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Abstract

Recently Reny and Wooders ([17]) showed that the intersecting collection of sets in Shapley's ([18]) generalization of the Knaster-Kuratowski-Mazurkiwicz Theorem could be chosen to be partnered as well as balanced. In this paper we provide a further extension by showing that the collection of sets can be chosen to be strictly balanced, implying the Reny-Wooders result. Our proof is topological, based on the Eilenberg-Montgomery fixed point Theorem. Reny and Wooders ([17]) also show that if the collection of partnered points in the intersection is countable, then at least one of them is minimally partnered. Here we show that if this collection is only assumed to be zero dimensional (or if the set of partnered and strictly balanced points is of dimension zero), then there is at least one strictly balanced and minimally partnered point in the intersection. The approach presented in this paper sheds a new geometric-topological light on the Reny-Wooders results.

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1 Introduction

A solution concept for a game (or economy) is said to be *partnered* if it exhibits no asymmetric dependencies between players. That is, whenever a player i needs the cooperation of player j or is dependent upon the actions of player j then j similarly depends on i. Partnership is a natural property to require of a solution concept. If a solution concept is not partnered, there is an opportunity for one player to demand a larger share of the surplus from another player. Thus, a payoff that is not partnered exhibits a potential for instability. Consider, for example, the two-person divide the dollar game. If the two players can agree on the division of the dollar between them, the dollar is divided between them according to the agreement. Any division giving the entire dollar to one player displays an asymmetric dependency since the player receiving the dollar needs the cooperation of the player getting nothing.

The definition of partnership is based on the notion of partnered collections of subsets of a finite set. Let N be a finite set, whose members are called players. A collection of coalitions, consisting of subsets of N, is partnered if each player i in N is in some coalition in the collection and whenever i is in all the coalitions containing player j then j is in all the coalitions containing player i. If i is in all the coalitions containing j we think of this as a situation where j "needs" i. To illustrate a partnered outcome for a game, we return to the divide the dollar example. An outcome in which one player receives the entire dollar is not partnered since the only coalition that can afford to give him the dollar is the two-player coalition, while the player getting nothing has an alternative coalition, the coalition consisting of himself alone. Thus, the player receiving the dollar needs the player receiving nothing but the player receiving nothing needs only himself.

The partnership property was originally introduced to study solution concepts of games and economies and has now been applied in a number of papers; see, for example, ([11], [12], [9], [1], [2], [3], [14], [16], [15]). More recently, Reny and Wooders ([17]) have extended Shapley's ([18]) generalization of the Knaster-Kuratowski-Mazurkiewicz Theorem by showing that the collection of sets satisfying the conclusion of the Theorem can be chosen to be partnered as well as balanced. Reny and Wooders ([17]) also show that if the intersection of a balanced and partnered collection satisfying the conclusion of the KKM Theorem contains at most countably many points, then at least one of these balanced collections is "minimally" partnered. A collection of subsets of a set N is minimally partnered if it is partnered and if for each player i there does not exist another player j such that j is in all the subsets containing player i. In other words, no one needs anyone else in particular.

In this paper, we provide further results using a topological approach, shedding new geometric-topological light on the results of Reny and Wooders. We obtain a further extension of Shapley's generalization of the K-K-M Theorem, showing that the collection of sets satisfying the conclusion of the Theorem can be chosen to be strictly balanced – the weights on the sets in the collection may be chosen to be all positive. This implies that the collection is partnered. Our argument involves the Eilenberg-Montgomery fixed point Theorem for set-valued mappings. (This is deeper than the Kakutani fixed point Theorem 'customarily' used in game theory and economics - see, however, Keiding ([7]).) Assuming that the set of partnered and balanced points is zero dimensional (weaker than countable), we obtain a stronger result: There is at least one point in the intersection of a strictly balanced and partnered collection of sets that is minimally partnered. We use a version of degree theory valid for set valued maps (correspondences), where the image of a point is not necessarily convex. (We are unaware of any previous application of this theory to game theory and mathematical economics.) Using similar methods we obtain the same conclusion, assuming that the closure of the set of strictly balanced (and hence partnered) points is of zero dimension.

