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Partial Equilibrium in Pure Exchange Economies

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PARTIAL EQUILIBRIUM IN PURE EXCHANGE ECONOMIES†

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Abstract

We formally investigate partial equilibrium in pure exchange economies with money. We start by identifying the logical implications of the *ceteris paribus* assumption. Then we define the partial equilibrium property (PEP) of a sector. Several characterizations of the PEP for economies with differentiable utility functions are given in Section 3. The case of ordinal utility functions is considered in Section 4. Section 5, the last section, is devoted to an investigation of the continuity of the PEP.

1 Introduction

The technique of partial equilibrium analysis of a single market is well known and widely used in economics (see, e.g., Lipsey and Steiner (1972)). For a lucid exposition and an historical survey of partial equilibrium analysis the reader is referred to Arrow and Hahn (1971, Section 1.4). However, as far as we know, there is no attempt to formally integrate partial equilibrium analysis into general equilibrium theory, that is, to find conditions in terms of the utility functions and the production sets that render it applicable in the Arrow-Debreu model of private ownership economy. This paper is a modest contribution in this direction.

Before we embark on a formal investigation of partial equilibrium we should clarify the logical implications of the *ceteris paribus* assumption. We quote from Lipsey and Steiner (1972, p.406): "All partial-equilibrium analyses are based on the assumption of *ceteris paribus.* Strictly interpreted, the assumption is that all other things in the economy are unaffected by any changes in the sector under consideration". In this work we restrict the use of the *ceteris paribus* assumption to situations where the rest of the economy is in equilibrium. Secondly, we decompose it into two complementary assumptions:

(i) The (prevailing) equilibrium in the rest of the economy is not destroyed by any changes in the sector.

(ii) The equilibrium in the rest of the economy can be combined with any equilibrium of the sector to yield an equilibrium of the entire economy.

Clearly, (i) follows from the foregoing interpretation of the *ceteris paribus* assumption. And (ii) precisely describes the use of the law of supply and demand in partial equilibrium analysis. We further remark that if the utility functions are differentiable and strictly concave, then (ii) implies (i).

Our approach is not completely general because we consider only pure exchange economies. More precisely, most of our results are obtained for pure exchange economies with money as modeled in Shapley and Shubik [1969]. We shall now describe our results. Let $L = \{1, ..., l\}$ be the set of commodities in a Shapley-Shubik economy. A market is simply a non-empty proper subset M of L. $T = L \setminus M$ is called the residual economy. A market M has the *partial equilibrium property* (PEP) if the following conditions are satisfied:

(i) Every competitive equilibrium in the residual economy is unaffected by reallocations in the market.

(ii) Every competitive equilibrium of the residual economy can be completed to an equilibrium for the entire economy by, simply, stabilizing the market.

It is easily proved that if the utility functions of the traders are separable with respect to the partition (M, T), then M has the PEP. Example 2.3 shows that separability is not a necessary condition. In Section 3 we characterize the PEP under various assumptions on the differentiability of the utility functions. If the utility functions are twice continuously differentiable then our conditions take a very simple form. Let $u^1, ..., u^n$ be the utility functions of the traders. Then the mixed derivatives $\frac{\partial u^i}{\partial x_j \partial x_h}, j \in M, h \in T, i = 1, ..., n,$ must vanish on a certain *subset* of the set of all feasible allocations which is determined by the equilibria of the residual economy. Notice that the foregoing mixed derivatives vanish everywhere if and only if $u^1, ..., u^n$ are separable with respect to the partition (M, T).

In Section 4 we consider the possibility of extending our results to pure exchange economies with nontransferable utilities. Finally, in Section 5, we investigate the approximate PEP. We find that the PEP is very robust.

2 Partial Equilibrium in Economies with Transferable Utilities

Let $N = \{1, ..., n\}$ be the set of traders and let R^l_+ be the commodity space. A pure exchange economy with transferable utilities (TU) (see Shapley and Shubik [1969, p.13]) is a 2n-tuple $E = < w^1, ..., w^n; u^1, ..., u^n >$ (where $w^i \in R^l_+$ is the initial endowment of trader i and $u^i : R^l_+ \rightarrow R$ is her utility function), that satisfies the following assumptions:

$$u^i \text{ is concave for } i \in N. \tag{2.1}$$

$$u^i \text{ is continuous for } i \in N. \tag{2.2}$$

$$w = w^1 + ... + w^n >> 0 \text{ (if } x \in R^l \text{ then } x >> 0 \text{ if } x_j > 0, j = 1, ..., l). \tag{2.3}$$

Let $E = < w^1, ..., w^n; u^1, ..., u^n >$ be an economy. Actually, there is a more detailed presentation of E when money is introduced explicitly. Let E_m be the economy E with money. Then $E_m = < \widehat{w}^1, ..., \widehat{w}^n; \widehat{u}^1, ..., \widehat{u}^n >$ where $\widehat{w}^i = < w^i, m^i > \in R^l_+ \times R$, and $\widehat{u}^i : R^l_+ \times R \rightarrow R$ satisfies $\widehat{u}^i(x, m) = u^i(x) + m$ for all $i \in N$. There is no need to introduce money explicitly in the sequel.

Let $E = < w^1, ..., w^n; u^1, ..., u^n >$ be a TU economy. $\mathbf{x} = < x^1, ..., x^n >$ is a feasible allocation for E if $x^i \in R^l_+$ for all $i \in N$, and $\sum_{i=1}^{n} x^i = w$. The set of all feasible allocations for E is denoted by $A = A(w)$. A competitive equilibrium (c.e.) for E is a pair $(\bar{\mathbf{x}}, p)$ that satisfies the following conditions:

$$\bar{\mathbf{x}} \in A(w) \text{ and } p \in R^l. \tag{2.4}$$

$$u^i(\bar{x}^i) - p.\, \bar{x}^i \geq u^i(x) - p.x \text{ for all } x \in R^l_+ \text{ and } i \in N. \tag{2.5}$$

If $(\bar{\mathbf{x}}, p)$ is a c.e., then $\bar{\mathbf{x}}$ is called a competitive allocation (c.a.).

Denote $U(x^1, ..., x^n) = \sum_{i=1}^{n} u^i(x^i)$ for $x^i \in R^l_+, i \in N$. Then $\bar{\mathbf{x}} = <\bar{x}^1, ..., \bar{x}^n>$ is a c.a. iff

$$\bar{\mathbf{x}} \in \arg\max_{\mathbf{x} \in A(w)} U(\mathbf{x}) \tag{2.6}$$

(see Shapley and Shubik [1969, p.14]).

We begin our study of partial equilibrium by describing the possible submarkets of the economy E. Denote by $L = \{1, ..., l\}$ the set of all commodities. A (proper) *submarket* of

L is a subset $M \subset L$, with $M \neq \emptyset$, L. Let M be a submarket and let $T = L \backslash M$. If $x \in R^l$ then x_M denotes the restriction of x to M; if $\mathbf{x} = <x^1, ..., x^n> \in R^{nl}$ is an allocation, then $\mathbf{x}_M = < x^1_M, ..., x^n_M >$. Also

$$A(w_M) = \{\mathbf{x}_M =< x^1_M, ..., x^n_M >| \, x^i_M \in R^M_+ \text{ for } i \in N \text{ and } \sum_{i=1}^{n} x^i_M = w_M\}. \tag{2.7}$$

Similar notations apply to the set T.

