ_U that satisfies NETPFG, AN, COV, SSRGP, CRGP, and BOUND.*

We postpone the proof of Theorem 2.1 and shall now prove several useful lemmata globally assuming that σ is a solution on some set Γ satisfying $\Gamma_U^{tb} \subseteq \Gamma \subseteq \Gamma_U$. The **standard** solution of a two-person game (N, v) is denoted by $x^{(N, v)}$ (i.e., $x_i^{(N, v)} = (v(\{i\}) - v(\{j\}) + v(N))/2$, where $N = \{i, j\}$). We start with the following simple result.

Lemma 2.2 *If σ satisfies COV, WRGP and BOUND, then σ satisfies PO.*

Proof: Let $(N, v) \in \Gamma$. If $|N| = 1$, then COV and BOUND imply that every member of $\sigma(N, v)$ is Pareto optimal. If $|N| \geq 2$, then WRGP applied to an arbitrary coalition $S \subseteq N$ of size 1 implies Pareto optimality of $\sigma(N, v)$. q.e.d.

For the remainder of this section we assume that the universe U of players contains at least three members, let us say 1, 2 and 3.

Lemma 2.3 *If σ satisfies NETPFG, COV, WRGP, CRGP, and BOUND and if $(N, v) \in \Gamma$ is the inessential game given by $v(S) = x(S)$ for some $x \in \mathbb{R}^N$, then $x \in \sigma(N, v)$.*

Proof: In the case $|N| = 2$ NETPFG, COV, and BOUND show that $\{x\} = \sigma(N, v)$. The fact that Γ contains every inessential two-person game in Γ_U together with WRGP implies the assertion for $|N| = 1$. If $|N| \geq 3$, then CRGP shows the assertion. q.e.d.

Lemma 2.4 *If $|U| \geq 5$ and σ satisfies NETPFG, COV, SSRGP, CRGP, and BOUND, then $\text{EXT } \mathcal{C}(N, v) \subseteq \sigma(N, v)$ for every $(N, v) \in \Gamma$ with $|N| = 2$.*

Here $\text{EXT } A$ denotes the set of extremal points of the convex subset A of some Euclidean space.

Proof: Let $(N, v) \in \Gamma$ satisfy $|N| = 2$, let us say $N = \{1, 5\}$, and $\mathcal{C}(N, v) \neq \emptyset$. It suffices to show that $(v(N) - v(\{5\}), v(\{5\})) \in \sigma(N, v)$. Without loss of generality we assume $v(\{1\}) = v(\{5\}) = 0$ (by COV) and $v(N) = 1$ (by Lemma 2.3 and COV). Let $\tilde{N} \subseteq U$ be a superset of N of cardinality 5, let us say $\tilde{N} = \{1, 2, 3, 4, 5\}$, and define for every $\alpha \in \mathbb{R}$

the game (\tilde{N}, w_α) , by

$$w_\alpha(S) = \begin{cases} \alpha - 4 \cdot |S|, & \text{if } \begin{cases} |S| \leq 2 \text{ and } S \notin \{\{2, 4\}, \{3, 4\}\} \\ \text{or } S \in \{\{2, 3, 5\}, \{1, 4, 5\}, \{1, 2, 3\}\} \end{cases} \\ \alpha - 4, & \text{if } S = \{2, 3, 4\} \\ \alpha - 1, & \text{if } |S| = 4 \\ 0, & \text{if } S \in \{\emptyset, \tilde{N}\} \\ \alpha, & \text{otherwise} \end{cases}$$

(In the present proof only $w = w_0$ is needed, but different values of the parameter α will be used in two other proofs.) Let $x = (0, 0, 0, 0, 0) \in \mathbb{R}^{\tilde{N}}$ and $u = w^{\{1, 2, 3, 4\}, x}$.

Claim 1: (\tilde{N}, w) is totally balanced.