We are very much indebted to Philip Reny for pointing out an inaccuracy in an earlier version and for other suggestions which led to improvements in the present paper.

2 Definitions and the Main Results

Let $N = \{1, 2, ..., n\}$ and let $\mathcal{P}$ be a collection of subsets of N. For each i in N let

$$\mathcal{P}_i = \{S \in \mathcal{P} : i \in S\}.$$

We say that $\mathcal{P}$ is *partnered* if for each i in N the set $\mathcal{P}_i$ is nonempty and for every i and j in N the following requirement is satisfied[1]:

[1]The concept of a partnered collection of sets was introduced in Maschler and Peleg ([11], [12]). They used the term "separating collection." We follow the terminology of

$$\text{if } \mathcal{P}_i \subseteq \mathcal{P}_j \text{ then } \mathcal{P}_j \subseteq \mathcal{P}_i;$$

i.e. if all subsets in $\mathcal{P}$ that contain i also contain j then all subsets containing j also contain i. Let $\mathcal{P}[i]$ denote the set of those $j \in N$ such that $\mathcal{P}_i = \mathcal{P}_j$. We say that $\mathcal{P}$ is *minimally partnered* if it is partnered and for each $i \in N$, $\mathcal{P}[i] = \{i\}$.

Let $\mathcal{N}$ denote the set of nonempty subsets of N. For any $S \in \mathcal{N}$ let e^S denote the vector in $\Re^n$ whose i^{th} coordinate is 1 if $i \in S$ and 0 otherwise. For ease in notation we denote $e^{\{i\}}$ by e^i.

Let Δ denote the unit simplex in $\Re^n$. For every $S \in \mathcal{N}$ define

$$\begin{aligned} \Delta^S &= conv\{e^i : i \in S\}, \text{ and} \\ m^S &= \frac{e^S}{|S|}, \end{aligned}$$

where "*conv*" denotes the convex hull and $|S|$ denotes the number of elements in the set S.

Let $\mathcal{B}$ be a collection of subsets of N. The collection is *balanced* if there exist nonnegative weights $\{\lambda^S\}_{S \in \mathcal{B}}$ such that

$$\sum_{S \in \mathcal{B}} \lambda^S e^S = e^N$$

and the collection is *strictly balanced* if all weights λ^S can be chosen to be positive. Observe that the collection $\mathcal{B}$ is balanced if and only if

$$m^N \in conv\{m^S : S \in \mathcal{B}\}.$$

Reny and Wooders ([17]) obtain the following two results.

Theorem A. (Reny and Wooders ([17]) Let $\{C^S : S \in \mathcal{N}\}$ be a collection of closed subsets of Δ such that

$$\cup_{S \subseteq T} C^S \supseteq \Delta^T \text{ for all } T \in \mathcal{N}. \tag{1}$$

Then there exists $x^* \in \Delta$ such that $\mathcal{S}(x^*) \equiv \{S \in \mathcal{N} : x^* \in C^S\}$ is balanced and partnered.

Bennett ([2]).

Theorem B. Reny and Wooders ([17]). Let $\{C^S : S \in \mathcal{N}\}$ be a collection of closed subsets of Δ satisfying (1). If the set $\{x^* \in \Delta : \mathcal{S}(x^*)$ is balanced and partnered$\}$ is at most countable, then at least one $x^* \in \Delta$ renders $\mathcal{S}(x^*)$ balanced and *minimally* partnered.

The next two Theorems will be used in our extension of Reny and Wooders' results.

Theorem 1. Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ such that:

$$F \text{ is upper} - \text{hemicontinuous}; \tag{2}$$

$$\begin{array}{c}\text{For all } \ x \in \mathcal{B} \ (:= \partial\Delta), \ \ F(x) \subseteq \mathcal{B} \text{ and } \ g(x) \notin F(x), \ \ \text{where } g \text{ is the} \\ \text{antipodal map with } m_N \text{ as the origin}, \ \ g : \Delta\backslash\{m_N\} \rightarrow \Delta\backslash\{m_N\};\end{array} \tag{3}$$

and

$$F \text{ assumes finitely many values.} \tag{4}$$

Then there exists $x \in \Delta$ such that $m_N \in rel\ int(F(x))$.