Now let $\mathbf{y}_M \in A(w_M)$. The *restricted* economy $E_T(\mathbf{y}_M)$ is defined by

$$E_T(\mathbf{y}_M) = < w^1_T, ..., w^n_T; u^1(\cdot, y^1_M), ..., u^n(\cdot, y^n_M) > \tag{2.8}$$

Clearly, $E_T(\mathbf{y}_M)$ satisfies our Assumptions (2.1) - (2.3). Thus, $\bar{\mathbf{x}}_T$ is a c.a. of $E_T(\mathbf{y}_M)$ iff

$$\bar{\mathbf{x}}_T \in \arg\max_{\mathbf{x}_T \in A(w_T)} U(\mathbf{x}_T, \mathbf{y}_M) \tag{2.9}$$

Also, similarly, each $\mathbf{x}_T \in A(w_T)$ defines a restricted economy

$$E_M(\mathbf{x}_T) = < w^1_M, ..., w^n_M; u^1(x^1_T, \cdot), ..., u^n(x^n_T, \cdot) > . \tag{2.10}$$

It is now possible to discuss partial equilibrium.

Let $\mathbf{y}_M \in A(w_M)$. Then we can choose a c.e. $(\mathbf{x}_T, p_T)$ of $E_T(\mathbf{y}_M)$, and consequently a c.e. $(\mathbf{x}_M, p_M)$ of $E_M(\mathbf{x}_T)$. M has the partial equilibrium property if it is always true that $((\mathbf{x}_T, \mathbf{x}_M), (p_T, p_M))$ is a c.e. of E, and, conversely, if $(\mathbf{x}, p)$ is a c.e. of E, then $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$. (Notice that if $(\mathbf{x}, p)$ is a c.e. of E, then $(\mathbf{x}_M, p_M)$ is a c.e. of $E_M(\mathbf{x}_T)$ by our definitions.) Formally, we introduce the following definition.

Definition 2.1 *The submarket M has the partial equilibrium property (PEP) if for every $\mathbf{y}_M \in A(w_M)$ and every pair $(\mathbf{x}, p)$, where $\mathbf{x} \in A(w)$ and $p \in R^l$, the following conditions are satisfied.*

$$\text{If } (\mathbf{x}, p) \text{ is a c.e. of } E, \text{ then } (\mathbf{x}_T, p_T) \text{ is a c.e. of } E_T(\mathbf{y}_M). \tag{2.11}$$

$$\text{If } (\mathbf{x}_T, p_T) \text{ is a c.e. of } E_T(\mathbf{y}_M) \text{ and } (\mathbf{x}_M, p_M) \text{ is a c.e. of } E_M(\mathbf{x}_T), \text{ then } (\mathbf{x}, p) \text{ is a c.e. of } E. \tag{2.12}$$

(2.11) stipulates that the stability of the residual market T is not affected by reallocations in M: The restriction of a c.e. of E to T is a c.e. of $E_T(\mathbf{y}_M)$ for every allocation

$\mathbf{y}_M \in A(w_M)$. (2.12) is the converse of (2.11): If T is stabilized with respect to (w.r.t.) an arbitrary allocation $\mathbf{y}_M$ in $A(w_M)$, and *after that* M is stabilized w.r.t. the c.a. in T, then the combined result is a c.e. in E. It is our view that definition 2.1 is suitable for the classical partial equilibrium analysis in economics because it allows us to stabilize M without destroying an already stable situation in T.The relationship between (2.11) and (2.12) is discussed in the Appendix.

If the utility functions of the traders are separable w.r.t. M, then M has the PEP.

Theorem 2.2 *If for each $i \in N$ there exist functions $u^i_j, j = 1,2, u^i_1 : R^T_+ \to R$ and $u^i_2 : R^M_+ \to R$, such that*

$$u^i(x) = u^i_1(x_T) + u^i_2(x_M) \text{ for all } x \in R^\ell_+ \tag{2.13}$$

then M has the PEP.

Proof: First notice that by (2.13) $u^i_j, i \in N, j = 1,2$, are continuous and concave. Now let $\mathbf{y}_M \in A(w_M), \mathbf{x} \in A(w)$ and $p \in R^\ell$. If $(\mathbf{x},p)$ is a c.e. of E then, by (2.5), $u^i(x^i) - p \cdot x^i \geq u(x) - p \cdot x$ for all $x \in R^\ell_+$ and $i \in N$. By (2.13)

$$u^i_1(x^i_T) - p_T \cdot x^i_T \geq u^i_1(x_T) - p_T \cdot x_T \text{ for all } x_T \in R^T_+ \text{ and i } \in N \tag{2.14}$$

Because $\mathbf{x}_T \in A(w_T)$, (2.14) implies that $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$. Thus, (2.11) has been proved.

Conversely, assume that $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$ and $(\mathbf{x}_M, p_M)$ is a c.e. of $E_M(\mathbf{x}_T)$. Then (2.14) holds and also

$$u^i_2(x^i_M) - p_M \cdot x^i_M \geq u^i_2(x_M) - p_M \cdot x_M \text{ for all } x_M \in R^M_+ \text{ and } i \in N. \tag{2.15}$$

Hence, by (2.13) - (2.15), $(\mathbf{x},p)$ is a c.e. of E. Thus, (2.12) has been proved. Q.E.D.

Condition (2.13) is not necessary for the PEP as is shown by the following example.

Example 2.3 *Let $N = \{1,2\}, l = 2, w^1 = w^2 = (1,1), u^1(x_1,x_2) = 2x_1 + 2x_2$, and $u^2(x_1,x_2) = \min(x_1,x_2)$. If $M = \{1\}$ (i.e., the submarket M consists of the first commodity), then M has the PEP. Indeed, the foregoing economy E has a unique c.e. $(\mathbf{x},p) = (((2,2),(0,0)),(2,2))$. Also, if $\mathbf{y}_M \in A(w_M)$ then the restricted economy $E_T(\mathbf{y}_M) = < 1,1; 2x_2 + 2y^1_1, \min(x_2, y^2_1) >$ has a unique equilibrium $((2,0),2)$. Similarly $((2,0),2)$ is the unique c.e. of every restricted economy $E_M(\mathbf{x}_T)$, where $\mathbf{x}_T \in A(w_T)$. Thus, (2.11) and (2.12) are satisfied. Finally, $u^2(x_1,x_2) = \min(x_1,x_2)$ is not separable.*

We conclude with the following lemma.

Lemma 2.4 *Assume that M has the PEP and let $\mathbf{y}_M \in A(w_M)$. If $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$, then $(\mathbf{x}_T, p_T)$ a c.e. of $E_T(\mathbf{z}_M)$ for every $\mathbf{z}_M \in A(w_M)$.*

Proof: If $(\mathbf{x}_M, p_M)$ is a c.e. of $E_M(\mathbf{x}_T)$, then, by (2.12), $((\mathbf{x}_T, \mathbf{x}_M), (p_T, p_M))$ is a c.e. of E. Thus, by (2.11) $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{z}_M)$ for every $\mathbf{z}_M \in A(w_M)$. Q.E.D.