Let $\emptyset \neq S \subseteq \tilde{N}$, $S \neq \tilde{N}$. It remains to show that the subgame (S, w) is balanced. We distinguish the following cases:

- (1) $|S| \leq 2$: The fact that $w(\{i\}) = -4$ for $i \in \tilde{N}$ and $v(S) \geq -4|S|$ shows balancedness in this case.
- (2) $w(S) = 0$: The fact that $w(T) \leq 0$ for $S \subseteq \tilde{N}$ shows balancedness in this case.
- (3) $S \in \{\{2, 3, 5\}, \{1, 4, 5\}, \{1, 2, 3\}\}$: Then (S, w) is inessential, thus the core is nonempty.
- (4) $S = \{2, 3, 4\}$: Then $(-4, -4, 4) \in \mathcal{C}(S, w)$ can easily be checked.
- (5) $S = \{1, 2, 3, 4\}$: Then $(1, -2, -2, 2) \in \mathcal{C}(S, w)$ holds true.
- (6) $S = \{1, 2, 3, 5\}$: Then $(1, -1, -1, 0) \in \mathcal{C}(S, w)$ holds true.
- (7) $S = \{1, 2, 4, 5\}$: Then $(-1, 2, -1, -1) \in \mathcal{C}(S, w)$ holds true.
- (8) $S = \{1, 3, 4, 5\}$: Then $(-1, 2, -1, -1) \in \mathcal{C}(S, w)$ holds true.
- (9) $S = \{2, 3, 4, 5\}$: Then $(-1, -1, 1, 0) \in \mathcal{C}(S, w)$ holds true.

Claim 2: $(\{1, 2, 3, 4\}, u)$ is totally balanced.

Indeed, u is balanced, because $u(S) \leq 0$ for every $S \subseteq \{1, 2, 3, 4\}$ and $u(\{1, 2, 3, 4\}) = 0$. Moreover, $u(S) \geq -4 \cdot |S|$, $u(\{i\}) = -4$ shows balancedness of one- and two-person subgames. If $S = \{1, 2, 4\}, \{1, 3, 4\}$, then $u(S) = 0$ and the subgame (S, u) is balanced. Finally, if $S = \{1, 2, 3\}$ or $S = \{2, 3, 4\}$, then $u(S) = -1$ and the vector $(1, -1, -1)$ or $(-1, -1, 1)$ respectively belongs to the core.

Now the proof can be completed. Claims 1 and 2 show that both, (\tilde{N}, w) and the reduced game $(\{1, 2, 3, 4\}, u)$, belong to Γ . We come up with $s_{ij}(x, w) = 0$ for $i, j \in \tilde{N}$ with

$i \neq j$, thus $x \in \sigma(\tilde{N}, u)$ by CRGP and Lemma 2.3. Let $y \in \mathbb{R}^{\{1,2,3,4\}}$ be given by $y = (1, -1, -1, 1)$. Then

$$s_{ij}(y, u) = 0 \quad \forall i, j \in \{1, 2, 3, 4\} \quad (2.1)$$

thus $y \in \sigma(\{1, 2, 3, 4\}, u)$ again by CRGP and Lemma 2.3. By RGPB $z = (y, 0) \in \sigma(\tilde{N}, w)$. The fact that $s_{51}(z, w) = 0 > -1 = s_{15}(z, w)$ finishes the proof, because it shows that the reduced game $(\{1, 5\}, w^{\{1,5\},z})$ is (N, v) and that $(1, 0) \in \sigma(N, v)$ by WRGP. **q.e.d.**

Lemma 2.5 *If $|U| \geq 5$ and σ satisfies NETPFG, COV, SSRGP, CRGP, and BOUND, then $\mathcal{C}(N, v) \subseteq \sigma(N, v)$ for every $(N, v) \in \Gamma$ satisfying $|N| = 2$.*

