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Preservation of Differences, Potential, Conservity

by

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Abstract

The potential approach for cooperative games was invented by Hart and Mas-Colell. In this paper now there is an extension with respect to a new characterizing property called conservity which gives a clear interpretation of the potential. The many analoga between game theory and physics are shown. The well known Shapley–Formula can be derived by the potential approach. Moreover the Banzhaf–Index can be uniquely characterized in this context. Eventually there are new proofs for Hart's and Mas-Colell's basic theorems using not the potential but the preservation of differences which is a little bit more elegant.

1 Preliminaries

Definition 1.1 *A cooperative game (with sidepayments) is a tuple* $\Gamma = (\Omega, v)$*, where*

$$\Omega \subset \mathbb{N}\backslash\{0\},\ |\Omega| < \infty \quad \text{set of players}$$
$$v:\ 2^{\Omega} \to \mathbf{R},\ v(\emptyset) = 0 \quad \text{characteristic function}$$

Remark. Elements of Ω are called players. Sets of players are coalitions. $v(S)$ is the worth which the coalition S can obtain by cooperation. Sometimes one may designates a characteristic function already as a game, if per definitionem the set of players can be recognized.

Now there is the question, how much utility shall be transfered to every single player.

Definition 1.2 *A solution concept is an operator, which assigns to every co-operative game* $\Gamma = (\Omega, K, f)$ *exactly one element of* $\mathbf{R}^{|\Omega|}$.

The main principle to solve a given game is not to consider just the fixed game but a whole family of games which are similar to the given one. One may have difficulties to deal with single elements of this family, but there is a simple connection between all these elements. And in some cases a solution is quite obvious.

One possible way to get a nice family of games is due to Lloyd S. Shapley[1]. The set of players is fixed, the characteristic function varies. The values of different characteristic functions f and g are been put together by the additivity axiom. Therefore the Shapley–Value may be viewed as a linear mapping on the set of all characteristic functions $F(\Omega_0)$ with fixed Ω_0. This set is together with point by point addition and multiplication of functions a real vector space. The normalized unanimity games form a basis. Shapley characterized his value as the linear continuation of the uniform distribution.

[1] cf. [8]

Corollary 1.3 [2] *Let* $\Gamma = (\Omega_0, v)$ *be a cooperative game and let* $\{\lambda_T\}_{T\in 2^\Omega\setminus\{\emptyset\}}$ *be the coefficients of expansion in the basis of the normalized unanimity games. Then the Shapley–Value* Ψ *for player* $i\in\Omega_0$ *is given by*

$$\Psi_i(\Omega_0, K_0, f) = \sum_{\substack{T\in 2_0^\Omega \\ i\in T}} \frac{1}{|T|}\lambda_T \tag{1}$$

2 Alternativa

In this section another family of games will be considered. The set of players varies, the characteristic function on the other side is "fixed". I.e., for a fixed cooperative game $\Gamma = (\Omega, v_0)$ one considers the set of subgames $G(\Gamma) := \{\Gamma^S := (S, v_{|2^S}) : S \subseteq \Omega\}$. It is possible to characterize the Shapley–Value on this family due to Hart and Mas-Colell[3].

2.1 Preservation of Differences

Definition 2.1 *Let* $\Gamma = (\Omega, v_0)$ *be arbitrary, but fixed and let* P *be a real valued operator on* $G(\Gamma)$. *For* $S \subseteq \Omega$ *and* $i\in S$ *the operator* ∇_i, *defined by*

$$\nabla_i P(\Gamma^S) := P(\Gamma^S) - P(\Gamma^{S-i}) \text{ [4]} \tag{2}$$

is called difference operator. The vector $\nabla_S := (\nabla_i)_{i\in S}$ *is called discrete gradient.*

Discrete differences are the analogon to differentiation in analysis.

Definition 2.2 *Let* $\Gamma = (\Omega, v_0)$ *be arbitrary, but fixed. A solution concept* Φ *preserves differences, if*

$$\nabla_i \Phi_j(\Gamma^S) = \nabla_j \Phi_i(\Gamma^S) \tag{3}$$

for all $S \subseteq \Omega$ *and all* $i, j\in S$, $i \neq j$.

[2] cf. [8]

[3] cf. [1], [2]

[4] $S - i$ is a short form for $S\setminus\{i\}$

Preservation of differences corresponds to the integrability condition in analysis. Myerson[5] has introduced this property as balanced contributions. Hart und Mas-Colell[6] point out, that one wants to preserve differences rather than ratios, since the resulting outcome should not depend on the choice of the origion of a player's utility scale. It is required, that the utility which player i can gain, if player j takes part, is equal to the profit which player j gets if player i joins. It will be shown in the following that the property of preservation of differences is a necessary property of the Shapley-Value.

2.2 Potential

Definition 2.3 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed. Then a solution concept Φ is called a discrete gradient field, if there exists a function $P : G(\Gamma) \rightarrow \mathbb{R}$ such that*

$$\Phi(\Gamma^S) = \nabla_S P(\Gamma^S) \tag{4}$$

for all $S \subseteq \Omega$. P is then called the potential operator, or shortly the potential, of the solution concept Φ.

For the present the potential is just a technical tool, but later on it gets a clear meaning.

Proposition 2.4 *A solution concept Φ is a discrete gradient field, if and only if it preserves differences.*

Proof. "$\Rightarrow$" Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed and let Φ be a discrete gradient field, i.e., for a certain potential $P : G(\Gamma) \rightarrow \mathbb{R}$ and for an arbitrary subset $S \subseteq \Omega$ and arbitrary $i, j \in S$, $i \neq j$ it is true that

$$\Phi_i(\Gamma^S) = \nabla_i P(\Gamma^S) \; ; \qquad \Phi_j(\Gamma^S) = \nabla_j P(\Gamma^S)$$

[5] cf. [3]

[6] cf. [2]

Therefrom

$$\begin{aligned}
\nabla_i \Phi_j(\Gamma^S) &= \nabla_i(\nabla_j P(\Gamma^S)) \\
&= \nabla_i(P(\Gamma^S) - P(\Gamma^{S-j})) \\
&= P(\Gamma^S) - P(\Gamma^{S-i}) - P(\Gamma^{S-j}) + P(\Gamma^{(S-j)-i})
\end{aligned}$$

and dito

$$\begin{aligned}
\nabla_j \Phi_i(\Gamma^S) &= \nabla_j(\nabla_i P(\Gamma^S)) \\
&= \nabla_j(P(\Gamma^S) - P(\Gamma^{S-i})) \\
&= P(\Gamma^S) - P(\Gamma^{S-j}) - P(\Gamma^{S-i}) + P(\Gamma^{(S-i)-j})
\end{aligned}$$

Therefore $\nabla_i \Phi_j(\Gamma^S) = \nabla_j \Phi_i(\Gamma^S)$ for all $S \subseteq \Omega$ and all $i, j \in S$, $i \neq j$.