3 Characterizations of the PEP when the Utility Functions are Differentiable

Let $E = < w^1, ..., w^n; u^1, ..., u^n >$ be a TU economy that satisfies (2.1) - (2.3). If $u^1, ..., u^n$ are differentiable on R^ℓ_+ , then several characterizations of the PEP are possible. Thus, we introduce the following assumption.

$$\frac{\partial u^i}{\partial x_j} \text{ is continuous on } R^\ell_+ \text{ for } i = 1, ..., n, \text{ and } j = 1, ..., \ell. \tag{3.1}$$

We first recall the price characterization of c.a.'s.

Lemma 3.1 $\bar{\mathbf{x}} \in A(w)$ *is a c.a. of* E *iff the following conditions are satisfied.*

$$\bar{x}^i_j > 0 \Longrightarrow \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i) \geq \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k), i, k = 1, ..., n, j = 1, ..., \ell. \tag{3.2}$$

Proof: *Necessity.* If $\mathbf{x}$ is a c.a., then there exists a price vector $p \in R^\ell$ such that for every $i \in N$

$$u(\bar{x}^i) - p.\ \bar{x}^i \geq u^i(x) - p.x \text{ for all } x \in R^\ell_+ . \tag{3.3}$$

If $\bar{x}^i_j > 0$ then, by (3.3), $p_j = \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i)$. Also, if $\bar{x}^i_j = 0$ then $p_j \geq \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i)$. Thus, (3.2) has been proved.

Sufficiency. Assume for an allocation $\bar{x} \in A(w)$ that (3.2) is satisfied. For every $1 \leq j \leq \ell$ there exists an $i \in N$ such that $\bar{x}^i_j > 0$ (see (2.3)). Hence, we may define $p_j = \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i)$. Because u^i is concave

$$u^i(x) - u^i(\bar{x}^i) \leq \text{ grad } u^i(\bar{x}^i).(x - \bar{x}^i) \leq p.(x - \bar{x}^i) \text{ for all } x \in R^\ell_+$$

(here grad $u^i(\bar{x}^i) =< \frac{\partial u^i}{\partial x^i_1}(\bar{x}^i), ..., \frac{\partial u^i}{\partial x^i_\ell}(\bar{x}^i) >$). Thus, by definition, $\bar{\mathbf{x}}$ is a c.a. Q.E.D.

An allocation $\mathbf{x} \in R^{n\ell}_+$ is interior if $x^i_j > 0$ for $i = 1, ..., n$ and $j = 1, ..., \ell$. Under the following assumption we obtain a simple characterization of c.a.'s.

$$\frac{\partial u^i}{\partial x_j} \text{ is continuous on } R^\ell_{++} \text{ for } i \in N \text{ and } j \in L, \tag{3.4}$$

and every c.a. of E is interior (here $R^\ell_{++} = \{x \in R^\ell \mid x >> 0\}$).

Corollary 3.2 *If (3.4) is true, then* $\bar{\mathbf{x}} \in A(w)$ *is a c.a. iff* $\bar{\mathbf{x}}$ *is interior and*

$$\frac{\partial u^i}{\partial x^i_j}(\bar{x}^i) = \frac{\partial u^1}{\partial x^1_j}(\bar{x}^1) \text{ for } i \in N \text{ and } j \in L. \tag{3.5}$$

Now let $M \subset L, M \neq \emptyset, L$, and $T = L \backslash M$. The following theorem is a consequence of Lemma 3.1.

Theorem 3.3 *If* E *satisfies (3.1) then* M *has the PEP iff for every* $\mathbf{y}_M \in A(w_M)$ *and every c.a.* $\bar{\mathbf{x}}_T$ *of* $E_T(\mathbf{y}_M)$ *the following conditions are satisfied:*

$$\bar{x}^i_j > 0 \Rightarrow \begin{cases} \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) = \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k_T, z^k_M) \text{ if } \bar{x}^k_j > 0 \\ \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) \geq \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k_T, z^k_M) \text{ if } \bar{x}^k_j = 0 \end{cases} \quad i, k \in N, j \in T, \mathbf{z}_M \in A(w_M) \tag{3.6}$$

Proof: *Necessity.* Assume that M has the PEP and let $\bar{\mathbf{x}}_T$ be a c.a. of $E_T(\mathbf{y}_M)$, where $\mathbf{y}_M$ is in $A(w_M)$. There exists $p_T \in R^T$ such that $(\bar{\mathbf{x}}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$. By Lemma 2.4 $(\bar{\mathbf{x}}_T, p_T)$ is a c.e. of $E_T(\mathbf{z}_M)$ for every $\mathbf{z}_M \in A(w_M)$. Hence (3.2) implies (3.6).

Sufficiency. Let $\mathbf{y}_M \in A(w_M), \bar{\mathbf{x}} \in A(w)$ and $p \in R^\ell$. Assume first that $(\bar{\mathbf{x}}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$ and $(\bar{\mathbf{x}}_M, p_M)$ is a c.e. of $E_M(\bar{\mathbf{x}}_T)$. Then, by (3.2)

$$\bar{x}^i_j > 0 \Longrightarrow \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) \geq \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k_T, y^k_M), i, k \in N \text{ and } j \in T. \tag{3.7}$$

$$\bar{x}^i_j > 0 \Longrightarrow \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i) \geq \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k), i, k \in N \text{ and } j \in M. \tag{3.8}$$

By (3.6) and (3.7)

$$\bar{x}^i_j > 0 \Longrightarrow \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i) \geq \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k), i, k \in N \text{ and } j \in T. \tag{3.9}$$

Clearly, (3.8) and (3.9) imply (3.2). Thus $(\bar{\mathbf{x}}, p)$ is a c.e. of E. Therefore, (2.12) has been proved.

Assume now that $(\bar{\mathbf{x}}, p)$ is a c.e. of E. Then (3.9) is true. By (3.6), (3.7) is true. Thus, $(\mathbf{x}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$. Therefore, (2.11) has been proved. Q.E.D.

We proceed to find a characterization of PEP under the following stronger conditions.

$$\text{If } \mathbf{y}_M \in A(w_M) \text{ then every c.a. } \bar{\mathbf{x}}_T \text{ of } E_T(\mathbf{y}_M) \text{ is interior} \tag{3.10}$$
$$\text{(i.e., } \bar{x}^i_j > 0 \text{ for } i \in N \text{ and } j \in T).$$

Corollary 3.4 *If E satisfies (3.1) and (3.10), then M has the PEP iff for every $\mathbf{y}_M$ in $A(w_M)$ and every c.a. $\bar{\mathbf{x}}_T$ of $E_T(\mathbf{y}_M)$ the following conditions are satisfied:*

$$\frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) = \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k_T, z^k_M), \; i,k \in N, j \in T \text{ and } \mathbf{z}_M \in A(w_M). \tag{3.11}$$

We now add the following assumption.

$$\begin{aligned} \frac{\partial^2 u^i}{\partial x_j \partial x_h}(x) &= \frac{\partial^2 u^i}{\partial x_j \partial x_h}(x_T, x_M) \text{ is continuous for } x_T >> 0, \\ x_M &\geq 0, i \in N, j \in T, \text{ and } h \in M. \end{aligned} \tag{3.12}$$

Corollary 3.5 *Assume (3.1), (3.10), and (3.12). Then M has the PEP iff for every $\mathbf{y}_M \in A(w_M)$ and every c.a. $\bar{\mathbf{x}}_T$ of $E_T(\mathbf{y}_M)$ the following conditions are satisfied:*

$$\frac{\partial^2 u^i}{\partial x^i_j \partial x^i_h}(\bar{x}^i_T, z) = 0 \text{ for } i \in N, \; j \in T, \; h \in M \text{ and } 0 \leq z \leq w_M. \tag{3.13}$$

Proof: *Necessity.* (3.6) and (3.10) imply (3.11). (3.10) - (3.12) yield (3.13).