Theorem 2. Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ satisfying (2), (4) and:

$$\text{For all } \ x \in \mathcal{B}, \ x \in \Delta^S \Rightarrow F(x) \subseteq \Delta^S \ (S \subseteq N). \tag{5}$$

Assume also that:

$$\text{The closure of the set } \{x : m_N \in rel\ int(F(x)\} \ \text{ is zero} - \text{dimensional.} \tag{6}$$

Then there exists $x \in \Delta$ such that $m_N \in int(F(x))$.
(Note that "*int*" means "interior in the topology on the hyperplane" $\sum_{i=1}^{n} x_i = 1$.)

Theorem 1 implies a strengthening of Theorem A of Reny and Wooders ([17]). Under somewhat different assumptions, Theorem 2 yields a stronger conclusion than those of Theorem B of Reny and Wooders ([17]).

Note that in the course of their proof of the KKMS Theorem, Shapley and Vohra ([19]) establish the following.

Proposition 1. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ satisfying (1). Then there is a homeomorphism φ of Δ into the interior of Δ and a correspondence F from Δ into the closed convex subsets of Δ satisfying (2), (4), (5) and such that

$$F(\varphi(x)) = conv\{m_S : x \in C^S\} \quad \text{for all} \quad x \in \Delta, \tag{7}$$

and

$$\text{if} \quad m_N \in F(x) \quad \text{then} \quad x \in \varphi(\Delta). \tag{8}$$

The following strengthening of Theorem A of Reny and Wooders ([17]) follows from Theorem 1 and Proposition 1:

Theorem 3. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ such that (1) is satisfied. Then there exists $x \in \Delta$ such that the collection $\{S : x \in C^S\}$ is partnered and strictly balanced.

Proof: Let F be a map satisfying the properties required in Proposition 1; from that Proposition there is such a map. Note that F satisfies condition (3). By Theorem 1 there exists $y \in \Delta$ such that $m_N \in rel\ int(F(y))$, and by (7) and (8) there exists $x \in \Delta$ $(x = \varphi^{-1}(y))$ such that

$$m_N \in rel\ int[conv\{m_S : x \in C^S\}]. \tag{9}$$

Clearly, $\Sigma := \{S : x \in C^S\}$ is balanced. Moreover, it is *strictly* balanced. In fact, let $S \in \Sigma$, $S \neq N$ (without loss of generality, $\Sigma \neq \{\mathrm{N}\}$) and let ℓ_S denote the line joining m_N and m_S. Then m_N is contained in the interior of the interval $\ell_S \cap conv\{m_S\}_{S\in\Sigma}$. Hence there exists an $a_S \in conv\{m_S\}_{S\in\Sigma}$ and positive numbers α_S, β_S such that $\alpha_S + \beta_S = 1$ and $m_N = \alpha_S m_S + \beta_S a_S$. We may average these equations with positive weights over $S \in \Sigma$, $S \neq N$ and obtain m_N as a convex combination of the points m_S, $S \in \Sigma$, with positive weights for each $S \neq N$.

If Σ is not partnered, then there exists $i, j \in N$ such that whenever $i \in S \in \Sigma$ also $j \in S$, but there exists $T \in \Sigma$ with $j \in T$, $i \notin T$. Then for all $y \in D(x) := conv\{m_S : x \in C^S\}$ we have $y_j \geq y_i$, but there exists $\widehat{y} \in D(x)$ such that $\widehat{y}_j > 0$ and $\widehat{y}_i = 0$ or $\widehat{y}_i > \widehat{y}_j$. Hence $m_N = (\frac{1}{n}, ..., \frac{1}{n})$ cannot satisfy (9)[2]. ∎

[2] The last part is not really needed, as strict balancedness implies partnership.