Proof: Let (N, v) be a balanced two-person game in Γ and let $x \in \mathcal{C}(N, v)$, let us say $N = \{1, 2\}$. By COV we can assume that $v(\{1\}) = v(\{2\}) = 0$ and without loss of generality we can assume $x_1 \leq x_2$. If $v(N) = 0$, then Lemma 2.3 finishes the proof. Therefore $v(N) = 1$ can be assumed by COV. Therefore $0 \leq x_1 \leq 1 - x_1 = x_2 \leq 1$ holds true. Take any player $i \in U \setminus N$, let us say $i = 3$, and define (\tilde{N}, u) with $\tilde{N} = \{1, 2, 3\}$ by

$$u(S) = \begin{cases} 1 - x_1, & \text{if } S = N, \\ 1, & \text{if } S = \tilde{N} \\ 0, & \text{otherwise} \end{cases}$$

Then (\tilde{N}, u) is totally balanced. With $y = (0, 1 - x_1, x_1) \in \mathbb{R}^{\tilde{N}}$ we come up with

$$u^{N,y}(\{1\}) = u^{N,y}(\{2\}) = 0, \quad u^{N,y}(N) = 1 - x_1, \quad (2.2)$$

$$u^{\{1,3\},y}(\{1\}) = u^{\{1,3\},y}(\{3\}) = 0, \quad u^{\{1,3\},y}(\{1,3\}) = x_1, \quad \text{and} \quad (2.3)$$

$$u^{\{2,3\},y}(\{2\}) = 1 - x_1, \quad u^{\{2,3\},y}(\{3\}) = 0, \quad u^{\{2,3\},y}(\{2,3\}) = 1, \quad (2.4)$$

thus $y \in \sigma(\tilde{N}, u)$ by Lemma 2.3, Lemma 2.4, COV, and CRGP. Again by Lemma 2.4 (or Lemma 2.3 in the trivial case that $x_1 = 0$) the preimputation which arises from $x_{\{1,3\}}$ by exchanging the components belongs to $\sigma(\{1, 3\}, u^{\{1,3\},y})$. RGPB shows that $z = (x_1, 1 - x_1, 0) \in \sigma(\tilde{N}, u)$ holds true, thus WRGP applied to the reduced game $(N, u^{N,z})$ shows that $x \in \sigma(N, u^{N,z})$. However, this reduced game is (N, v) . **q.e.d.**

Lemmata 2.2, ..., 2.5 will be used in the proof of Theorem 2.1 as well as in the next section. The following result only applies in the current section.

Lemma 2.6 *If $|U| \geq 5$, $\Gamma = \Gamma_U$, and σ satisfies AN, COV, SSRGP, CRGP, and BOUND, then*

$$\sigma(N, v) = \emptyset$$

for every two-person game $(N, v) \in \Gamma_U$ satisfying $\mathcal{C}(N, v) = \emptyset$.

Proof: Assume, on the contrary, that there is $x \in \sigma(N, v)$. Without loss of generality $N = \{1, 2\}$ can be assumed. Moreover, by COV, we can assume that $v(\{1\}) = v(\{2\}) = 0$ and $v(N) = -1$ hold true. Without loss of generality $x_1 \leq x_2 = 1 - x_1$ (by PO).

Step 1: $x^{(N,v)} \in \sigma(N, v)$.

With $\tilde{N} = \{1, 2, 3\}$ we define a game (\tilde{N}, w) by

$$w(S) = \begin{cases} -x_1, & \text{if } S = \{1\}, N \\ -x_2, & \text{if } |S| = 2 \text{ and } S \neq N \\ x_1 - 2x_2, & \text{if } |S| = 1 \text{ and } S \neq \{1\} \\ 0, & \text{if } S = \emptyset, \tilde{N} \end{cases}$$

COV, AN, and CRGP imply that $y^0 = (0, 0, 0) \in \sigma(\tilde{N}, w)$. Indeed, a straightforward computation shows that

$$s_{12}(y^0, w) = s_{13}(y^0, w) = s_{23}(y^0, w) = s_{12}(x, v) = -x_1$$

and

$$s_{21}(y^0, w) = s_{31}(y^0, w) = s_{32}(y^0, w) = s_{21}(x, v) = -x_2$$

hold true.