Sufficiency. Let $\mathbf{y}_M \in A(w_M)$ and let $\bar{\mathbf{x}}_T$ be a c.a. of $E_T(\mathbf{y}_M)$. (3.13) implies

$$\frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) = \frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, z) \text{ for } i \in N, j \in T \text{ and } 0 \leq z \leq w_M. \tag{3.14}$$

Because $\bar{x}_T$ is an interior c.a. of $E_T(\mathbf{y}_M)$, the following conditions are satisfied.

$$\frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M) = \frac{\partial u^k}{\partial x^k_j}(\bar{x}^k_T, y^k_M) \text{ for } i,k \in N, \text{ and } j \in T. \tag{3.15}$$

Clearly, (3.14) and (3.15) imply (3.11). Q.E.D.

We now make the following assumption:

$$E \text{ has a unique c.a. } \hat{\mathbf{x}} \tag{3.16}$$

Under (3.16) we obtain a simpler characterization of the PEP.

Theorem 3.6 *Assume (3.1) and (3.16). Then M has the PEP if for every $\mathbf{y}_M \in A(w_M)$ and every c.e. $(\bar{\mathbf{x}}_T, p_T)$ of E_T $(\mathbf{y}_M)$ there exist $\bar{\mathbf{x}}_M \in A(w_M)$ and $p_M \in R^M$ such $((\bar{\mathbf{x}}_T, \bar{\mathbf{x}}_M), (p_T, p_M))$ is a c.e. of E.*

Proof: Let $\mathbf{y}_M \in A(w_M), \bar{\mathbf{x}} \in A(w)$, and $p \in R^\ell$. Assume first that $(\bar{\mathbf{x}}_T, p_T)$ is a c.e. of $E_T(\mathbf{y}_M)$ and $(\bar{\mathbf{x}}_M, p_M)$ is a c.e. of $E_M(\bar{\mathbf{x}}_T)$. By our assumption there exist $\mathbf{z}_M \in A(w_M)$ and $q_M \in R^M$ such that $((\bar{\mathbf{x}}_T, \mathbf{z}_M), (p_T, q_M))$ is a c.e. of E. By (3.16), $\bar{\mathbf{x}}_T = \hat{\mathbf{x}}_T$ and $\mathbf{z}_M = \hat{\mathbf{x}}_M$. Thus, $(\bar{\mathbf{x}}_T, p_T)$ is a c.e. of $E_T(\hat{\mathbf{x}}_M)$. Also, $(\bar{\mathbf{x}}_M, p_M)$ is a c.e. of $E_M(\hat{\mathbf{x}}_T)$. Hence $\bar{\mathbf{x}}_M = \hat{\mathbf{x}}_M$ and $((\bar{\mathbf{x}}_T, \bar{\mathbf{x}}_M), (p_T, p_M))$ is the c.e. of E (by (2.3) and (3.1) each c.a. determines its vector of competitive prices). Thus, (2.12) has been proved.

Now assume that $(\bar{x}, p)$ is the c.e. of E. Then $\bar{\mathbf{x}}_T = \hat{\mathbf{x}}_T$. We claim that $(\bar{\mathbf{x}}_T, p_T)$ is the c.e. of $E_T(\mathbf{y}_M)$. Indeed, assume on the contrary that $(\mathbf{z}_T, q_T)$ is a c.a. of $E_T(\mathbf{y}_M)$ and $(\mathbf{z}_T, q_T) \neq (\hat{\mathbf{x}}_T, p_T)$. By our assumption there exists $\mathbf{z}_M \in A(w_M)$ and $q_M \in R^M$ such that $((\mathbf{z}_T, \mathbf{z}_M), (q_T, q_M))$ is the c.e. of E. As $(\mathbf{z}_T, q_T) \neq (\hat{\mathbf{x}}_T, p_T)$, the desired contradiction has been obtained. Thus, (2.11) has been proved. Q.E.D.

The following assumption implies (3.16).

$$u^i \text{ is strictly concave on } R^\ell_+ \text{ for } i \in N \tag{3.17}$$

(3.17) enables us to obtain the following direct characterization of PEP.

Theorem 3.7 *Assume (3.1), and (3.17). Then M has the PEP iff*

$$\hat{x}^i_j > 0 \Rightarrow \frac{\partial u^i}{\partial x^i_j}(\hat{x}^i_T, y^i_M) \geq \frac{\partial u^k}{\partial x^k_j}(\hat{x}^k_T, z^k_M) \text{ for } i, k \in N, j \in T, \text{ and } \mathbf{y}_M, \mathbf{z}_M \in A(w_M) \tag{3.18}$$

Proof: *Necessity.* Clearly, $\hat{x}_T$ is a c.a. of $E_T(\mathbf{y}_M)$ for every $\mathbf{y}_M \in A(w_M)$ (see Lemma 2.4). Hence, (3.6) implies (3.18).

Sufficiency. Let $y_M \in A(w_M)$. By (3.18)

$$\hat{x}^i_j > 0 \Rightarrow \frac{\partial u^i}{\partial x^i_j}(\hat{x}^i_T, y^i_M) \geq \frac{\partial u^k}{\partial x^k_j}(\hat{x}^k_T, y^k_M)\ ,\ i, k \in N \text{ and } j \in T. \tag{3.19}$$

Hence, by Lemma 3.1 $\hat{\mathbf{x}}_T$ is a c.a. of $E_T(\mathbf{y}_M)$. By (3.17) $\hat{\mathbf{x}}_T$ is the unique c.a. of $E_T(\mathbf{y}_M)$ (see (2.9)). If p_T is the price vector which is associated with $\hat{\mathbf{x}}_T$, then (for $j \in T$) $p_j = \frac{\partial u^i}{\partial x^i_j}(\hat{x}^i_T, y^i_M)$ for all i such that $\hat{x}^i_j > 0$. By (3.18), $p_j = \frac{\partial u^i}{\partial x^i_j}(\hat{x}^i)$ for all i such that $\hat{x}^i_j > 0$ (for $j \in T$). Thus, $(\hat{\mathbf{x}}_T, p_T)$ can be extended to the c.e. $(\hat{\mathbf{x}}, p)$ of E. By Theorem 3.6 M has the PEP. Q.E.D.

Finally, we characterize the PEP for strictly concave and twice continuously differentiable utility functions when the c.a. is interior.