It is well known (Maschler, Peleg, and Shapley ([9])) that there exist partnered collections which are not balanced. Let Σ be such a collection for an n-person game. Then $\{N\} \cup \Sigma$ is balanced and partnered, but $m_N \notin rel\ int[conv\{m_S\}_{S\in\Sigma\cup\{N\}}]$. One may choose Σ to be minimally partnered. Then $\Sigma \cup \{N\}$ is balanced and minimally partnered, but $m_N \notin rel\ int[conv\{m_S\}_{S\in\Sigma\cup\{N\}}]$. Thus, our Theorem 1 is a strengthening of Theorem A in ([17]).

The following consequence of Theorem 2 is related to Theorem B of Reny and Wooders ([17]).

Theorem 4. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ such that (1) is satisfied. Assume that the closure of the set $\{x : \{S : x \in C^S\}$ is strictly balanced$\}$ is zero-dimensional. Then there exists $x \in \Delta$ such that the collection $\{S : x \in C^S\}$ can be chosen to be minimally partnered and strictly balanced.

Proof of Theorem 4. Assume that the closure of the set $\{x : \{S : x \in C^S\}$ is strictly balanced and partnered$\}$ is zero-dimensional. Let F be the map whose existence is stated in Proposition 1. Nore in particular that F satisfies (5). Hence there exists (by Theorem 2) $x \in \Delta$ such that $m_N \in int(D(x))$ [where $D(x) = conv[m_S : x \in C^S)]$. If $\Sigma = \{S : x \in C^S\}$ is not minimally partnered, then there exists a pair i, j such that for every $S \in \Sigma$ either i and j both belong to S, or neither belongs. Hence for all $y \in D(x)$, $y_i = y_j$. Thus $int(D(x))$ is empty, a contradiction. ∎

Comparing our Theorem 4 with Reny and Wooders ([17]) Theorem B it appears that neither is stronger than the other. While a countable set may be dense (and hence have closure of positive dimension) a set of dimension zero may be uncountable (for example, a Cantor set on a line). Note also that the statement "$m_N \in int(D(x))$" is stronger than the conclusion of Theorem 4. The statement means that every hyperplane through m_N (except for $\sum_{i=1}^n x_i = 1$) has vectors e^S with $x \in C^S$ on both sides.

Our final result is a proper strengthening of Theorem B of Reny and Wooders ([17]). As a formulation for correspondences (similar to Theorems 1 and 2) is cumbersome, we state here the result only for closed coverings.

Theorem 5. Let $\{C^S\}_{S\subseteq N}$ be a closed covering of Δ such that (1) is satisfied. If the set $\{x^* \in \Delta : \mathcal{S}(x^*)$ is balanced and partnered$\}$ is zero dimensional, then at least one $x^* \in \Delta$ renders $\mathcal{S}(x^*)$ strictly balanced and minimally partnered. In fact, $m_N \in int\ [conv\{m_S\}_{S \in \mathcal{S}(x^*)}]$.

3 Proofs of Theorems 1, 2 and 5

To prove Theorem 1, note that if y is not in the relative interior of a convex set K, then removing an open ball $B(y, \delta)$ of radius δ centered at y from K results in a nonempty closed contractible set (i.e., a set homeomorphic to a simplex of a certain dimension) for $\delta > 0$ sufficiently small. It follows from (2) and (4) that if $m_N \in rel\ int(F(x))$ for *no* $x \in \Delta$, then there exists a $\delta > 0$ such that $F(x) \backslash B(m_N, \delta)$ is nonempty and contractible for all $x \in \Delta$. Moreover the openness of $B(m_N, \delta)$ implies that the correspondence $x \to F(x) \backslash B(m_N, \delta)$ is upper-hemicontinuous. Let h denote the usual radial retraction of the punctured simplex $\Delta \backslash \{m_N\}$ onto $\mathcal{B}$. Then $h(F(x) \backslash B(m_N, \delta))$ is contractible for all x, and the same is true of $g(h(F(x) \backslash B(m_N, \delta)))$, where g is the antipodal map as given in (3). Clearly the correspondence $x \to g(h(F(x) \backslash B(m_N, \delta)))$ is upper hemicontinuous. By the Eilenberg-Montgomery fixed point theorem ([5]) every upper-hemicontinuous correspondence mapping the simplex into the collection of its non-empty, closed, and contractible subsets has a fixed point. Hence there exists a point $x^* \in \Delta$ such that $x^* \in g(h(F(x^*) \backslash B(m_N, \delta)))$. In particular, $x^* \in \mathcal{B}$. By assumption (3) $F(x^*) \subseteq \mathcal{B}$. But on $\mathcal{B}$, h is the identity. Hence, $x^* \in g(F(x^*) \backslash B(m_N, \delta)) \subseteq g(F(x^*))$, contradicting (3). ∎