With $a = x_2 - x_1$ and using AN and COV we come up with $(a, -a) \in \sigma(N, w^{N,y^0})$. Putting $y = (a, -a, 0)$ we get $y \in \sigma(\tilde{N}, w)$ by RGPB.

Note that

$$s_{13}(y, w) = s_{31}(y, w) = -x_1. \quad (2.5)$$

Equation (2.5) (together with WRGP, AN and COV) shows Claim 1.

Step 2: Now the proof can be finished. Let \tilde{N} , w_α , x, y, z be defined as in the proof of Lemma 2.4 and put $\bar{\alpha} = 1/2$. Then (compare with (2.1))

$$s_{ij}(x, w_{\bar{\alpha}}) = s_{kl}(y, u_{\bar{\alpha}}) = 1/2 \quad \forall i, j \in \tilde{N}, k, l \in \{1, 2, 3, 4\} \text{ with } i \neq j, k \neq l,$$

where $u_{\bar{\alpha}} = w_{\bar{\alpha}}^{\{1,2,3,4\},x}$. CRGP and Step 1 imply $x \in \sigma(\tilde{N}, w_{\bar{\alpha}})$ and $y \in \sigma(\{1, 2, 3, 4\}, u_{\bar{\alpha}})$, thus $z \in \sigma(\tilde{N}, w_{\bar{\alpha}})$ by RGPB. The fact that

$$s_{51}(z, w_{\bar{\alpha}}) = 1/2 > -1/2 = s_{15}(z, w_{\bar{\alpha}})$$

shows that $(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$ is inessential. By WRGP the restriction $z_{\{1,5\}}$ is a member of $\sigma(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$, thus, for every $\beta > 0$,

$$(1 + \beta, 1 - \beta)/2 \in \sigma(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$$

by COV. This observation contradicts BOUND.

q.e.d.

Proof of Theorem 2.1: As shown in Section 1 the core satisfies the desired properties. In order to show the uniqueness part, let σ be a solution on Γ_U which satisfies all properties. It suffices to show that σ coincides with the core on two-person games. Let $(N, v) \in \Gamma_U$ satisfy $|N| = 2$. In view of Lemma 2.5 it remains to show that $\sigma(N, v) \subseteq \mathcal{C}(N, v)$ is true. Assume the contrary. If $\mathcal{C}(N, v) = \emptyset$, then Lemma 2.6 completes the proof. If

$\mathcal{C}(N, v) \neq \emptyset$, then we can assume that $N = \{1, 2\}$. With $\tilde{N} = \{1, 2, 3\}$ we define (\tilde{N}, u) by $u(S) = v(S \cap N)$. Choose $x \in \sigma(N, v) \setminus \mathcal{C}(N, v)$ and observe that $(x^{(N,v)}, 0) \in \sigma(\tilde{N}, u)$ by CRGP and Lemma 2.5. Therefore $y = (x, 0) \in \sigma(\tilde{N}, u)$ by RGPB. By WRGP $y_{\{1,3\}} \in \sigma(\{1, 3\}, u^{\{1,3\},v})$. However, the fact that this reduced game is **not** balanced directly leads to a contradiction to Lemma 2.6. q.e.d.

3 Totally Balanced Games

This section shows that suitable modifications of Theorem 2.1 are true for certain subsets of Γ_U .

Theorem 3.1 *If $|U| \geq 5$, then the core is the unique solution on Γ_U^{tb} that satisfies NETPFG, COV, SSRGP, CRGP, and BOUND.*

Proof: Uniqueness remains to be shown. Let σ be a solution that satisfies NETPFG, COV, SSRGP, CRGP, and BOUND, thus PO. In view of Lemma 2.5 it suffices to show that σ is a subsolution of the core. By WRGP it suffices to show this assertion for two-person games. A careful inspection of the proof of Theorem 2.1 shows that, given the contrary, there is a nonbalanced two-person game in Γ_U^{tb} which is not the case. q.e.d.