"$\Leftarrow$" Let $\nabla_i \Phi_j(\Gamma^S) = \nabla_j \Phi_i(\Gamma^S)$ for all $S \in \Omega$ and all $i, j \in S$, $i \neq j$. Then one defines recursively

$$P(\Gamma^S) = P(\Gamma^{S-i}) + \Phi_i(\Gamma^S) \tag{5}$$

for all $S \subseteq \Omega$ and all $i \in S$. By induction one can see that this expression is well defined.

Basis of induction. Let $|S| = 1$ and set $P(\Gamma^{\emptyset}) = 0$ then $P(\Gamma^{\{i\}}) = \Phi_i(\Gamma^{\{i\}})$ is well defined for arbitrary $i \in \Omega$.

Induction hypothesis. Let equation (5) be well defined for all $S \subset \Omega$ with $|S| \leq n$ for arbitrary, but fixed $n \in \mathbb{N}$, $n < |\Omega|$.

Induction step. $n \to n+1$. Let $S \subseteq \Omega$ with $|S| = n+1$ be given and let $i, j \in S$ be arbitrary. Then

$$\begin{aligned}
P(\Gamma^{S-i}) + \Phi_i(\Gamma^S) &\overset{n.V.}{=} P(\Gamma^{S-i}) + \Phi_j(\Gamma^S) - \Phi_j(\Gamma^{S-i}) + \Phi_i(\Gamma^{S-j}) \\
&\overset{IV}{=} P(\Gamma^{S-i}) + \Phi_j(\Gamma^S) - (P(\Gamma^{S-i}) - P(\Gamma^{(S-i)-j})) \\
&\quad + P(\Gamma^{S-j}) - P(\Gamma^{(S-j)-i}) \\
&= P(\Gamma^{S-j}) + \Phi_j(\Gamma^S)
\end{aligned}$$

Per definitionem therefore $\nabla_S P(\Gamma^S) = \Phi(\Gamma^S)$ for all $S \subseteq \Omega$. □

The find out of a potential is the turn back of the preservation of differences and corresponds to integration in analysis. Uniquenes can be obtained up to an additive constant. In fact only differences of a potential are important.

2.3 Conservity

Definition 2.5 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed and let S_a, S_e be two arbitrary coalitions in Ω. A finite sequence $(S_1, \cdots, S_n)$ of coalitions in Ω is called a way from S_a to S_e, if the following holds*

- $S_1 = S_a,\ S_n = S_e$
- $\forall i = 1, \cdots, n\ \exists j = j^{S_i}_{S_{i+1}} \in \Omega : \qquad j^{S_i}_{S_{i+1}} = (S_i \cup S_{i+1}) \backslash (S_i \cap S_{i+1})$

Definition 2.6 *Let $\Gamma = (\Omega, v_0)$ be a cooperative game, Φ a solution concept, S_a, S_e two coalitions in Ω and $(S_1, \cdots, S_n)$ a way from S_a to S_e. Then*

$$W(\Phi, S_a, S_e, (S_1, \cdots, S_n)) := \sum_{i=1}^{n-1} \delta^{S_i}_{S_{i+1}}\ \Phi_{j^{S_i}_{S_{i+1}}}(\Gamma^{S_i \cup S_{i+1}}) \tag{6}$$

with

$$\delta^{S_i}_{S_{i+1}} := \begin{cases} +1 & ;\ S_i \subset S_{i+1} \\ -1 & ;\ S_i \supset S_{i+1} \end{cases}$$

is called the expenditure of Φ for the pair (S_a, S_e) with respect to the way $(S_1, \cdots, S_n)$.

The expenditure has a clear meaning. Imagine the players of a given game $\Gamma = (\Omega, v_0)$ meet together at a certain place. The worthes of all coalitions have been settled. Moreover one has come in terms with a solution concept Φ.

A certain master of the game now has the possibility to pay off an arbitrary player $i \in \Omega$ according to the solution concept and send him away afterwards. Thereby the situation has changed: the new set of players is $\Omega - i$. Cooperation with player i is not possible anymore, because he is not present. All the other coalitions can obtain by cooperation the same worthes as before. But the payoffs to the remaining players according to the agreed solution concept have been changed. The master can now pay off another player $j \in (\Omega - i)$ according to the new calculation and send him away.

This procedure might go on. Of course the master can fetch a player from outside and bring him into the game while the master demands exactly the amount which this player will get by the solution concept Φ afterwards according to the new situation.

In this sense $W(\Phi, S_a, S_e, (S_1, \cdots, S_n))$ is exactly the amount of utility which has to be transfered to the master, just to bring the coalition S_e into the game according to the way $(S_1, \cdots, S_n)$ while starting with the coalition S_a. The choice of the sign is just an agreement with the author.

The master himself may be viewed as a "*deus ex machina*". Every arbitrary coalition $S \subseteq \Omega$ can be the master if their players for example want to play the game on their own or want to gain utility on closed ways.