Theorem 3.8 *Assume (3.12), (3.17), and* $\widehat{\mathbf{x}}_T >> 0$ *(here* $\widehat{\mathbf{x}}$ *is the unique c.a. of* E*). Then* M *has the PEP iff*

$$\frac{\partial^2 u^i}{\partial x^i_j \partial x^i_h}(\widehat{x}^i_T, z) = 0 \textit{ for } i \in N, j \in T, h \in M, \textit{ and } 0 \le z \le w_M. \tag{3.20}$$

Proof: *Necessity.* If M has the PEP, then (3.18) implies (3.20) because of (3.12) and $\widehat{\mathbf{x}}_T >> 0$.

Sufficiency. By (3.20) $\frac{\partial u^i}{\partial x^i_j}(\widehat{x}^i) = \frac{\partial u^i}{\partial x^i_j}(\widehat{x}^i_T, z)$ for $i \in N$, $j \in T$, and $0 \le z \le w_M$. Because $\widehat{\mathbf{x}}_T$ is an interior c.a. of $E_T(\widehat{x}_M)$, $\frac{\partial u^i}{\partial x^i_j}(\widehat{x}^i) = \frac{\partial u^k}{\partial x^k_j}(\widehat{x}^k)$ for $i, k \in N$ and $j \in T$. Hence (3.18) is satisfied. Q.E.D.

4 Partial Equilibrium in Pure Exchange Economies With Nontransferable Utilities

Let $E = < w^1, ..., w^n; u^1, ..., u^n >$ be a pure exchange economy with non-transferable utilities (NTU). In this section we make the following assumptions.

$$u^i : R^\ell_+ \to R \text{ is quasi-concave for } i \in N \tag{4.1}$$

$$u^i \text{ is strictly increasing on } R^\ell_+ \text{ for all } i \in N \tag{4.2}$$

$$u^i \text{ is continuous for } i \in N. \tag{4.3}$$

$$w^i \neq 0 \text{ for all } i \in N \text{ and } w = \sum_{i=1}^{n} w^i >> 0. \tag{4.4}$$

Let $M \subset L = \{1, ..., \ell\}$, with $M \neq L, \emptyset$, and $T = L \backslash M$. In this section we will always assume that both M and T have at least two members. Let $\mathbf{y}_M \in A(w_M)$ and $\delta_T \in R^N$. We define the *restricted economy with transfer* $\delta_T, E_T(\mathbf{y}_M, \delta_T)$, as follows. The initial endowment of trader i in $E_T(\mathbf{y}_M, \delta_T)$ is w^i_T, and her utility function is $u^i(\cdot, y^i_M)$. In addition i must transfer the amount δ^i_T of income from the residual market T to the submarket M (all price vectors are normalized such that the sum of the coordinates is 1). If $\mathbf{x}_T \in A(w_T)$ and $\delta_M \in R^N$ then $E_M(\mathbf{x}_T, \delta_M)$ is defined similarly. Let $\mathbf{y}_M \in A(w_M)$ and $\delta_T \in R^N$. $(\bar{\mathbf{x}}_T, \bar{p}_T)$ is a *competitive equilibrium* (c.e.) of $E_T(\mathbf{y}_M, \delta_T)$ if

$$\bar{\mathbf{x}}_T \in A(w_T) \text{ and } \sum_{i \in T} \bar{p}_j = 1. \tag{4.5}$$

For each $i \in N$

$$\bar{x}^i_T \in \underset{x^i_T \in B^i(\bar{p}_T, \delta^i)}{\arg\max} \; u^i(x^i_T, y^i_M) \tag{4.6}$$

where $B^i(\bar{p}_T, \delta^i_T) = \{x^i_T \in R^T_+ \,|\bar{p}_T \cdot x^i_T \leq \bar{p}_T \cdot w^i_T - \delta^i_T\}$

For the sake of completeness we recall that $(\bar{\mathbf{x}}, \bar{p})$ is a c.e. of E if

$$\bar{\mathbf{x}} \in A(w) \text{ and } \sum_{j=1}^{\ell} \bar{p}_j = 1 \tag{4.7}$$

$$\text{For each } i \in N \; \bar{x}^i \in \underset{x^i \in B^i(\bar{p}, 0)}{\arg\max} \; u^i(x^i) \tag{4.8}$$

By Assumptions (4.1) - (4.4) E has at least one c.e.. Also, if $(\bar{\mathbf{x}}, \bar{p})$ is a c.e. of E, then $\bar{p}_j > 0$ for $j = 1, ..., \ell$.

We now are able to discuss the PEP for NTU economies. As far as we can see only a weaker notion (as compared to Definition 2.1) is definable here. First we introduce some notations.

Let $(\bar{\mathbf{x}}, \bar{p})$ be a c.e. of E. We denote $\alpha(\bar{p}) = \sum_{j \in T} \bar{p}_j$. Clearly $0 < \alpha(\bar{p}) < 1$. Thus $\widehat{p}_T = \bar{p}_T / \alpha(\bar{p})$ and $\widehat{p}_M = p_M/(1-\alpha(p))$ are well defined normalized price vectors. Also we denote $\delta_T^i = \widehat{p}_T.w_T^i - \widehat{p}_T.x_T^i$ and $\delta_M^i = \widehat{p}_M.w_M^i - \widehat{p}_M.\, x_M^i$ for $i \in N$, and by δ_T and $\delta_M$the corresponding vectors. We claim that $\alpha(\bar{p})\delta_T + (1-\alpha(\bar{p}))\delta_M = 0$. Indeed

$$\alpha(\bar{p})\ [\widehat{p}_T.w_T^i - \widehat{p}_T.\, \bar{x}_T^i] + (1-\alpha(\bar{p}))[\widehat{p}_M.w_M^i - \widehat{p}_M.\, \bar{x}_M^i] =$$

$$\bar{p}_T\ .w_T^i - \bar{p}_T\ .\ \bar{x}_T^i + \bar{p}_M\ .w_M^i - \bar{p}_M\ .\ \bar{x}_M^i = \bar{p}\ .w^i - \bar{p}\ .\ \bar{x}^i = 0 \text{ for all } i \in N$$

Everything has been prepared for the following definition.

Definition 4.1 *The submarket M has the weak PEP (WPEP) if for every $\mathbf{y}_M \in A(w_M), \mathbf{y}_M >> 0_M$, and for every c.e. $(\bar{\mathbf{x}}, \bar{p})$ of E, the following condition is satisfied*

$$(\bar{\mathbf{x}}_T, \widehat{p}_T) \text{ is a c.e. of } E_T(\mathbf{y}_M, \delta_T) \tag{4.9}$$

Notice that it is always true that $(\bar{x}_M, \widehat{p}_M)$ is a c.e. of $E_M(\bar{\mathbf{x}}_T, \delta_M)$. Intuitively, (4.9) means that no reallocation $\mathbf{y}_M$ in the submarket M, that may result in a disequilibrium in M, affects the equilibrium situation in the residual economy $E_T(\mathbf{y}_M, \delta_T)$. Thus, the residual economy remains in equilibrium till the market M returns to equilibrium.