For the proof of Theorem 2 we need degree theory as extended for correspondences (see, for example, Lloyd ([10]), 115–120). Actually, a stronger version is needed, where the values are not necessarily convex (see, for example, Borisovich ([4])). In our case the values assumed by the correspondence are contractible and compact, so that the Begle-Vietoris mapping theorem ([5], [6], [20]) is applicable and may serve as a basis for degree theory.

It follows from (5) and a simple homotopy argument that

$$d(F, int(\Delta), m_N) = 1. \tag{10}$$

Denote by $\overline{X}$ the closure of the set $\{x : m_N \in rel\ int(F(x))\}$. By assumption, $\overline{X}$ is zero-dimensional. This means that for every $\epsilon > 0$, the set $\overline{X}$ may be covered by a finite number of disjoint open sets whose diameter is less than ϵ. Let $\{D_{i,m}\}_{i=1}^{p_m}$ denote such a collection of sets with *diam*

$(D_{i,m}) < \frac{1}{m}$, $\overline{X} \subseteq \cup_{i=1}^{P_m} D_{i,m}$ and $D_{i,m} \cap D_{j,m} = \emptyset$ for $i \neq j$. Then $D_{i,m} \cap \overline{X}$ is both open and closed in $\overline{X}$, so that $\partial D_{i,m} \cap \overline{X} = \emptyset$.

With δ as in the proof of Theorem 1, set $\delta(x) = \min[dist(x, \overline{X}), \delta]$, and define an upper-hemicontinuous correspondence G by

$$G(x) = F(x) \backslash B(m_N, \delta(x)). \tag{11}$$

Then $m_N \notin G(x)$ if $x \notin \overline{X}$. It follows from (5) (compare (11)) that

$$d(G, int(\Delta), m_N) = 1. \tag{12}$$

By construction, $m_N \notin G(y)$ for all $y \in \partial D_{i,m}$, $1 \leq i \leq P_m$. Hence $d(G, D_{i,m}, m_N)$ is well defined and

$$\sum_{i=1}^{P_m} d(G, D_{i,m}, m_N) = d(G, int(\Delta), m_N). \tag{13}$$

It follows from (13) and (12) that there exists $i_0 = i_0(m)$ such that $d(G, D_{i_0(m),m}, m_N) \neq 0$. By compactness there exists $\overline{x} \in \overline{X}$ and a sequence $D_{i_0(m),m}$ of neighborhoods (with $D_{i_0(m),m} \cap \overline{X}$ compact) such that $\overline{x} = \cap_{m=1}^{\infty} D_{i_0(m),m}$. For each m, $d(G, D_{i_0(m),m}, m_N) \neq 0$ implies the existence of an $(n-1)$-dimensional ball B_m centered on m_N such that

$$B_m \subseteq \cup_{x \in D_{i_0(m),m}} G(x) \subseteq \cup_{x \in D_{i_0(m),m}} F(x). \tag{14}$$

Set $\underline{a}_i = e^i - m_N$, $1 \leq i \leq n$. Fix for a moment $\underline{a}_j$ for an index $1 \leq j \leq n$. By (4) there exists a positive number δ_j such that if $m_N + \epsilon \underline{a}_j \in F(y)$ for a certain $y \in \Delta$ and a positive ϵ (no matter how small), then $m_N + \delta_j \underline{a}_j \in F(y)$. By (14) there exists a sequence x^m converging to $\overline{x}$ and a sequence of positive real numbers ϵ_m such that $m_N + \epsilon_m \underline{a}_j \in F(x^m)$. Hence $m_N + \delta_j \underline{a}_j \in F(x^m)$. By the upper-hemicontinuity $m_N + \delta_j \underline{a}_j \in F(\overline{x})$. The convexity of $F(\overline{x})$ and the spanning property of $\underline{a}_1, ..., \underline{a}_n$ imply that m_N is an interior point of $F(\overline{x})$. ∎