Definition 2.7 *Under the assumption of the preceeding definition a solution concept Φ is called conservative, if the expenditure is equal to zero for every closed way, i.e., for every pair (S, S), $S \subseteq \Omega$ and every way from S to S.*

Definition 2.8 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed and let S_a, S_e be two coalitions in Ω. A way $(S_1, \cdots, S_n)$ from S_a to S_e is called rigorously climbing, if $S_i \subset S_{i+1}$ for all $i = 1, \cdots, n-1$.*

Equivalence Proposition 2.9 *For a solution concept Φ it is equivalent:*

(*i*) *Φ is conservative.*

(*ii*) *The expenditure W is independent of the way.*

(*iii*) *The expenditure W is independent of the way for every way which is starting in $\emptyset$ and rigorously climbing.*

(*iv*) *Φ preserves differences.*

Proof. "$(i) \Rightarrow (ii)$" Let S_a, S_e be two arbitrary coalitions in Ω and let $(S_1, \cdots, S_n)$, $(\tilde{S}_1, \cdots, \tilde{S}_m)$ be two arbitrary ways from S_a to S_e. Then $(T_1, \cdots, T_{n+m-1}) := (S_1, \cdots, S_n, \tilde{S}_{m-1}, \cdots, \tilde{S}_1)$ is by assumption a closed way with $\sum_{i=1}^{n+m-1} \delta_{T_{i+1}}^{T_i} \; \Phi_{j_{T_{i+1}}^{T_i}} (\Gamma^{T_i \cup T_{i+1}}) = 0$. Therefrom holds

$$\begin{aligned}
\sum_{i=1}^{n-1} \delta_{S_{i+1}}^{S_i} \; \Phi_{j_{S_{i+1}}^{S_i}} (\Gamma^{S_i \cup S_{i+1}}) &= \sum_{i=1}^{n-1} \delta_{T_{i+1}}^{T_i} \; \Phi_{j_{T_{i+1}}^{T_i}} (\Gamma^{T_i \cup T_{i+1}}) \\
&= -\sum_{i=n}^{n+m-2} \delta_{T_{i+1}}^{T_i} \; \Phi_{j_{T_{i+1}}^{T_i}} (\Gamma^{T_i \cup T_{i+1}}) \\
&= -\sum_{i=m}^{2} \delta_{\tilde{S}_{i-1}}^{\tilde{S}_i} \; \Phi_{j_{\tilde{S}_{i-1}}^{\tilde{S}_i}} (\Gamma^{\tilde{S}_i \cup \tilde{S}_{i-1}}) \\
&= \sum_{i=1}^{m-1} (-\delta_{\tilde{S}^i}^{\tilde{S}_{i+1}}) \; \Phi_{j_{\tilde{S}_i}^{\tilde{S}_{i+1}}} (\Gamma^{\tilde{S}_{i+1} \cup \tilde{S}_i}) \\
&= \sum_{i=1}^{m-1} \delta_{\tilde{S}_{i+1}}^{\tilde{S}_i} \; \Phi_{j_{\tilde{S}_{i+1}}^{\tilde{S}_i}} (\Gamma^{\tilde{S}_i \cup \tilde{S}_{i+1}})
\end{aligned}$$

"$(ii) \Rightarrow (iii)$" clear

"$(iii) \Rightarrow (iv)$" Let $S \subseteq \Omega$ be arbitrary and let the expenditure of Φ be independent of the way for every way which is starting in $\emptyset$ and rigorously climbing. Then for arbitrary $i, j \in S$, $i \neq j$ it is true that

$$\begin{aligned}
&\Phi_i(\Gamma^S) + \Phi_j(\Gamma^{S-i}) + \sum_{l \in (S-i-j)} \Phi_l(\Gamma^{\{k \in (S-i-j):\ k \leq l\}}) \\
= \; &\Phi_j(\Gamma^S) + \Phi_i(\Gamma^{S-j}) + \sum_{l \in (S-i-j)} \Phi_l(\Gamma^{\{k \in (S-i-j):\ k \leq l\}})
\end{aligned}$$

Thus $\Phi_i(\Gamma^S) + \Phi_j(\Gamma^{S-i}) = \Phi_j(\Gamma^S) + \Phi_i(\Gamma^{S-j})$. And therefrom $\nabla_j \Phi_i(\Gamma^S) = \nabla_i \Phi_j(\Gamma^S)$ for all $S \subseteq \Omega$ and all $i, j \in S$, $i \neq j$.

"$(iv) \Rightarrow (i)$" Let Φ preserve differences. Then by proposition 2.4 there exists a potential $P : G(\Gamma) \to \mathbb{R}$, such that for every $S \subseteq \Omega$ always $\nabla_S P(\Gamma^S) = \Phi(\Gamma^S)$. Now let $(S_1, \cdots, S_n)$ be an arbitrary closed way with $S_1 = S_n$. Then it is true that

$$\begin{aligned}\sum_{i=1}^{n-1} \delta_{S_{i+1}}^{S_i}\, \Phi_{j_{S_{i+1}}^{S_i}}(\Gamma^{S_i \cup S_{i+1}}) &= \sum_{i=1}^{n-1} \delta_{S_{i+1}}^{S_i} \left(P(\Gamma^{S_i \cup S_{i+1}}) - P(\Gamma^{(S_i \cup S_{i+1}) - j_{S_{i+1}}^{S_i}})\right) \\ &= \sum_{i=1}^{n-1} P(\Gamma^{S_{i+1}}) - P(\Gamma^{S_i}) \\ &= P(\Gamma^{S_n}) - P(\Gamma^{S_1}) \\ &= 0\end{aligned}$$

Therefore Φ is conservative. □

Remark. If Φ is a conservative solution concept then it is true that $W(\Phi, S_a, S_e, (S_1, \cdots, S_n) = P(S_e) - P(S_a)$. Differences of the potential characterize in this sense the expenditure.

In classical mechanics conservative forces imply conservation of energy. This theorem is highly important in physics. It means that the whole mechanic energy is the same at every time.

In this game theoretical context one has a similar property of conservative solution concepts which can be described informally. If one deals with a solution concept which is not conservative, then there exists at least one closed way for which the expenditure is positive. This utility is deprived from the players of the grand coalition. A repetition might be done such that more and more utility is deprived from the players. For conservative solution concepts on the other side the whole (transferable) utility of the players is the same at every time.

Corollary 2.10 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed and let Φ be a conservative solution concept, then a potential $P : G(\Gamma) \to \mathbf{R}$ is given by*

$$P(\Gamma^S) = \sum_{i \in S} \Phi_i(\Gamma^{\{j \in S:\ j \leq i\}}) \tag{7}$$

Proof. It is obvious that P is a real valued operator on $G(\Gamma)$. Now let $S \subseteq \Omega$ and $i \in S$ be arbitrary, then holds

$$\nabla_i P(\Gamma^S) \;=\; P(\Gamma^S) - P(\Gamma^{S-i})$$

$$
\begin{aligned}
&= \sum_{l \in S} \Phi_l(\Gamma^{\{j \in (S):\ j \leq l\}}) - \sum_{l \in S-i} \Phi_l(\Gamma^{\{j \in (S-i):\ j \leq l\}}) \\
&= \Phi_i(\Gamma^S) + \sum_{l \in S-i} \Phi_l(\Gamma^{\{j \in (S-i):\ j \leq l\}}) - \sum_{l \in S-i} \Phi_l(\Gamma^{\{j \in (S-i):\ j \leq l\}}) \\
&= \Phi_i(\Gamma^S)
\end{aligned}
$$

This is always true because of the independence of the way of the expenditure of Φ for every $S \subseteq \Omega$ and every $i \in S$ by proposition 2.9. □

Set $P(\Gamma^{\emptyset}) = 0$, then the potential $P(\Gamma^S)$ has the following meaning: It describes exactly the amount of utility which has to be transfered to the master just to bring the members of the coalition S step by step into the game. In this sense $(-P(S))$ is the potential ability of the coalition S to obtain utility (from the master). It is the game theoretical analogon to the potential energy in physics.

