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The Nash Solution as a von Neumann–Morgenstern Utility Function on Bargaining Games

by

Anke Gerber

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University of Bielefeld

33501 Bielefeld, Germany

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Abstract

In this paper we prove that the Nash solution is a risk neutral von Neumann-Morgenstern utility function on the class of pure bargaining games. Our result is a generalization of the result of Roth [3] to bargaining games with status quo $\neq 0$.

1 Notation

Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ ($n \in I\!N$) be a permutation. Then π defines a mapping $\pi : I\!R^n \to I\!R^n$ which we denote by the same symbol via

$$(\pi(x))_i := x_{\pi^{-1}(i)}, \; x \in I\!R^n.$$

π also induces a mapping $\pi : \mathcal{P}(I\!R^n) \to \mathcal{P}(I\!R^n)$ (again denoted by the same symbol) via

$$\pi(A) := \{\pi(x) | \; x \in A\}, \; A \subset I\!R^n.$$

Let $* : I\!R^n \times I\!R^n \to I\!R^n$ be defined by

$$x * y := (x_1 y_1, \ldots, x_n y_n), \; (x, y) \in I\!R^n \times I\!R^n.$$

$(I\!R^n_{++}, *)$ is a group in the algebraic sense. For $x \in I\!R^n_{++}$ we denote the inverse element with respect to $*$ by x^{-1} $\left(x^{-1} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)\right)$. Of course, $*$ also induces a mapping $* : I\!R^n \times \mathcal{P}(I\!R^n) \to \mathcal{P}(I\!R^n)$ (again denoted by the same symbol) via

$$x * A := \{x * y | \; y \in A\}, \; x \in I\!R^n, \; A \subset I\!R^n.$$

For $x \in I\!R^n$ let $L_x : I\!R^n \to I\!R^n$ be defined by $L_x(y) = y + x$ ($y \in I\!R^n$). Then L_x defines a mapping $L_x : \mathcal{P}(I\!R^n) \to \mathcal{P}(I\!R^n)$ (denoted by the same symbol) via

$$L_x(A) := \{L_x(y) | \; y \in A\}, \; A \subset I\!R^n.$$

2 Definitions

Definition 2.1 *(A, s) is an* **n-person (pure) bargaining game** ($n \geq 2$) *if*

1. $A \subset I\!R^n, s \in A$.

2. *A is convex and closed.*

3. $\{x \in A|\ x \geq s\}$ *is bounded (therefore compact).*

4. *A is comprehensive, i. e.* $x \in A,\ y \leq x \Rightarrow y \in A$.

Let $H = \{(A,s)|\ (A,s)$ *is an n-person bargaining game*$\}$. *For* $(A,s) \in H$ *let* $R_{A,s} = \{i \in N|\ \exists x \in A$ *s.t.* $x_i > s_i\}$ *be the set of strategic positions, where* $N = \{1,\ldots,n\}$. i *is called dummy for* $(A,s) \in H$ *if* $i \notin R_{A,s}$.

Definition 2.2 $\phi : H \longrightarrow I\!R^n$ *is called a* **solution** *if*

$$\phi(A,s) \in A \quad \forall (A,s) \in H.$$

Definition 2.3 $\nu : H \longrightarrow I\!R^n$ *is called* **Nash solution**, *if* ν *is a solution and if* ν *obeys the following conditons:*

1. ν *is* **stable under affine transformations of utility**, *i.e. let* $(A,s) \in H, a \in I\!R^n_{++}, b \in I\!R^n$, *and let* $(B,t) \in H$ *be defined by* $t := a*s+b, B := a*A+b$, *then* $\nu(B,t) = a*\nu(A,s)+b$.

2. ν *is* **independent of irrelevant alternatives**, *i.e. let* (A,s), $(B,t) \in H, A \subset B, s = t,\ \nu(B,t) \in A$, *then* $\nu(A,s) = \nu(B,t)$.

3. ν *is* **symmetric**, *i.e. let* $(A,s) \in H, \pi$ *a permutation of* $\{1,\ldots,n\}$ *and let* $(B,t) \in H$ *be defined by* $t := \pi(s), B := \pi(A)$, *then* $\nu(B,t) = \pi(\nu(A,s))$.