Remark 3.2 *Theorem 3.1 remains valid, if Γ_U^{tb} is replaced by any superset of Γ_U^{tb} which does not contain nonbalanced two-person games. Examples are Γ_U^b , the set of all balanced superadditive games, and the set of all superadditive games in Γ_U .*

4 Independence and Modifications of the Axioms

The following examples show that the properties used in Theorems 2.1 and 3.1 are logically independent. We show that these results are not valid, if $|U| = 4$, and simultaneously we show the independence of SRGP and SSRGP.

Example 4.1 *Define $\sigma^0(N, v) = \mathcal{C}(N, v) \cap \mathcal{PK}(N, v)$. It is well-known (see, e.g., Sudhölter (1993)) that the prekernel coincides with the prenucleolus for games with at most three players. Therefore $|\sigma^0(N, v)| \leq 1$ in the case that $|N| \leq 3$ is satisfied. In view of this fact σ^0 satisfies RGPB on any subset Γ of Γ_U , whenever $|U| \leq 4$. It satisfies NETPFG, AN, COV, CRGP, and BOUND. On every superset of Γ_U^{tb} the solution σ^0 satisfies WRGP and it satisfies RGP on Γ_U . In the case that $|U| \geq 5$ this solution satisfies all properties except SSRGP or SRGP.*

From now on we assume that the universe U of players contains at least five members. The empty solution Φ ($\Phi(N, v) = \emptyset \forall (N, v) \in \Gamma_U$) satisfies all properties except NETPFG.

Example 4.2 Choose two distinct players, let us say 1 and 2, of U and define σ^1 on Γ_U by

$$\sigma^1(N, v) = \begin{cases} \{x^{(N, v)}\}, & \text{if } N = \{1, 2\} \text{ and } C(N, v) = \emptyset \\ C(N, v), & \text{otherwise} \end{cases}$$

Then σ^1 satisfies NETPFG, COV, SRGP, and BOUND.

Claim: σ^1 satisfies CRGP.

Let $N \subseteq U$ with $|N| \geq 3$ and (N, v) be a game with $C(N, v) \neq \emptyset$. Moreover, let $x \in X(N, v)$ satisfy $x_S \in \sigma^1(S, v^{S, x})$, whenever $S \subseteq N$ with $|S| = 2$. If $\{1, 2\} \not\subseteq N$, then $x \in C(N, v)$ by definition. If $S = \{1, 2\} \subseteq N$, it suffices to show that $C(S, v^{S, x}) \neq \emptyset$. Assume the contrary. Then $s_{12}(x, v) = s_{21}(x, v) > 0$. Let $s_{12}(x, v)$ be attained by T . If $T = N \setminus \{2\}$, then $s_{j2}(x, v) \geq e(T, x, v) > 0$ for every $j \in N \setminus \{1, 2\}$, which is impossible. Otherwise there is some player $j \in N \setminus (T \cup \{2\})$ and $s_{1j}(x, v) \geq e(T, x, v) > 0$ yields a contradiction.

We conclude that σ^1 satisfies all properties of Theorem 2.1 except AN.

Example 4.3 for every game (N, v) let the "equal treatment vector" $x^{et, v} \in \mathbb{R}^N$ be given by $x_i^{et, v} = v(N)/|N|$. Then σ^2 , defined by

$$\sigma^2(N, v) = \begin{cases} \{x^{et, v}\}, & \text{if } x^{et, v} \in C(N, v) \\ \emptyset, & \text{otherwise} \end{cases}$$

satisfies all properties except COV.

Example 4.4 The solution σ^3 defined by $\sigma^3 = \mathcal{PN}(N, v) \cap C(N, v)$ shows the independence of CRGP.