3 Determination

The well known Shapley–Formula can be derived by the aid of a potential.

Proposition 3.1 *Let $\Gamma = (\Omega, v)$ be a cooperative game. Then the Shapley-Value Ψ for $i \in \Omega$ is given by*

$$
\Psi_i(\Gamma) = \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(|\Omega| - |R|)!\ (|R| - 1)!}{|\Omega|!} \Big(v(R) - v(R - i)\Big) \tag{8}
$$

Proof. In the following n respectively r denote the cardinality of the set Ω respectively R. Furthermore let $P : G(\Gamma) \to \mathbf{R}$ be a potential of the Shapley-Value Ψ. Then

$$
\Psi_i(\Gamma) = \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{1}{\binom{n}{r}} \Psi_i(\Gamma^R) - \sum_{\substack{R \subset \Omega \\ i \in R}} \frac{1}{\binom{n}{r}} \Psi_i(\Gamma^R)
$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{1}{\binom{n}{r}} \Psi_i(\Gamma^R) - \sum_{\substack{R \subset \Omega \\ i \in R}} (n-r) \frac{(n-r-1)!\ r!}{n!} \Psi_i(\Gamma^R)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{1}{\binom{n}{r}} \Psi_i(\Gamma^R) - \sum_{\substack{R \subseteq \Omega \\ i \in R}} \sum_{j \in R-i} \frac{(n-r)!\ (r-1)!}{n!} \Psi_i(\Gamma^{R-j})$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left(r\Psi_i(\Gamma^R) - \sum_{j \in R-i} \Psi_i(\Gamma^{R-j}) \right)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left(\Psi_i(\Gamma^R) + \sum_{j \in R-i} \Psi_i(\Gamma^R) - \Psi_i(\Gamma^{R-j}) \right)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left(\Psi_i(\Gamma^R) \right.$$

$$\left. + \sum_{j \in R-i} P(\Gamma^R) - P(\Gamma^{R-i}) - (P(\Gamma^{R-j}) - P(\Gamma^{(R-j)-i})) \right)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left(\sum_{j \in R} P(\Gamma^R) - P(\Gamma^{R-j}) \right.$$

$$\left. - \sum_{j \in R-i} P(\Gamma^{R-i}) - P(\Gamma^{(R-i)-j}) \right)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left((\Phi(\Gamma^R))(R) - (\Phi(\Gamma^{R-i}))(R-i) \right)$$

$$= \sum_{\substack{R \subseteq \Omega \\ i \in R}} \frac{(n-r)!\ (r-1)!}{n!} \left(v(R) - v(R-i) \right)$$

□

Remark. This proposition 3.1 can also be verified by theorem 4.2. One can directly show that a solution concept defined by (8) is efficient and preserves differences.

4 Theorem

The symmetry in the partial differences may be viewed as a partial difference equation analog to a partial differential equation in physics. With an additional constraint which is always necessary there is a unique solution namely the Shapley–Value. Hart and Mas-Colell[7] on the other side use the potential for their proof.

Definition 4.1 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed. A solution concept Φ is efficient, if for all $S \subseteq \Omega$ holds*

$$(\Phi(\Gamma^S))(S) = v_0(S) \tag{9}$$

Theorem 4.2 *There exists exactly one solution concept Φ which is efficient and preserves differences. This Φ is equal to the Shapley–Value Ψ.*

Proof. First let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed and let $\{\lambda_T\}_{T \in 2^\Omega \setminus \{\emptyset\}}$ be the coefficients of expansion in the basis of the normalized unanimity games. Because of corollary 1.3 it is true that

$$\begin{aligned}(\Psi(\Gamma^S))(S) &= \sum_{i \in S} \sum_{\substack{T \in 2^S \\ i \in T}} \frac{1}{|T|} \lambda_T \\ &= \sum_{T \in 2^S \setminus \{\emptyset\}} \lambda_T \\ &= v_0(S)\end{aligned}$$

[7] cf. [1], citehart2

Hence Ψ is efficient. For arbitrary $S \subseteq \Omega$ and arbitrary $i, j \in S$, $i \neq j$ it is true again because of corollary 1.3

$$\Psi_i(\Gamma^S) = \sum_{\substack{T \in 2^S \\ i \in T}} \frac{1}{|T|} \lambda_T$$

and dito

$$\Psi_i(\Gamma^{S-j}) = \sum_{\substack{T \in 2^{(S-j)} \\ i \in T}} \frac{1}{|T|} \lambda_T$$

Therefore for the difference it is true that

$$\begin{aligned} \nabla_j \Psi_i(\Gamma^S) &= \sum_{\substack{T \in 2^S \\ i \in T}} \frac{1}{|T|} \lambda_T - \sum_{\substack{T \in 2^S \\ i \in T \\ j \notin T}} \frac{1}{|T|} \lambda_T \\ &= \sum_{\substack{T \in 2^S \\ i,j \in T}} \frac{1}{|T|} \lambda_T \qquad (10) \end{aligned}$$

and mutatis mutandis

$$\nabla_i \Psi_j(\Gamma^S) = \sum_{\substack{T \in 2^S \\ i,j \in T}} \frac{1}{|T|} \lambda_T \qquad (11)$$

A comparison between (10) and (11) completes the proof of existence. One still has to show uniqueness. Let Φ be an arbitrary solution concept which is efficient and preserves differences. Moreover let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed. Then $\nabla_j \Phi_i(\Gamma^S) = \nabla_i \Phi_j(\Gamma^S)$ for arbitrary $S \subseteq \Omega$ and arbitrary $i, j \in S$, $i \neq j$. Now one sums up both sides over all $j \in (S - i)$.

$$\sum_{j \in (S-i)} \nabla_j \Phi_i(\Gamma^S) = \sum_{j \in (S-i)} \nabla_i \Phi_j(\Gamma^S)$$

hence

$$|S|\Phi_i(\Gamma^S) - \sum_{j\in(S-i)} \Phi_i(\Gamma^{S-j}) = (\Phi(\Gamma^S))(S) - (\Phi(\Gamma^{S-i}))(S-i)$$

and eventually because of effiency of Φ

$$\Phi_i(\Gamma^S) = \frac{1}{|S|}\Big(v_0(S) - v_0(S-i) + \sum_{j\in(S-i)} \Phi_i(\Gamma^{S-j})\Big) \tag{12}$$

By induction one can see that $\Phi_i(\Gamma^S) = \Psi_i(\Gamma^S)$.