In order to characterize WPEP we make the following assumptions.

$$\text{Every c.e. of } E \text{ is interior.} \tag{4.10}$$

(4.10) is guaranteed, for example, by the following condition.

$$u^i \text{ is interior, that is, if } x \in R^{\ell}_{++}, y \in R^{\ell}_{+}, \text{ and } y_j = 0 \text{ for some } j, \text{ then } u^i(x) > u^i(y) \tag{4.11}$$

The second assumption is:

$$\frac{\partial^2 u^i(x)}{\partial x_j \partial x_h} \text{ is continuous on } R^{\ell}_{++} \text{ for } i = 1, ..., n, j, h = 1, ..., \ell. \tag{4.12}$$

We have prepared everything for the formulation of the following Theorem.

Theorem 4.2 *The market M has the WPEP if for each c.e. $(\bar{\mathbf{x}},\bar{p})$ of E the following conditions are satisfied:*

$$\frac{\partial u^i}{\partial x^i_j}(\bar{x}^i_T, y^i_M)/\,\bar{p}_j \;=\; \frac{\partial u^i}{\partial x^i_k}(\bar{x}^i_T, y^i_M)/\,\bar{p}_k, \quad j,k \in T, i \in N, \qquad (4.13)$$
$$\text{and } 0 \;<\; < y^i_M \leq w_M$$

For every $i \in N$ the Hessian of the Lagrangian

$$L^i(z^i_T,\lambda) = u^i(z^i_T, y^i_M) - \lambda[\widehat{p}_T.w^i_T - \delta^i_T - \widehat{p}_T.z^i_T] \qquad (4.14)$$

is negative definite at $\bar{x}^i_T$ subject to the constraint $\widehat{p}_T.z^i_T = 0$, for all values of the parameter $0 << y^i_M \leq w_M$.

Proof: For $i \in N$ $\bar{x}^i_T \in \arg\max_{x^i_T \in B^i(\widehat{p}_T,\delta^i_T)} u^i(x^i_T, y^i_M)$ if (4.13) and (4.14) are satisfied. Because $\bar{\mathbf{x}}_T \in A(w_T)$, (4.9) is satisfied. Q.E.D.

Remark 4.3 *Clearly, if M has the WPEP then (4.13) holds and $L^i(z^i_T,\lambda)$ has a negative semi-definite Hessian at $\bar{x}^i_T$, subject to the constraint $\widehat{p}_T.z^i_T = 0$. WPEP can be somewhat strengthened in the following way.*

Definition 4.4 *The submarket M has the PEP if*

$$M \text{ has the WPEP.} \qquad (4.15)$$

For each c.e. $(\bar{\mathbf{x}},\bar{p})$ of E and for each c.e. $(\mathbf{x}^*_M, p^*_M)$ of $E_M(\bar{\mathbf{x}}_T,\delta_M)$ the pair $((\bar{\mathbf{x}}_T\ \mathbf{x}^*_M),\tilde{p})$ is a c.e. of E, where (4.16)

$$\tilde{p} \;=\; (1-\alpha(\bar{p}))(0_T, p^*_M) + \alpha(\bar{p})(\widehat{p}_T, 0_M).$$

(4.16) has the following interpretation. If after a perturbation the market M is stabilized at a new equilibrium $(\mathbf{x}^*_M, p^*_M)$ (and not at the old c.e. of $E_M(\bar{\mathbf{x}}_T,\delta_M),(\bar{x}_M,\widehat{p}_M))$, then the new situation $((\bar{\mathbf{x}}_T, x^*_M),\tilde{p})$ is also a c.e. of E.

Unfortunately, we do not know how to characterize PEP for NTU economies.

5 Approximation of TU Pure Exchange Economies with the PEP

We consider the following class of economies

$$\mathcal{E} = \{E \mid E =< w^1, ..., w^n; u^1, ..., u^n >\}$$

where

$$\sum_{i=1}^{n} w^i << (K, ..., K) \in R^{\ell}_{++} \text{ and } w^i >> 0 \text{ for all } i \in N; \tag{5.1}$$

$$u^i \text{ is strictly concave for } i \in N; \tag{5.2}$$

$$\frac{\partial u^i}{\partial x_j} \text{ is continuous on } R^{\ell}_{+} \text{ for all } i \in N \text{ and } j \in L. \tag{5.3}$$

Further let $E =< w^1, ..., w^n; u^1; ..., u^n >, \mathbf{y}_M \in A(w_M)$, and $\mathbf{x}_T \in A(w_T)$. Then, by (5.2) each of the economies $E, E_T(\mathbf{y}_M)$, and $E_M(\mathbf{x}_T)$ has a unique c.a..

Our last assumption is

$$\text{For all } E =< w^1, ..., w^n; u^1, ...u^n > \text{ in } \mathcal{E}, \mathbf{y}_M \in A(w_M), \tag{5.4}$$

and $\mathbf{x}_T \in A(w_T)$ the c.a. of each of the economies $E, E_T(\mathbf{y}_M)$, and $E_M(\mathbf{x}_T)$ is interior.

Let again $E \in \mathcal{E}$. By (5.2), (5.3), and (5.4) E has a unique c.e. $(\mathbf{x}, p)$ which we shall denote by $(\mathbf{x}(E), p(E))$. Every restricted economy that will be considered will also have a unique c.e. which will be denoted similarly.

We shall use the following metric d on $\mathcal{E}$.

Let $E =< w^1, ..., w^n; u^1, ..., u^n >$ and $E_* =< w^1_*, ..., w^n_*; u^1_*, ..., u^n_* >$ be members of $\mathcal{E}$. Then

$$d(E, E_*) = \max_{1 \le i \le n} [\| w^i - w^i_* \|, \max_{0 \le x \le \widehat{K}} | u^i(x) - u^i_*(x) |] \tag{5.5}$$

where $\widehat{K} = (K, ..., K)$. It is now possible to formulate and prove the following continuity result in our model.

Lemma 5.1 *If $E(k), k = 1, 2, ...,$ and E are in $\mathcal{E}$, and $d(E(k), E) \to 0$, then $\mathbf{x}(E(k)) \to \mathbf{x}(E)$ and $p(E(k)) \to p(E)$*

Proof: We first prove that $\mathbf{x}(E(k)) \to \mathbf{x}(E)$. By (5.1) the sequence $(\mathbf{x}(E(k))$ is bounded. Let $(\mathbf{x}(E(k_j)))$ be a convergent subsequence of $(\mathbf{x}(E(k)))$. We have to prove that $\mathbf{x}(E(k_j)) \to \mathbf{x}(E)$. Assume, on the contrary, that $\mathbf{x}(E(k_j)) \to \mathbf{y}$ and $\mathbf{y} \neq \mathbf{x}(E)$. Because $\mathbf{x}(E(k_j)) \in A(w(E(k_j))), j = 1, 2, ..., \mathbf{y} \in A(w)$. (If $E \in \mathcal{E}, E =< w^1, ..., w^n; u^1, ..., u^n >$, then we shall denote, if necessary for clarity, $w^i = w^i(E)$ and $\mathrm{u}^i = u^i(E), i = 1, ..., n$.). Because $\mathbf{y} \neq \mathbf{x}(E)$ and (5.2)

$$\sum_{i=1}^{n} u^i(x^i(E)) > \sum_{i=1}^{n} u^i(y^i) \tag{5.6}$$

Denote $E(k_j) = \tilde{E}(j) =< \tilde{w}_j^1, ..., \tilde{w}_j^n; \tilde{u}_j^1, ..., \tilde{u}_j^n >, j = 1, 2, ...$ and define $\mathbf{z}_j \in R_+^{n\ell}$ by

$$z_{jh}^i = \frac{\tilde{w}_{jh}}{w_h(E)} x_h^i(E), i = 1, ..., n, h = 1, ..., \ell.$$

Clearly, $\mathbf{z}_j \in A(\tilde{w}_j), j = 1, 2, ...$ and $\mathbf{z}_j \to \mathbf{x}(E)$. By (5.5) $(\tilde{u}_j^i)$ converges uniformly to $u^i, i = 1, ..., n$. Therefore

$$\sum_{i=1}^{n} \tilde{u}_j^i(z_j^i) \to \sum_{i=1}^{n} u^i(x^i(E)) \tag{5.7}$$

From (5.6) and (5.7) we obtain that, for j sufficiently large $\sum_{i=1}^{n} \tilde{u}_j^i(z_j^i) > \sum_{i=1}^{n} \tilde{u}_j^i(x^i(\tilde{E}(j)))$, which is the desired contradiction.