4. ν *is* **pareto optimal**, *i.e. let* $(A,s) \in H, x \in A, x \geq \nu(A,s)$, *then* $x = \nu(A,s)$.

5. ν *is* **individually rational**, *i.e.* $\nu(A,s) \geq s\ \forall (A,s) \in H$.

The following result ist due to Nash [2].

Theorem 2.4 *There is a unique Nash solution on H which is defined by*

$$\nu(A,s) = \begin{cases} argmax\{\prod_{i\in R_{A,s}}(x_i - s_i)|\ x \in A, x \geq s\} & , \quad R_{A,s} \neq \emptyset \\ s & , \quad R_{A,s} = \emptyset \end{cases},$$

$(A,s) \in H$.

3 Assumptions and first results

The set of positions in bargaining games is given by $N \times H$.

Definition 3.1 *A set M is called* **mixture set** *if for any $a, b \in M$ and any $0 \leq p \leq 1$ we can associate another element of M denoted by $[pa;(1-p)b]$ which is called a lottery between a and b such that the following conditions are fulfilled:*

1. $[1a;0b] = a\ \forall a,b \in M$.
2. $[pa;(1-p)b] = [(1-p)b;pa]\ \forall a,b \in M, 0 \leq p \leq 1$.
3. $[q[pa;(1-p)b];(1-q)b] = [pqa;(1-pq)b]\ \forall a,b \in M, 0 \leq p,q \leq 1$.

Let M be the mixture set generated by $N \times H$ and let $\succeq$ be a preference relation on M which fulfills the following axioms:

Axiom 1 (Continuity) *Let $a,b,c \in M$. Then the following sets are closed:* $\{p|[pa;(1-p)b] \succeq c,\ 0 \leq p \leq 1\}$ *and* $\{p|c \succeq [pa;(1-p)b],\ 0 \leq p \leq 1\}$.

Axiom 2 (Independence) *Let $a, a' \in M, a \sim a', b \in M$. Then*

$$\left[\frac{1}{2}a;\frac{1}{2}b\right] \sim \left[\frac{1}{2}a';\frac{1}{2}b\right].$$

For $A \subset I\!R^n$ let

$$compH(A) = \{x | \ \exists y \in A, x \leq y\}$$

be the comprehensive hull of A.

The following sets will play a significant role in the analysis:

$$D^R = compH\left(\left\{x \in I\!R^n \left| \sum_{i \in R} x_i \leq 1, x_i \geq 0 \ (i \in R), x_j = 0 \ (j \notin R)\right.\right\}\right),$$

$(\emptyset \neq R \subset N)$, and

$$D_x = compH(\{x\}).$$

The two axioms for $\succeq$ guarantee the existence of an expected utility function θ that represents $\succeq$ (see [1]). θ is unique up to affine transformations. We choose the following normalization: Let $e = (1, \ldots, 1) \in I\!R^n$ and $i \in N$ arbitrary. Then

$$\begin{aligned} \theta(i, (D_e, e)) &= 1 \\ \theta(i, (D_{-e}, -e)) &= -1 \end{aligned}$$

We assume that $\succeq$ obeys the following conditions which will be explained later on:

1. $(i, (A, s)) \sim (\pi(i), (\pi(A), \pi(s))) \ \forall (i, (A, s)) \in N \times H, \pi$ a permutation of N.

2. Let $x, s, t \in I\!R^n, i \in N, 0 \leq p \leq 1$, s.t. $x_i = ps_i + (1-p)t_i$. Then
$$(i, (D_x, x)) \sim [p\,(i, (D_s, s))\,; (1-p)\,(i, (D_t, t))]\,.$$

3. Let $(A, s) \in H, a \in I\!R^n_{++}, a_i \geq 1$ and $t = a * s, B = a * A$. Then
$$(i, (A, s)) \sim \left[\frac{1}{a_i}\,(i, (B, t))\,; \left(1 - \frac{1}{a_i}\right)(i, (D_0, 0))\right].$$