Basis of induction. Let $S \subseteq \Omega$ with $|S| = 1$ be arbitrary, thus $S = \{i\}$ for arbitrary $i \subseteq \Omega$, then by corollary 1.3

$$\Phi_i(\Gamma^{\{i\}}) = \lambda_{\{i\}} = \Psi_i(\Gamma^{\{i\}})$$

Induction hypothesis. For arbitrary, but fixed $n \in \mathbb{N}\setminus\{0\}$, $n < |\Omega|$ Let $\Phi_i(\Gamma^S) = \Psi_i(\Gamma^S)$ for all $S \subseteq \Omega$ with $|S| = n$ and all $i \in S$.

Induction step. $n \to n+1$. Let $S \subseteq \Omega$ with $|S| = n+1$ be arbitrary. Then by assumption it is true in equation (12) for arbitrary $i \in S$

$$\begin{aligned}
\Phi_i(\Gamma^S) \overset{IV}{=} & \frac{1}{|S|}\Big(\sum_{T\in 2^S\setminus\{\emptyset\}} \lambda_T - \sum_{T\in 2^{S-i}\setminus\{\emptyset\}} \lambda_T\Big) \\
& + \frac{1}{|S|} \sum_{j\in(S-i)} \sum_{\substack{T\in 2^{(S-j)} \\ i\in T}} \frac{1}{|T|}\lambda_T \\
= & \frac{1}{|S|} \sum_{\substack{T\in\cap 2^S \\ i\in T}} \lambda_T \\
& + \frac{1}{|S|} \sum_{\substack{T\in 2^S \\ T\neq S \\ i\in T}} (|S| - |T|)\, \frac{1}{|T|}\lambda_T
\end{aligned}$$

$$= \sum_{\substack{T \in 2^S \\ T=S}} \frac{1}{|T|}\lambda_T + \frac{1}{|S|} \sum_{\substack{T \in 2^S \\ T \neq S \\ i \in T}} \lambda_T$$

$$+ \sum_{\substack{T \in 2^S \\ T \neq S \\ i \in T}} \left(\frac{1}{|T|} - \frac{1}{|S|}\right) \lambda_T$$

$$= \sum_{\substack{T \in 2^S \\ i \in T}} \frac{1}{|T|}\lambda_T$$

$$= \Psi_i(\Gamma^S)$$

Thus Φ is equal to the Shapley–Value Ψ. □

The Shapley–Value can be uniquely characterized for fixed $\Gamma = (\Omega, v_0)$ on the set $G(\Gamma)$. Because Γ is arbitrary, one has an axiomatization on the class of all cooperative games. If one wishes to axiomatize the Shapley–Value directly on the class of all cooperative games, then the demand on efficiency and preservation of differences can be "weakened".

Corollary 4.3 *A solution concept Φ is equal to the Shapley–Value Ψ, if and only if the following two properties are valid.*

(i) $\forall \Gamma = (\Omega, v): \qquad ((\Phi(\Gamma))(\Omega) = v(\Omega)$

(ii) $\forall \Gamma = (\Omega, v)\ \forall i, j \in \Omega,\ i \neq j: \qquad \nabla_i \Phi_j(\Gamma) = \nabla_j \Phi_i(\Gamma)$

Proof. "$\Leftarrow$" For every $\Gamma = (\Omega, v)$ the Shapley–Value Ψ is efficent and preserves differences by theorem 4.2 particularly for the grand coalition Ω.

"$\Rightarrow$" Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed. Then every subgame $\Gamma^S \in G(\Gamma)$ can particularly be viewed as a new cooperative game. By assumption the efficiency and preservation of differences are valid for Γ^S. □

In a slightly different way Hart and Mas-Colell[8] found the potential and the preservation of differences. The efficiency property is binded in their definitions of the potential. But if one wants to have reasonable interpretations, one should have the auxilary condition of effiency. The results concerning the conservaty and the physically interpretations are essential and new. The many analoga to physics are here really clear[9].

5 Miszellaneum

Preservation of differences alone is not sufficient to uniquely characterize the Shapley–Value. The Banzhaf–Index is a solution concept for cooperative games which is especially for simple games in parliaments very useful.[10]. It is possible to characterize it uniquely by preservation of differences and an additional constraint.[11]

Definition 5.1 *A solution concept Ξ defined by*

$$\Xi_i(\Omega, v) := 2^{(1-|\Omega|)} b_i(\Omega) \qquad \forall i \in \Omega \tag{13}$$

$$\text{with } b_i(\Omega) := \sum_{\substack{T \subseteq \Omega \\ i \in T}} v(T) - v(T-i) \tag{14}$$

is called Banzhaf-Index.

Definition 5.2 *Let $\Gamma = (\Omega, v_0)$ be arbitrary, but fixed. A solution concept Φ contributes the marginalities, if for all $S \subseteq \Omega$ it is true that*

$$(\Phi(\Gamma^S))(S) = 2^{(1-|S|)} \sum_{i \in S} b_i(S) \tag{15}$$

[8]cf. [1], [2]

[9]cf. appendix

[10]cf. [5]

[11]This theorem has independently been derived by A. Ostmann and P. Sudhölter

Theorem 5.3 *A solution concept Φ is equal to the Banzhaf-Index Ξ, if and only if it contributes the marginalities and preserves differences.*