We shall now prove that $p(E(k)) \to p(E)$. By (5.4) $\mathbf{x}(E)$ is interior. Therefore there exists $\varepsilon > 0$ such that

$$B = \{\mathbf{x} =< x^1, ..., x^n > \in R_+^{n\ell} \mid \| x^i - x^i(E) \| \leq \varepsilon, i = 1, ..., n\} \subset R_{++}^{n\ell}$$

Clearly, $\mathbf{x}(E(k)) \in B$ for k sufficiently large, and $\mathbf{x}(E(k)) \to \mathbf{x}(E)$ as we have already proved. By (5.3) and (5.4) $p(E(k)) = \text{grad } u_k^1(x^1(E(k)))$, where $u_k^1 = u^1(E(k)), k = 1, 2, ...$ By Rockafellar [1970, Theorem 25.7], grad u_k^1 converges uniformly on $B_1 = \{x^1 \in R_+^\ell \mid$ there exists $\mathbf{x} =< x^1, x^2, ..., x^n > \in B\}$ to grad u^1. Hence, $p(E(k)) \to p(E)$. Q.E.D.

Let $E \in \mathcal{E}$ and let $\varepsilon > 0$. $\mathbf{x} \in A(w)$ is an *ε-approximate* c.a. of E if $\| \mathbf{x} - \mathbf{x}(E) \| < \varepsilon$. Let $\mathbf{x} \in A(w)$ and $\delta > 0$. x is a δ-c.a. of E if there exists $p \in R^\ell$ such that

$$u^i(x^i) - p.x^i \geq u^i(x) - p.x - \delta \text{ for all } i \in N \text{ and } x \in R_+^\ell. \tag{5.8}$$

In our model approximate c.a.'s are almost competitive. More precisely,

Lemma 5.2 *Let $E \in \mathcal{E}$. For every $\delta > 0$ there exists $\varepsilon > 0$ such that every ε-approximate c.a. of E is a δ-c.a.*

Proof: Let $\delta > 0$. Denote $p = p(E)$ and $\widehat{\mathbf{x}} = \mathbf{x}(E)$. There exists $\varepsilon > 0$ such that if $\mathbf{x} \in A(w)$ and $\| \mathbf{x} - \widehat{\mathbf{x}} \| < \varepsilon$ then $\sum_{i=1}^{n} u^i(x^i) > \sum_{i=1}^{n} u^i(\widehat{x}^i) - \delta$. Now let $\mathbf{x}$ be an ε-approximate equilibrium. We claim that the pair $(\mathbf{x}, p)$ satisfies (5.8). Indeed, assume on the contrary, that there exist $i \in N$ and $x \in R^{\ell}_{+}$ such that

$$u^i(x^i) - p.x^i < u^i(x) - p.x - \delta \tag{5.9}$$

(5.9) implies

$$u^i(x^i) - p.x^i < u^i(\widehat{x}^i) - p.\widehat{x}^i - \delta \tag{5.10}$$

Also, for every $h \in N \backslash \{i\}$

$$u^h(x^h) - p.x^h \leq u^h(\widehat{x}^h) - p.\widehat{x}^h \tag{5.11}$$

(5.10) and (5.11) imply that $\sum_{i=1}^{n} u^i(x^i) < \sum_{i=1}^{n} u^i(\widehat{x}^i) - \delta$, which is the desired contradiction. Q.E.D.

We are now able to define approximate PEP.

Definition 5.3 *Let $E \in \mathcal{E}$ and let $\varepsilon > 0$. M has the ε-PEP w.r.t. E if for all $\mathbf{y}_M \in A(w_M)$ and all $\mathbf{x} \in A(w)$ the following conditions are satisfied.*

If $\mathbf{x}$ is a c.a. of E, then $\mathbf{x}_T$ is an ε-approximate c.a. of $E_T(\mathbf{y}_M)$. (5.12)

If $\mathbf{x}_T$ is a c.a. of $E_T(\mathbf{y}_M)$, and $\mathbf{x}_M$ is a c.a. of $E_M(\mathbf{x}_T)$, (5.13)
then $\mathbf{x} = (\mathbf{x}_T, \mathbf{x}_M)$ is an ε-approximate c.a. of E.

Our first result shows that the PEP is approximately robust.

Theorem 5.4 *Let $E \in \mathcal{E}$, let M has the PEP w.r.t. E, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $E_* \in \mathcal{E}$ and $d(E_*, E) < \delta$, then M has the ε-PEP w.r.t. E_*.*

Proof: Assume, on the contrary, that there exists a sequence $E(k) \in \mathcal{E}, k = 1, 2, ...$, such that $d(E, E(k)) \to 0$ and M does not have the ε-PEP w.r.t. $E(k)$. We have to distinguish the following two possibilities.

-

$$\begin{aligned} & \text{There exist sequences} \\ E(k_j), j &= 1, 2, ... \text{ and } \mathbf{y}_M(k_j) \in A(w_M(E(k_j))) \text{ such that} \\ & \mathbf{x}_T(E(k_j)) \text{ is not an } \varepsilon\text{-approximate c.a. of} \\ E_T(\mathbf{y}_M(k_j)), j &= 1, 2, ... \end{aligned} \tag{5.14}$$

In order to simplify our notations we shall assume $k_j = j, j = 1, 2, ...$ and $\mathbf{y}_M(j) \to \mathbf{y}_M$. By our assumptions $E_T(\mathbf{y}_M(j)) \to E_T(\mathbf{y}_M)$. Let $\mathbf{x} = \mathbf{x}(E)$. Then $\mathbf{x}_T$ is the c.a. of $E_T(\mathbf{y}_M)$ because M has the PEP w.r.t. E. By Lemma 5.1 $\mathbf{x}_T(E(j)) \to \mathbf{x}_T$ and $\mathbf{x}_T(E_T(\mathbf{y}_M(j))) \to \mathbf{x}_T$. Hence, $\mathbf{x}_T(E(j)) - \mathbf{x}_T(E_T(\mathbf{y}_M(j))) \to 0$, which is the desired contradiction.