4. Let $(i, (A, s)), (j, (B, t)) \in N \times H$ and $(i, (A, s)) \sim (j, (B, t))$. Then
$$(i, L_x(A, s)) \sim (j, L_x(B, t)),$$
where $L_x : H \longrightarrow H$ (same symbol) is defined by
$$L_x(A, s) = (L_x(A), L_x(s))\,, \ (A, s) \in H.$$

5. (a) Let i be dummy in $(A,0) \in H$. Then

$$(i,(A,0)) \sim (i,(D_0,0)).$$

(b) If i is no dummy in $(A,0) \in H$ then

$$(i,(A,0)) \succ (i,(D_0,0)).$$

(c) Let $x,y,s,t \in I\!R^n, x \geq s, y \geq t, i \in N$. Let $x_i = y_i$. Then

$$(i,(D_x,s)) \sim (i,(D_y,t)).$$

(d) Let $\emptyset \neq R \subset N, R \neq \{i\}$. Then

$$\left(i,\left(D^{\{i\}},0\right)\right) \succ \left(i,\left(D^R,0\right)\right).$$

6. Let $x \geq 0, D_x \subset B \subset C, x$ pareto optimal in C and $(i,(D_x,0)) \sim (i,(C,0))$. Then

$$(i,(D_x,0)) \sim (i,(B,0)).$$

With the exception of condition 2. the assumptions are already known from Roth [3] [1]. Let us give a short interpretation of all conditions always keeping in mind that the utility of a bargaining position is directly correlated to the player's expectation about what she will gain in the game. 1. reflects the fact that a player's gain is independent of her name. 2. and 3. are two different forms of neutrality towards ordinary risk (in contrast to the strategic risk of playing a certain position in a bargaining game). 4. is some kind of invariance towards richness. Being very sloppy one could say that the utility gain does not depend on the status quo. This is a very strict assumption. One can easily think of situations in which this condition is violated. 5. is self explaining and very intuitive. One could object against 5.(c) , however, claiming that a higher status quo outcome for i should imply a higher utility even when the maximum amount that can be achieved by i is the same.

[1] Condition 5.(b) does not appear in Roth [3]. In my opinion it is crucial for establishing that the "certain equivalent" of playing a bargaining game is strictly positive. It is not clear how Roth can prove the claim using only the conditions given in his paper.

In general this is clearly true. In the situation of 5.(c) however, the final outcome x, y, respectively, is sure (there is no "strategic risk"). 6. expresses independence of irrelevant alternatives. Condition 1. directly implies that $\forall i \in N$

$$\begin{aligned}\theta(i,(D_e,e)) &= 1,\\ \theta(i,(D_{-e},-e)) &= -1.\end{aligned}$$

Lemma 3.2 *Let $x, s \in I\!R^n, x \geq s, i \in N$. Then*

$$\theta(i,(D_x,s)) = x_i.$$

Proof: From condition 5.(c) we get $\theta(i,(D_x,s)) = \theta(i,(D_x,x))$. Therefore it suffices to prove the claim for the case $x = s$.
Case 1: $|s_i| \leq 1$.
Let $p = \frac{1}{2}(s_i + 1)$. Then $s_i = p1 + (1-p)(-1)$ and condition 2. implies

$$\begin{aligned}\theta(i,(D_s,s)) &= p\theta(i,(D_e,e)) + (1-p)\theta(i,(D_{-e},-e))\\ &= p1 + (1-p)(-1)\\ &= 2p-1\\ &= s_i.\end{aligned}$$

Case 2: $s_i > 1$.
Let $p = \frac{2}{s_i+1}$. Then $1 = ps_i + (1-p)(-1)$ and condition 2. implies:

$$\begin{aligned}\theta(i,(D_e,e)) &= p\theta(i,(D_s,s)) + (1-p)\theta(i,(D_{-e},-e))\\ \Longleftrightarrow 1 &= p\theta(i,(D_s,s)) + (1-p)(-1)\\ \Longrightarrow \theta(i,(D_s,s)) &= \frac{2-p}{p} = s_i.\end{aligned}$$