Proof. "$\Rightarrow$" Let $\Gamma = (\Omega, v)$ be arbitrary, but fixed. The Banzhaf–Index contributes the marginalities per definitionem. One still has to proof the preservation of differences. For arbitrary $S \subseteq \Omega$ and $i, j \in S$, $i \neq j$ it is true that $\Xi_i(\Gamma^S) - \Xi_i(\Gamma^{S-j}) = 2^{(1-|S|)}\,(b_i(S) - 2b_i(S-j))$. Hence

$$
\begin{aligned}
b_i(S) - 2b_i(S-j) &= \sum_{\substack{T\subseteq S \\ i\in T}} v(T) - v(T-i) \\
&\quad - 2 \sum_{\substack{T\subseteq (S-j) \\ i\in T}} v(T) - v(T-i) \\
&= \sum_{\substack{T\subseteq S \\ i\in T \\ j\in T}} v(T) - v(T-i) \\
&\quad - \sum_{\substack{T\subseteq S \\ i\in T \\ j\notin T}} v(T) - v(T-i) \\
&= \sum_{\substack{T\subseteq S \\ i\in T \\ j\in T}} v(T) - \sum_{\substack{T\subseteq S \\ i\notin T \\ j\in T}} v(T) \\
&\quad - \sum_{\substack{T\subseteq S \\ i\in T \\ j\notin T}} v(T) + \sum_{\substack{T\subseteq S \\ i\notin T \\ j\notin T}} v(T)
\end{aligned}
$$

Mutatis mutandis $b_i(S) - 2b_i(S-j) = b_j(S) - 2b_j(S-i)$ and thus $\nabla_j \Xi_i(\Gamma^S) = \nabla_i \Xi_j(\Gamma^S)$.

"$\Leftarrow$" Let $\Gamma = (\Omega, v_0)$ be an arbitrary, but fixed cooperative game and let $\nabla_j \Phi_i(\Gamma^S) = \nabla_i \Phi_j(\Gamma^S)$ for all $S \subseteq \Omega$ and all $i, j \in S$, $i \neq j$. Then one sums up both sides over all $j \in (S-i)$.

$$\sum_{j\in(S-i)} \nabla_j \Phi_i(\Gamma^S) = \sum_{j\in(S-i)} \nabla_i \Phi_j(\Gamma^S)$$

therefrom

$$|S|\ \Phi_i(\Gamma^S) - \sum_{j\in(S-i)} \Phi_i(\Gamma^{S-j}) = (\Phi(\Gamma^S))(S) - (\Phi(\Gamma^{S-i}))(S-i)$$

and eventually, because Φ contributes the marginalities

$$\begin{aligned}
2^{(|S|-1)}\ |S|\ \Phi_i(\Gamma^S) &= \sum_{j\in S} b_j(S) - 2 \sum_{j\in(S-i)} b_j(S-i) + 2 \sum_{j\in(S-i)} b_i(S-j) \\
&= b_i(S) + \sum_{j\in(S-i)} \sum_{\substack{T\subseteq S \\ j\in T}} v(T) - v(T-j) \\
&\quad -2 \sum_{j\in(S-i)} \sum_{\substack{T\subseteq(S-i) \\ j\in T}} v(T) - v(T-j) \\
&\quad +2 \sum_{j\in(S-i)} \sum_{\substack{T\subseteq(S-j) \\ i\in T}} v(T) - v(T-i) \\
&= b_i(S) + \sum_{j\in(S-i)} \Bigg(\sum_{\substack{T\subseteq S \\ j\in T}} v(T) - \sum_{\substack{T\subseteq S \\ j\notin T}} v(T) \\
&\quad -2 \sum_{\substack{T\subseteq S \\ j\in T \\ i\notin T}} v(T) + 2 \sum_{\substack{T\subseteq S \\ j\notin T \\ i\notin T}} v(T) \\
&\quad +2 \sum_{\substack{T\subseteq S \\ j\notin T \\ i\in T}} v(T) - 2 \sum_{\substack{T\subseteq S \\ j\notin T \\ i\notin T}} v(T) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= b_i(S) + \sum_{j\in(S-i)} \Bigg(\sum_{\substack{T\subseteq S\\ j\in T\\ i\in T}} v(T) + \sum_{\substack{T\subseteq S\\ j\in T\\ i\notin T}} v(T) \\
&\quad - \sum_{\substack{T\subseteq S\\ j\notin T\\ i\in T}} v(T) - \sum_{\substack{T\subseteq S\\ j\notin T\\ i\notin T}} v(T) \\
&\quad -2 \sum_{\substack{T\subseteq S\\ j\in T\\ i\notin T}} v(T) - 2 \sum_{\substack{T\subseteq S\\ j\notin T\\ i\in T}} v(T)\Bigg) \\
&= b_i(S) + \sum_{j\in(S-i)} \Bigg(\sum_{\substack{T\subseteq S\\ j\in T\\ i\in T}} v(T) - \sum_{\substack{T\subseteq S\\ j\in T\\ i\notin T}} v(T) \\
&\quad + \sum_{\substack{T\subseteq S\\ j\notin T\\ i\in T}} v(T) - \sum_{\substack{T\subseteq S\\ j\notin T\\ i\notin T}} v(T)\Bigg) \\
&= b_i(S) + \sum_{j\in(S-i)} \Bigg(\sum_{\substack{T\subseteq S\\ i\in T}} v(T) - \sum_{\substack{T\subseteq S\\ i\notin T}} v(T)\Bigg) \\
&= b_i(S) + |S-i| \sum_{\substack{T\subseteq S\\ i\in T}} v(T) - v(T-i) \\
&= |S| b_i(S)
\end{aligned}
$$

Thus $\Phi_i(\Gamma^S) = 2^{(1-|S|)} b_i(S) = \Xi_i(\Gamma^S)$ for arbitrary $S \subseteq \Omega$ and $i \in S$. □

6 Consistency

The Shapley–Value can also be characterized on the family of cooperative games where both the set of players Ω and the characteristic function f varies. Again in this family there are games for which a solution is quite obvious, namely the two person games. The connection between the elements of this family is given by a property which is called consistency.

6.1 The Reduced Game

Definition 6.1 *Let $\Gamma = (\Omega, v)$ be a cooperative game and let Φ be a solution concept. For arbitrary $T \subseteq \Omega$, $T \neq \emptyset$ there is function $v_T^\Phi : 2^T \to \mathbb{R}$ defined by*

$$v_T^\Phi(S) := v(S \cup T^c) - \sum_{i \in T^c} \Phi_i \left(S \cup T^c, v_{|2^{(S \cup T^c)}} \right) \text{ }^{12} \tag{16}$$

The tuple $\Gamma_T = (T, v_T^\Phi)$ is called the reduced game with respect to T.