Remark: Theorems 1 and 2 may be generalized to contractible non-convex sets. Inspection of the proof of Theorem 1 shows that the condition (4) may be replaced by the assumption that there existes a positive number δ such that the sets $F(x) \backslash B(m_N, \delta)$ are nonempty and contractible for all $x \in \Delta$.

Similarily, Theorem 2 is true if (4) is replaced by the property that for every $\underline{a} \neq 0$ there exists a $\delta > 0$ such that if both m_N and $m_N + \epsilon \underline{a} \in F(x)$ for any $x \in \Delta$ and $\epsilon > 0$ then $m_N + \delta \underline{a} \in F(x)$.

To prove Theorem 5, we follow Reny-Wooders ([17]) (inspired by [3]) and set

$$c_{ij}(x) = \min_{\{S: i \notin S, j \in S\}} dist(x, C^S) \tag{15}$$

for $x \in \Delta$, $1 \leq i \leq j \leq n$,

$$c_{ii}(x) = 0 \text{ for } x \in \Delta,\ 1 \leq i \leq n, \tag{16}$$

$$\eta_i(x) = \sum_{j=1}^{n} [c_{ij}(x) - c_{ji}(x)] \text{ for } x \in \Delta,\ 1 \leq i \leq n, \tag{17}$$

$$\delta(x) = min[\sum_{i=1}^{n} |\eta_i(x)|, \delta] \text{ for } x \in \Delta, \tag{18}$$

where the positive number δ is chosen so that the set $F(x) \backslash B(m_N, \delta)$ (for F as in Proposition 1) is nonempty and contractible for all $x \in \Delta$, (compare the proof of Theorem 1), and if $m_N \notin F(x)$, then $B(m_N, \delta) \cap F(x) = \emptyset$. Then $\delta(x)$ is a non-negative continuous function on Δ. Define the correspondence $H(x)$ by

$$H(x) = F(x) \backslash B(m_N, \delta(\varphi^{-1}(x))) \text{ for } x \in \varphi(\Delta), \tag{19}$$

$$H(x) = F(x) \text{ for } x \notin \varphi(\Delta). \tag{20}$$

(Contrast with the definition of G(x) in (11).) The choice of δ and (8) imply that $H(x)$ is upper-hemicontinuous. Let X denote the set $\{x^* \in \Delta : \mathcal{S}(\varphi^{-1}(x^*))$ is balanced and partnered.

We claim that if $x \notin X$ then $m_N \notin H(x)$. In fact, if $m_N \in H(x)$ then $m_N \in F(x)$. Thus (8) implies that $x \in \varphi(\Delta)$. Hence $H(x)$ is given by (19). It follows that $\mathcal{S}(\varphi^{-1}(x))$ is balanced and $\delta(\varphi^{-1}(x)) = 0$. But according to a Lemma of Bennett-Zame ([3]), as adapted by Reny-Wooders ([17]), if $\eta_i(y) = 0$ for all $i = 1, ..., n$ (as implied by $\delta(y) = 0$), then $\mathcal{S}(y)$ is partnered. Hence $x = \varphi(y) \in X$.

Set now $Y = \{x \in \Delta : m_N \in H(x)\}$. Then Y is a closed subset of X, hence a closed zero-dimensional set. Note that the correspondence $H(x)$ satisfies the conditions of Theorem 1, except for (4). But by the remark made after the proof of Theorem 2, the conclusion of Theorem 1 holds for the correspondence H (see also (18)). Hence there exists a point $x \in \Delta$ such that $m_N \in$ $rel\ int(H(x))$, and in particular Y is not empty.