Case 3: $s_i < -1$.
Let $p = \frac{-2}{s_i-1}$. Then $-1 = ps_i + (1-p)1$ and condition 2. implies:

$$\begin{aligned}\theta(i,(D_{-e},-e)) &= p\theta(i,(D_s,s)) + (1-p)\theta(i,(D_e,e))\\ \Longleftrightarrow -1 &= p\theta(i,(D_s,s)) + 1 - p\\ \Longrightarrow \theta(i,(D_s,s)) &= \frac{p-2}{p} = s_i.\end{aligned}$$

□

Lemma 3.3 *Let i be dummy in $(A,s) \in H$. Then*

$$\theta(i,(A,s)) = s_i.$$

Proof: For $s = 0$ the claim is true by virtue of condition 5.(a) and Lemma 3.2. For $s \neq 0$ i is dummy in $L_{-s}(A,s) = (L_s(A),0)$ as well. Therefore $\theta(i, L_{-s}(A,s)) = 0 = \theta(i,(D_0,0))$. Finally condition 4. and Lemma 3.2 imply that $\theta(i,(A,s)) = \theta(i,(D_s,s)) = s_i$.

□

Lemma 3.4 *Let $(A,s) \in H, a \in I\!R^n_{++}, t = a * s, B = a * A$. Then*

$$\theta(i,(B,t)) = a_i\theta(i,(A,s)).$$

Proof: Case 1: $a_i \geq 1$.
Condition 3. implies:

$$\theta(i,(A,s)) = \frac{1}{a_i}\theta(i,(B,t)) + \left(1 - \frac{1}{a_i}\right) \underbrace{\theta(i,(D_0,0))}_{=0 \text{ by Lemma 3.2}}$$

Case 2: $a_i < 1$.
$\Rightarrow \frac{1}{a_i} > 1$. We have $s = a^{-1} * t, A = a^{-1} * B$. Then by Case 1:

$$\theta(i,(A,s)) = \frac{1}{a_i}\theta(i,(B,t)).$$

□

Define the "certain equivalent" $f(r)$ for a strategic position in the game $(D^R,0), \emptyset \neq R \subset N, |R| = r$, by

$$\left(i,\left(D^R,0\right)\right) \sim \left(i,\left(compH\left(f(r)D^{\{i\}}\right),0\right)\right), \ (i \in R).$$

$f(r)$ is well defined because of the continuity axiom for $\succeq$, condition 1., 5.(b) and 5.(d). From 5.(b) we get that $f(r) > 0\ \forall r$ and from condition 5.(d) it follows that $f(r) < 1$ for $r > 1$. $f(1) = 1$ is trivial. $\theta\left(i, \left(D^R, 0\right)\right) = f(r)$ since $D^{\{i\}} = D_{e_i}$,[2] $compH\left(f(r)D^{\{i\}}\right) = f(r)D^{\{i\}}$ for $f(r) > 0$ and Lemma 3.4 applies. $\succeq$ reflects neutrality towards strategic risk if $f(r) = \frac{1}{r}$, it reflects risk aversity if $f(r) < \frac{1}{r}$ and it is risk preferring if $f(r) > \frac{1}{r}\ \forall r$.

Lemma 3.5 *Let* $B_R = compH\left(\{y \geq 0 | \sum_{i \in R} b_i y_i \leq 1, y_j = 0\ (j \notin R)\}\right)$, $b_i > 0\ (i \in R)$. *Then*

$$\theta\left(i, (B_R, 0)\right) = \frac{f(r)}{b_i}.$$

Proof: $B_R = a * D^R$, where $a_i = \frac{1}{b_i}\ (i \in R), a_j = 1\ (j \notin R)$. Lemma 3.4 implies:

$$\theta\left(i, (B_R, 0)\right) = \frac{1}{b_i}\theta\left(i, \left(D^R, 0\right)\right) = \frac{f(r)}{b_i}.$$

□

4 The main result

Theorem 4.1 *Let* $(A, s) \in H$ *and* $R_{A,s}$ *be the set of strategic positions for* (A, s). *Let* $k \in R_{A,s}$, $q \in \mathbb{R}^n_+$ *s.t.* $q_k = f(r)$, *and* $\sum_{i \in R} q_i = 1$ *and let*

$$x = argmax\left\{ \prod_{i \in R_{A,s}} (x_i - s_i)^{q_i} | x \geq s, x \in A \right\}.$$

Then

$$\theta\left(k, (A, s)\right) = x_k.$$

[2] $e_i \in \mathbb{R}^n$ denotes the *ith* unit vector.