Example 4.5 The solution which assigns to every game its set of preimputations shows the independence of BOUND in Theorem 2.1. This solution is bounded, if it is restricted to one-person games. In order to show the independence of BOUND in Theorem 3.1 we choose two distinct players, let us say 1 and 2, in U and define the solution σ^4 on Γ_U^{tb} by

$$\sigma^4(N, v) = \begin{cases} C(N, v), & \text{if } \{1, 2\} \not\subseteq N \\ X(N, v) \setminus \{x^{(N, v)}\}, & \text{if } N = \{1, 2\} \text{ and } (N, v) \text{ is inessential} \\ \emptyset, & \text{otherwise} \end{cases}$$

Then σ^4 satisfies NETPFG, COV, SSRGP, and BOUND¹, i.e., boundedness for one-person games. In order to show that it satisfies CRGP, it suffices to choose $(N, v) \in \Gamma_U^{tb}$ with $\{1, 2\} \subseteq N$, $\{1, 2\} \neq N$ and to show that there is no $x \in X(N, v)$ with $x_S \in \sigma^4(S, v^{S, x})$. Assume, on the contrary, that there is a preimputation x with the required property. Then the reduced game w.r.t. the coalition $\{1, 2\}$ and x is inessential, thus $s_{12}(x, v) = -s_{21}(x, v) \neq 0$, let us say $s_{12}(x, v) > 0$ (otherwise exchange the rôles of 1 and 2). Let $S \subseteq N$ be any coalition attaining $s_{12}(x, v)$. Two cases may occur:

- (1) $S = N \setminus \{2\}$: Then $s_{1j}(x, v) \geq e(S, x, v) > 0$ for every $j \in N \setminus \{1, 2\}$, thus $x_T \notin \mathcal{C}(T, v^{T,x})$, where $T = \{1, j\}$, which is impossible.
- (2) $S \neq N \setminus \{2\}$: Then $s_{j2}(x, v) \geq e(S, x, v) > 0$ for every $j \in N \setminus S \cup \{2\}$, thus the proof can be finished as before.

Remark 4.6 It is possible to relax BOUND. Indeed, this property is only used in three proofs. The first occurrence can be located in the proof of Lemma 2.2. In fact, only BOUND^1 is needed here. BOUND secondly occurs in the proof of Lemma 2.3 and it is used to show that the standard solution belongs to the solution when applied to any two-person inessential game. BOUND is thirdly used in the proof of Lemma 2.6, actually in the form of BOUND^{2f} , i.e., boundedness, if the solution is restricted to two-person flat games. If $|U| \geq 2$, then Lemma 2.2 remains true, if BOUND^1 is replaced by BOUND^{2f} and NETPFG and RGPB are added. Thus BOUND can be replaced by BOUND^{2f} in both Theorems. Moreover, the following two results will be proved in Section 5.

Theorem 4.7 Theorem 3.1 is valid, if BOUND is replaced by AN and BOUND^1 .

Note that the axioms that occur in Theorem 4.7 are logically independent by the preceding examples and the following example.

Example 4.8 Let σ^5 on Γ_U be defined by

$$\sigma^5(N, v) = \begin{cases} X^*(N, v) \setminus X(N, v), & \text{if } |N| \leq 2 \\ \emptyset, & \text{otherwise} \end{cases}$$

On every set of $\Gamma_U^{\text{tb}} \subseteq \Gamma \subseteq \Gamma_U$ the solution σ^5 satisfies NETPFG, COV, AN, SSRGP and CRGP. However, it does not satisfy BOUND^1 .

However, it is possible to replace BOUND by AN, if a stronger version of CRGP is used. A solution σ on a set Γ of games satisfies **CRGP***, if the phrase " $x \in X(N, v)$ " in the definition of CRGP (see Definition 1.2(3)) is replaced by " $x \in X^*(N, v)$ ".

Theorem 4.9 If $|U| \geq 5$ and if Γ is a set of games with $\Gamma_U^{\text{tb}} \subseteq \Gamma \subseteq \Gamma_U$ which does not contain any nonbalanced two-person game, then the core is the unique solution on Γ that satisfies NETPFG, COV, SSRGP, **CRGP***, and AN.