We can now continue the proof as in the proof of Theorem 2, with Y replacing $\overline{X}$ and H replacing G. We conclude that there exists $\overline{x} \in Y$ such that m_N is an interior point of $F(\overline{x})$. As in the proof of Theorem 4, this implies the existence of $x \in \Delta$ ($x = \varphi^{-1}(\overline{x})$) such that $m_N \in int(D(x))$, from which all the assertions of Theorem 5 follow. ∎

Remark: Note that $\overline{x} \in Y$ implies $\delta(\varphi^{-1}(\overline{x})) = 0$ or $\eta_i(\varphi^{-1}\overline{x})) = 0$ for all $1 \leq i \leq n$. Thus the "net credits" ([3], [16]) of each player at x are zero. However, one does not need all the assumptions of Theorem 5 for the non-emptiness of Y. For this the assumptions of Theorem 3 suffice. (This observation was made in response to a suggestion by Philip Reny.)

References

[1] Albers, W. (1979) "Core and kernel variants based on imputations and demand profiles" in *Game Theory and Related Topics*, O. Moeschlin and D. Pallaschke, eds. North Holland, Amsterdam.

[2] Bennett, E. (1983) "The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games," *International Journal of Game Theory* 12, 1–28.

[3] Bennett, E. and W.R. Zame (1988) "Bargaining in cooperative games," *International Journal of Game Theory* 17, 279–300.

[4] Borosovitch, Y.G., B.D. Gelman, A.D. Myshkis, and V.V. Obukhovski (1980) "Topological methods in the fixed-point theory of multi-valued maps," *Russian Math. Surveys* 35 (1), 65–143.

[5] Eilenberg, S. and D. Montgomery (1946) "Fixed point theorems for multi-valued transformations," *American Journal of Mathematics* 68, 214–22.

[6] Hocking, J.G. and G.S. Young (1961) "*Topology*," Addison-Wesley, Reading MA.

[7] Keiding, H. (1985) "On the existence of equilibrium in social systems with coordination," *Journal of Mathematical Economics* 14, 105–111.

[8] Lloyd, N.G. (1978) "*Degree Theory*," Cambridge University Press, Cambridge.

[9] Maschler, M., B. Peleg and L.S. Shapley (1972) "The kernal and bargaining set for convex games," *International Journal of Game Theory* 1, 73–93.

[10] Lloyd, N.G. "*Degree Theory*," Cambridge University Press, Cambridge.

[11] Maschler, M. and B. Peleg (1966) "A characterization, existence proof and dimension bounds for the kernel of a game," *Pacific Journal of Mathematics* 18, 289–328.

[12] Maschler, M. and B. Peleg (1967) "The structure of the kernel of a cooperative game," *SIAM Journal of Applied Mathematics* 15, 569–604.

[13] Page, F.H., Jr. and M.H. Wooders (1994b) "The partnered core of an economy and the partnered competitive equilibrium," University of Alabama Working Paper.

[14] Reny, P.J., E. Winter and M.H. Wooders (1993) "The partnered core of a game with side payments," University of Western Ontario Discussion Paper.

[15] Reny, P.J. and M.H. Wooders (1995) "Credible threats of secession, partnerships, and commonwealths," to appear in *Understanding Strategic Interaction; Essays in Honor of Reinhard Selten*.

[16] Reny, P.J. and M.H. Wooders (1993a) "The partnered core of a game without side payments," *Journal of Economic Theory* (to appear).

[17] Reny, P.J. and M.H. Wooders (1993b) "An extension of the KKMS Theorem," Revised as University of Alabama Department of Finance, Economics, and Legal Studies Working Paper No. 248 (1995).

[18] Shapley, L.S. (1973) "On balanced games without side payments," in *Mathematical Programming*, Hu, T.C. and S.M. Robinson (eds), Academic Press, New York.

[19] Shapley, L.S. and R. Vohra (1991) "On Kakutani's fixed point Theorem, the KKMS Theorem and the core of a balanced game," *Economic Theory* 1, 108–116.

[20] Spanier, E.H. (1966) "*Algebraic Topology*," McGraw Hill, New York.