Proof: Case 1: $s = 0$.
To simplify the notation we use R instead of $R_{A,0}$. Let $c = \prod_{i \in R} x_i^{q_i} \geq \prod_{i \in} y_i^{q_i}$ $\forall y \in A, y \geq 0$ and let $E = \{y \geq 0 | \prod_{i \in R} y_i^{q_i} \geq c\} = \{y \geq 0 | \sum_{i \in R} q_i \log y_i \geq \log c\}$. E and A are convex sets with intersection x. Therefore there exists a plane T that seperates E and A. As T is tangent to E in x, T is given by

$$T = \left\{ z \left| \sum_{i \in R} \frac{q_i}{x_i} z_i = \sum_{i \in R} \frac{q_i}{x_i} x_i = \sum_{i \in R} q_i = 1 \right. \right\}.$$

Let $B = compH\left(\left\{ z \geq 0 \left| \sum_{i \in R} \frac{q_i}{x_i} z_i \leq 1, z_j = 0 \ (j \notin R) \right. \right\}\right)$. Then clearly $A \subset B$ since T seperates E and A and

$$\theta(k, (B, 0)) = f(r) \frac{x_k}{q_k} = x_k = \theta\left(k, (D_x, x)\right)$$

by Lemma 3.2 and 3.5. Therefore $(k, (B, 0)) \sim (k, (D_x, x))$. As $D_x \subset A \subset B$ condition 6. implies $(k, (D_x, 0)) \sim (k, (A, 0)) \Rightarrow \theta(k, (A, 0)) = x_k$.
Case 2: $s \neq 0$.
Again we use R instead of $R_{A,s}$. Let

$$\begin{aligned} x &= argmax \left\{ \prod_{i \in R} (y_i - s_i)^{q_i} \left| y \geq s, y \in A \right. \right\} \\ \Rightarrow x - s &= argmax \left\{ \prod_{i \in R} y_i^{q_i} \left| y \geq 0, y \in L_{-s}(A) \right. \right\}. \end{aligned}$$

Case 1 implies:

$$\theta\left(k, L_{-s}(A, s)\right) = \theta\left(k, \left(L_{-s}(A), 0\right)\right) = x_k - s_k,$$

i.e. $(k, L_{-s}(A, s)) \sim (k, (D_{x-s}, 0))$. By condition 4.:

$$\theta\left(k, L_s\left(L_{-s}(A, s)\right)\right) \sim \left(k, L_s\left(D_{x-s}, 0\right)\right).$$

$$\Rightarrow \theta\left(k, (A, s)\right) = \theta\left(k, (D_x, s)\right) = x_k.$$

□

Corollary 4.2 *If $\succeq$ is neutral towards strategic risk ($f(r) = \frac{1}{r} \ \forall r$) θ is equal to the Nash solution.*

Proof: Let $(A, s) \in H, R_{A,s}$ the set of strategic positions for (A, s). Let $k \in R_{A,s}$ and $x = argmax\{\prod_{i \in R_{A,s}} (y_i - s_i)^{q_i} | \; y \geq s, y \in A\}$, where $q_k = \frac{1}{r}$ and $\sum_{i \in R_{A,s}} q_i = 1$. Choose $q_i = \frac{1}{r} \; \forall i \in R_{A,s}$. Then $\theta(k, (A, s)) = x_k = \nu_k(A, s)$ $(k \in R_{A,s})$. For $k \notin R_{A,s}$ i.e. k dummy in (A, s) we get $\theta(k, (A, s)) = s_k = \nu_k(A, s)$ by condition 4., 5.(a) and Lemma 3.2. Especially $\theta(k, (A, s)) = s_k \; \forall k \in N$ if $R_{A,s} = \emptyset$.

□

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