Remark. If Φ is efficient, then one can simplify equation (16)

$$v_T^\Phi(S) = \sum_{i \in S} \Phi_i \left(S \cup T^c, v_{|2^{(S \cup T^c)}} \right) \tag{17}$$

Definition 6.2 *A solution concept Φ is consistent, if for every cooperative game $\Gamma = (\Omega, v)$ and every coalition $T \subseteq \Omega$, $T \neq \emptyset$ it is true that*

$$\Phi_i(T, v_T^\Phi) = \Phi_i(\Omega, v) \qquad \forall i \in T \tag{18}$$

The reduced game for the Shapley–Value was found by Hart and Mas-Colell[13]. There is the following motivation. Let $\Gamma = (\Omega, v)$ be a cooperative game and let Φ be a solution concept. For every coaliton T which is discontented with Φ one can define a reduced game in the following sense. Every discontented coalition $S \subseteq T$ can play the subgame $\Gamma^{S \cup T^c}$ together with the contented players of T^c. The worth of coalition S in this game is then $v(S \cup T^c)$. The players in T^c have

[12] $\Phi(\Omega, v)$ is short for $\Phi((\Omega, v))$

[13] cf. [1], [2]

to be paid off according to Φ because they always agree to the distribution. The worth of the coalition S in the reduced game is then $v(S \cup T^c) - (\Phi(\Gamma^{S \cup T^c}))(T^c)$. If Φ is efficient then this simplifies to $(\Phi(\Gamma^{S \cup T^c}))(S)$ which is exactly that what S can obtain on his own in the subgame $\Gamma^{S \cup T^c}$.

The solution concept Φ is now called consistent, if it gives for every discontented coalition $T \subseteq \Omega$ the same payoffs to the players in T in the reduced game $\Gamma_T = (T, v_T^\Phi)$ as before in the original game $\Gamma = (\Omega, v)$.

Definition 6.3 *A solution concept Φ is standard for two person games, if for every cooperative game $\Gamma = (\Omega, v)$ with $|\Omega| = 2$ it is true that*

$$\Phi_i(\Omega, v) = v(\{i\}) + \frac{1}{2}(v(\{i, j\}) - v(\{i\}) - v(\{j\})) \qquad \forall i, j \in \Omega \tag{19}$$

6.2 Theorems

Hart and Mas-Colell[14] have proofed the following two propositions by the use of a potential. Here it is shown that one can do it by preservation of differences as well which is little bit more elegant.

Proposition 6.4 *The Shapley–Wert Ψ is standard for two person games and consistent.*

Proof. Let $\Gamma = (\Omega, v)$ be a cooperative game and Ψ the Shapley–Value. Then by corollary 1.3 Ψ is standard for two person games. By induction on the cardinality of the subsets of Ω one can show the consistency in equation (18), because Ψ is efficient by theorem 4.2.

Basis of induction. Let $i \in \Omega$ be arbitrary and $T = \{i\}$. Then $v_T^\Psi(\emptyset) = 0$ and $v_T^\Psi(\{i\}) = \Psi_i(\Omega, v)$. Therefore per definitionem $\Psi_i(T, v_T^\Psi) = \Psi_i(\Omega, v)$.

Induction hypothesis. For arbitrary, but fixed $l \in \mathbb{N} \setminus \{0\}$, $l < |\Omega|$ let equation (18) be true for all $T \subset \Omega$ with $|T| \leq l$.

[14] cf. [1], [2]

Induction step. $l \to l+1$. Let $T \subseteq \Omega$ be arbitrary with $|T| = l+1$. By therorem 4.2 Ψ preserves differences, i.e., $\nabla_j \Psi_i(\Omega, v) = \nabla_i \Psi_j(\Omega, v)$ for all $i, j \in T$, $i \neq j$. One sums up both sides over all $j \in (T - i)$.

$$\sum_{j \in (T-i)} \nabla_j \Psi_i(\Omega, v) = \sum_{j \in (T-i)} \nabla_i \Psi_j(\Omega, v)$$

hence

$$\begin{aligned} |T| \Psi_i(\Omega, v) &= \sum_{j \in (T-i)} \Psi_i(\Omega - j, v_{|2^{(\Omega - j)}}) \\ &\quad + (\Psi(\Omega, v))(T) - (\Psi(\Omega - i, v_{|2^{(\Omega - i)}}))(T - i) \end{aligned} \tag{20}$$

Mutatis mutandis

$$\begin{aligned} |T| \Psi_i(T, 2^T, v_T^\Psi) &= \sum_{j \in (T-i)} \Psi_i(T - j, (v_T^\Psi)_{|2^{(T-j)}}) \\ &\quad + (\Psi(T, v_T^\Psi))(T) \\ &\quad - (\Psi(T - i, (v_T^\Psi)_{|2^{(T-i)}}))(T - i) \end{aligned}$$

Because of efficiency it is true that

$$(\Psi(T, v_T^\Psi))(T) = v_T^\Psi(T) = (\Psi(\Omega, v))(T)$$

By assumption of induction follows

$$\begin{aligned} |T| \Psi_i(T, v_T^\Psi) &= \sum_{j \in (T-i)} \Psi_i(\Omega - j, v_{|2^{(\Omega - j)}}) \\ &\quad + (\Psi(\Omega, v))(T) \\ &\quad - (\Psi(\Omega - i, v_{|2^{(\Omega - i)}}))(T - i) \end{aligned} \tag{21}$$

A comparison between (20) and (21) completes the proof. □

Proposition 6.5 *Let Φ be a solution concept which is standard for two person games and consistent. Then Φ is equal to the Shapley–Value Ψ.*

Proof. By corollary 4.3 it is sufficient to show the efficiency and preservation of Φ only for the grand coalition. By induction over the cardinality of the grand coalition one can show the effiency.

Basis of induction. Let $|\Omega| = 1$. Consider the two person game $(\{i,j\}, 2^{\{i,j\}}, \tilde{v})$, where $\tilde{v}(\{i\}) = v(\{i\})$ and $\tilde{v}(S) = 0$ otherwise. Then $v = \tilde{v}^{\Phi}_{\{i\}}$ and hence

$$\begin{aligned}\Phi_i(\Omega, v) &= \Phi_i(\{i\}, \tilde{v}^{\Phi}_{\{i\}}) \\ &= \Phi_i(\{i,j\}, \tilde{v}) \\ &= v(\{i\})\end{aligned}$$

Also for $|\Omega| = 2$ Φ is efficient because of the property of being standard.