In order to show the impact of NETPFG we now describe all solutions that satisfy the remaining properties. For proofs see Section 5. Let $\text{int } \mathcal{C}$ denote the interior of the core, i.e.,

$$\text{int } \mathcal{C}(N, v) = \{x \in \mathcal{C}(N, v) \mid e(S, x, v) < 0 \forall \emptyset \neq S \neq N\}.$$

Note that $\text{int } \mathcal{C}$ satisfies all properties of the theorems except NETPFG.

Theorem 4.10 Let $|U| \geq 5$. The unique solutions that satisfy AN, COV, SSRGP, CRGP, and BOUND on Γ_U or Γ_U^{fb} respectively, are Φ , int \mathcal{C} , and \mathcal{C} .

Note that AN cannot be dropped as a condition of Theorem 4.10 even in the case of totally balanced games as the following example shows.

Example 4.11 Choose distinct players, let us say 1 and 2, of U . The solution σ^6 , defined by

$$\sigma^6(N, v) = \begin{cases} \mathcal{C}(N, v), & \text{if } N \subseteq \{1, 2\} \\ \emptyset, & \text{otherwise} \end{cases}$$

satisfies all axioms of Theorem 4.10 except AN.

5 Appendix

Proof of Theorem 4.7: Let σ satisfy the required axioms. In view of Remark 4.6 it suffices to show that $\sigma(N, v)$ is bounded, whenever $(N, v) \in \Gamma_U$ is a flat two-person game. Assume the contrary. Let (N, v) be the flat game and let us assume $N = \{1, 2\}$ for simplicity. By AN, COV and NETPFG

$$\sigma(N, v) \supseteq X(N, v) \setminus \{(0, 0)\}.$$

Choose $x \in \sigma(N, v)$ satisfying $x_1 < 0 < x_2 = -x_1$. This can be done by AN and PO. Let $(\tilde{N}, w), y^0, y$ be defined as in the proof of Claim 1 of Lemma 2.6. It is easy to check that this game is totally balanced, thus $y^0 \in \sigma(\tilde{N}, w)$ by COV, CRGP, and AN. Therefore $y \in \sigma(\tilde{N}, w)$ by RGPB. However, the reduced game $(\{1, 3\}, w^{\{1, 3\}, y})$ is not balanced. q.e.d.

Proof of Theorem 4.9: Let σ satisfy the required axioms. It suffices to show that $\sigma(N, v)$ is Pareto optimal for every flat two-person game $(N, v) \in \Gamma_U$. Assume the contrary and let (N, v) be a flat game, let us say $N = \{1, 2\}$, and let $x \in \sigma(N, v)$ satisfy $x_2 \neq -x_1$, i.e., $x_2 < -x_1$. By AN we can assume $x_1 \leq x_2$. Take $i \in U \setminus N$, let us say $i = 3$, and let $\tilde{N} = \{1, 2, 3\}$ and let (\tilde{N}, w) be defined by

$$w(S) = \begin{cases} -x_1, & \text{if } S = \{1\}, N \\ -x_2, & \text{if } |S| = 2 \text{ and } S \neq N \\ \min\{x_1 - x_2, x_1 - 2x_2\}, & \text{if } S = \{2\}, \{3\} \\ -x_1 - x_2, & \text{if } S = \tilde{N} \\ 0, & \text{if } S = \emptyset \end{cases}$$

Then (\tilde{N}, w) is balanced, because $(-x_1, 0, -x_2) \in \mathcal{C}(\tilde{N}, w)$. Moreover, it is easy to check that this game is totally balanced. With $y = (0, 0, 0)$ the reduced two-person games are flat, thus AN, COV, CRGP*, and the equalities

$$s_{12}(y, w) = s_{13}(y, w) = s_{23}(y, w) = -x_1 = s_{12}(x, v)$$

and

$$s_{21}(y, w) = s_{31}(y, w) = s_{32}(y, w) = -x_2 = s_{21}(x, v)$$

show that $y \in \sigma(\tilde{N}, w)$. By WRGP $0 \in \sigma(\{3\}, w^{\{3\}, y})$. By COV $t \in \sigma(\{3\}, w^{\{3\}, y})$ is true for every $t < w^{\{3\}, y}(\{3\}) = -x_1 - x_2 > 0$, thus $y^t = (0, 0, t) \in \sigma(\tilde{N}, w)$ by RGPB. Choose $t < x_1 - x_2$ and observe that (N, w^{N, y^t}) is not balanced. q.e.d.