Induction hypothesis. For arbitrary, but fixed $n \in \mathbb{N} \setminus \{0\}$ let Φ be efficient for all $\Gamma = (\Omega, v)$ with $|\Omega| \leq n$.

Induction step. $n \rightarrow n+1$. Let $\Gamma = (\Omega, v)$ be an arbitrary cooperative game with $|\Omega| = n+1$ and let $l \in \Omega$ be arbitrary, then

$$\begin{aligned}(\Phi(\Gamma))(\Omega) = \sum_{i \in \Omega} \Phi_i(\Omega, v) &= \Phi_l(\Omega, v) + \sum_{i \in \Omega - l} \Phi_i(\Omega, v) \\ &= \Phi_l(\Omega, v) + \sum_{i \in \Omega - l} \Phi_i(\Omega - l, v^{\Phi}_{\Omega - l}) \\ &\overset{IV}{=} \Phi_l(\Omega, v) + v^{\Phi}_{\Omega - l}(\Omega - l) \\ &\overset{n.V.}{=} v(\Omega)\end{aligned}$$

The preservation of differences can be proofed by induction, too.

Basis of induction. For cooperative games $\Gamma = (\Omega, v)$ with players, $\Phi(\Omega, v)$ is equal to $\Psi(\Omega, v)$ by corollary 1.3 and hence preseves differences because of theorem 4.2.

Induction hypothesis. Let $n \in \mathbb{N} \setminus \{0\}$ be arbitrary, but fixed. Then $\Phi(\Omega, v)$ preserves differences for every cooperative game $\Gamma = (\Omega, v)$ with $|\Omega| \leq n$.

Induction step. $n \rightarrow n+1$. Let $\Gamma = (\Omega, v)$ be an arbitrary cooperative game with $|\Omega| = n+1 \geq 3$. By assumption of induction the preservation of differences has only to be shown for Ω and arbitrary $i, j \in \Omega$. Let $l \in \Omega$, $l \neq i, j$, then

$$\begin{aligned}\Phi_i(\Omega, v) - \Phi_j(\Omega, v) &= \Phi_i\left(\Omega - l, v^{\Phi}_{\Omega - l}\right) \\ &\quad - \Phi_j\left(\Omega - l, v^{\Phi}_{\Omega - l}\right) \\ &\overset{IV}{=} \Phi_i\left(\Omega - l - j, (v^{\Phi}_{\Omega - l})_{|2^{(\Omega - l - j)}}\right)\end{aligned}$$

$$
\begin{aligned}
& -\Phi_j\left(\Omega - l - i, (v^{\Phi}_{\Omega - l})_{|2^{(\Omega - l - i)}}\right) \\
= \; & \Phi_i\left(\Omega - l - j, (v_{|2^{(\Omega - j)}})^{\Phi}_{(\Omega - l - j)}\right) \\
& -\Phi_j\left(\Omega - l - i, 2^{(\Omega - l - i)}, (v_{|2^{(\Omega - i)}})^{\Phi}_{(\Omega - l - i)}\right) \\
= \; & \Phi_i\left(\Omega - j, v_{|2^{(\Omega - j)}}\right) \\
& -\Phi_j\left(\Omega - i, v_{|2^{(\Omega - i)}}\right)
\end{aligned}
$$

Thus $\nabla_i \Phi(S, v_{|2^S}) = \nabla_j \Phi(S, v_{|2^S})$ for all $\Gamma = (\Omega, v)$ and all $S \subseteq \Omega$, $i, j \in S$, $i \neq j$.
□

7 Appendix

Here the analoga between game theory and physics can easily be seen on a table. A force in physics cooresponds to solution concept in game theory. If the integrability condition respectively the preservation of differences is fulfilled then there exists in both cases a potential. The different partial derivatives respectively differences form the force respectively the solution concept. Conservative forces are very important in physics. Examples are the graviational force, the Coulomb force, the force of linear harmonic oszillator and so on. They imply that the work is independent of the way. Similarly for conservative solution concepts the expenditure is independent of the way. Eventually one can search for an easy way to calculate the potential. It might be a task for future numerical analysis to find for every game such a way and hence to give an algorithm for computing the Shapley-Value.

Physics	**Game Theory**
$F : \mathbb{R}^n \to \mathbb{R}^n$	$\Phi : G(\Gamma) \to \mathbb{R}^{\|\Omega\|}$
$\frac{\partial v_j}{\partial x_i} = \frac{\partial v_i}{\partial x_j}$ $\forall i, j = 1, \cdots, n$	$\nabla_i \Phi_j(\Gamma^S) = \nabla_j \Phi_i(\Gamma^S)$ $\forall S \subseteq \Omega,\ \forall i \in S,\ i \neq j$
$\exists V : \mathbb{R}^n \to \mathbb{R}$ grad $V(x) = F(x)$	$\exists P : G(\Gamma) \to \mathbb{R}$ $\nabla_S P(\Gamma^S) = \Phi(\Gamma^S)$
$\gamma : [\alpha, \beta] \to \mathbb{R}^n$ is a way from a to e, if • $\gamma(\alpha) = a,\ \gamma(\beta) = e$ • γ piecewise continously differentiable	$(S_1, \cdots, S_n)$ is a way from S_a to S_e, if • $S_1 = S_a,\ S_n = S_e$ • $S_{i+1} = S_i \pm j_i$
$\oint_\gamma F(x)\, dx = 0$ for every closed way	$\sum_{i=1}^{n-1} \delta_{S_i}^{j_i}\ \Phi_{j_i}(\Gamma^{S_i \cup S_{i+1}}) = 0$ for every closed way
$\int_\alpha^\beta F(\gamma(t))\gamma'(t)\, dt$ $= V(e) - V(a)$ independent of γ	$\sum_{i=1}^{n-1} \delta_{S_i}^{j_i}\ \Phi_{j_i}(\Gamma^{S_i \cup S_{i+1}})$ $= P(S_e) - P(S_a)$ independent of $(S_1, \cdots, S_n)$
$V(x) = \int_0^x F(\tilde{x})\, d\tilde{x}$ is unique up to an additive constant	$P(S) = \sum_{i \in S} \Phi_i(\Gamma^{\{j \in S:\ j \leq i\}})$ is unique up to an additive constant

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