Proof of Theorem 4.10: The three solutions satisfy the axioms. Let σ be a solution with the desired properties and let us assume that σ is neither the empty solution nor the core.

Claim 1: σ is a subsolution of $\text{int } \mathcal{C}$.

By our assumption we obtain for every two-person game (N, v) :

- (1) If (N, v) is flat or if $\mathcal{C}(N, v) = \emptyset$, then $\sigma(N, v) = \emptyset$. (see the proof of Lemma 2.6)
- (2) If $\text{int } \mathcal{C}(N, v) \neq \emptyset$, then $\emptyset \neq \sigma(N, v) \subseteq \text{int } \mathcal{C}(N, v)$. (by our assumption, see the proof of Theorem 2.1)

Claim 1 follows from CRGP.

Let (N, v) ($N = \{1, 2\}$) be the unanimity game and choose $x \in \sigma(N, v)$ satisfy $x_1 \leq 1 - x_1 = x_2$ (this can be done by PO and AN).

Claim 2: $x^{(N, v)} \in \sigma(N, v)$.

Let $(\tilde{N}, w), y^0, a$ be defined as in the proof of Lemma 2.6. Then $y^1 = (a, 0, -a) \in \sigma(\tilde{N}, w)$ by AN, COV, and SSRGP. However, $y_{\{2,3\}}^1$ is the standard solution of the reduced game $(\{2, 3\}, w^{\{2,3\}, y^1})$ which is isomorphic to (N, v) .

Claim 3: $\text{int } \mathcal{C}(N, v) \subseteq \sigma(N, v)$.

Let $(\tilde{N}, w_{\tilde{\alpha}})$ and x, y be defined as in the proof of Lemma 2.4 and let $\bar{x} \in \text{int } \mathcal{C}(N, v)$. By PO $\bar{x}_2 = 1 - \bar{x}_1$ and by AN we can assume that $\bar{x}_1 \leq \bar{x}_2$. Moreover, $\bar{x}_1 \geq 0$ holds true by Claim 1. Put $y^\beta = \beta \cdot y$ for $\beta \in \mathbb{R}$ and let $\tilde{\alpha} = -\bar{x}_1$, $\tilde{\beta} = 1 - 2\bar{x}_1 = \bar{x}_2 - \bar{x}_1$. Then $(\tilde{N}, w_{\tilde{\alpha}})$ is totally balanced, because $\tilde{\alpha} \leq 0$. CRGP and Claim 2 imply $x \in \sigma(\tilde{N}, w_{\tilde{\alpha}})$. With $u_{\tilde{\alpha}} = w_{\tilde{\alpha}}^{\{1,2,3,4\}, x}$ we obtain that $(\{1, 2, 3, 4\}, u_{\tilde{\alpha}})$ is totally balanced, thus $y^{\tilde{\beta}} \in \sigma(\{1, 2, 3, 4\}, u_{\tilde{\alpha}})$ by the same reasons. Finally $z^{\tilde{\beta}} = (y^{\tilde{\beta}}, 0) \in \sigma(\tilde{N}, w_{\tilde{\alpha}})$ holds true by RGPB. However,

$$s_{51}(\tilde{N}, w_{\tilde{\alpha}}) = \tilde{\alpha}, \quad s_{15}(\tilde{N}, w_{\tilde{\alpha}}) = \tilde{\alpha} - \tilde{\beta} = -x_2,$$

thus COV and WRGP imply Claim 3.

The proof is finished by CRGP. q.e.d.

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