$$\begin{aligned} & \text{There exist sequences } E(k_j) \text{ and} \\ \mathbf{y}_M(k_j) &\in A(w_M(E(k_j))), j = 1, 2, ...\text{such that} \\ & (\mathbf{x}_T(E_T(\mathbf{y}_M(k_j))), \mathbf{x}_M(E_M(\mathbf{x}_T(E_T(\mathbf{y}_M(k_j)))))) \\ & \text{is not an } \varepsilon\text{-approximate c.a. of} \\ E(k_j), j &= 1, 2, ... \end{aligned} \tag{5.15}$$

Again, we simplify our notations by assuming $k_j = j, j = 1, 2, ...$, and $\mathbf{y}_M(k_j) \to \mathbf{y}_M$. Then $\mathbf{x}_T(E_T(\mathbf{y}_M(j))) \to \mathbf{x}_T(E_T(\mathbf{y}_M))$ by Lemma 5.1. Denote $\mathbf{x}_T(j) = \mathbf{x}_T(E_T(\mathbf{y}_M(j))), j = 1, 2, ...$ Then, again by Lemma 5.1, $\mathbf{x}_M(E_M(\mathbf{x}_T(j))) \to \mathbf{x}_M(E_M(\mathbf{x}_T(E_T(\mathbf{y}_M))))$. Because M has the PEP w.r.t. E

$$(\mathbf{x}_T(E_T(\mathbf{y}_M)), \mathbf{x}_M(E_M(\mathbf{x}_T(E_T(\mathbf{y}_M))))) = \mathbf{x}(E).$$

Hence

$$(\mathbf{x}_T(j), \mathbf{x}_M(E_M(\mathbf{x}_T(j)))) - \mathbf{x}(E(j)) \to 0$$

because $\mathbf{x}(E(j)) \to \mathbf{x}(E)$ by Lemma 5.1. Thus, the desired contradiction has been obtained. Q.E.D.

Our final result is an approximation theorem for the PEP. Its precise formulation is as follows.

Theorem 5.5 *Let $E \in \mathcal{E}$ and $M \subset L, M \neq L, \emptyset$. If $E(k) \in \mathcal{E}, k = 1, 2, ..., d(E(k), E) \to 0, M$ has the $\varepsilon(k)$-PEP w.r.t. $E(k)$, and $\varepsilon(k) \to 0$, then M has the PEP w.r.t. E.*

Proof: Let $\mathbf{y}_M \in A(w_M)$ and $\mathbf{x} = \mathbf{x}(E)$. By Remark A.2 we only have to prove

$$(\mathbf{x}_T(E_T(\mathbf{y}_M)), \mathbf{x}_M(E_M(\mathbf{x}_T(E_T(\mathbf{y}_M))))) = \mathbf{x}(E) \tag{5.16}$$

Define $\mathbf{y}_M(k)$ by

$$y^i_{Mh}(k) = \frac{w_h(E(k))}{w_h(E)} y^i_{Mh}, i = 1, ..., n, h \in M, k = 1, 2, ...$$

Clearly, $\mathbf{y}_M(k) \to \mathbf{y}_M$. Therefore, by (5.5), $E_T(k)(\mathbf{y}_M(k)) \to E_T(\mathbf{y}_M)$. Denote $\mathbf{x}_T(k) = \mathbf{x}_T(E_T(k)(\mathbf{y}_M(k)))$. Then, by Lemma 5.1, $\mathbf{x}_T(k) \to \mathbf{x}_T(E_T(\mathbf{y}_M))$. Hence, also

$$\mathbf{x}_M(E_M(k)(\mathbf{x}_T(k))) \to \mathbf{x}_M(E_M(\mathbf{x}_T(E_T(\mathbf{y}_M))))$$

By our assumptions $\mathbf{x}(E(k)) \to \mathbf{x}(E)$ and $\| (\mathbf{x}_T(k), \mathbf{x}_M(E_M(k)(\mathbf{x}_T(k))) - \mathbf{x}(E(k)) \| < \varepsilon(k), k = 1, 2, ...$

Therefore (5.16) is true. Q.E.D.

Remark 5.6 *The first part of Assumption (5.1) is not essential for the results of this section. However, it enables us to use Definition (5.5) which allows relatively uncomplicated proofs. Clearly, (5.1) does not restrict the applicability of our results. Hence we adopt it.*

A Appendix: The Independence of (2.11) and (2.12)

We now show by means of an example that (2.12) does not imply (2.11).

Example A.1 *Let* $E =< w^1, w^2; u^1, u^2 >$ *where* $w^1 = w^2 = (1,1), u^1(x_1, x_2) = \min(x_1, x_2, 1)$, *and* $u^2(x_1, x_2) = (x_1 + x_2)/2$. *The set of c.e.'s of E is given by*

$$\{(((t,t),(2-t,2-t)),(\frac{1}{2},\frac{1}{2})) \mid 0 \leq t \leq 1\}$$

Let $M = \{1\}$. If $\mathbf{y}_M = (y_1^1, y_1^2) \in A(w_M)$, then

$$E_T(\mathbf{y}_M) =< 1, 1; \min(x_2, y_1^1, 1), (y_1^2 + x_2)/2 >$$

Let $\mathbf{y}_M \in A(w_M)$ satisfy $0 < y_1^1 < 1$. If $(\mathbf{x}, p)$ is a c.e., $(\mathbf{x}, p) = (((t,t),(2-t,2-t)),(\frac{1}{2},\frac{1}{2}))$ and $t \neq y_1^1$, then $(\mathbf{x}_T, p_T) = ((t, 2-t), \frac{1}{2})$ is *not* a c.e. of $E_T(\mathbf{y}_M)$. Thus, (2.11) is violated.

Now let $\mathbf{y}_M = (y_1^1, y_1^2) \in A(w_M)$. The c.e. of $E_T(\mathbf{y}_M)$ is $(\mathbf{x}_T, \mathbf{p}_T) = (t, 2-t), \frac{1}{2})$, where $t = \min(y_1^1, 1)$. Also, $E_M(\mathbf{x}_T) =< 1, 1; \min(x_1, t), \frac{x_1}{2} + 1 - \frac{t}{2} >$. Hence, its c.e. is given by $(\mathbf{x}_M, p_M) = ((t, 2-t), \frac{1}{2})$. Clearly, $((\mathbf{x}_M, \mathbf{x}_T), (p_M, p_T))$ is a c.e. of E. Thus, (2.12) is satisfied.

Remark A.2 *Let* $E =< w^1, ..., w^n; u^1, ..., u^n >$ *satisfy (3.1) and (3.16) (i.e.,* $u^1, ..., u^n$ *are differentiable and E has a unique c.a.). Then, by Theorem 3.6, (2.12) implies (2.11). All our results, except Theorem 3.3 and its Corollaries, are obtained under Assumptions (3.1) and (3.16).*

The next example shows that (2.11) does not imply (2.12).

Example A.3 *Let* $E =< w^1, w^2; u^1, u^2 >$ *where* $w^1 = w^2 = (1,1)$ *and* $u^1(x_1, x_2) = u^2(x_1, x_2) = \min[-(x_1-1)^2, -(x_2-1)^2]$. *The c.e. is given by* $(((1,1),(1,1)),(0,0))$. *Let* $M = \{1\}$. *If* $\mathbf{y}_M = (y_1^1, y_1^2)$ *is in* $A(w_M)$, *then*

$$E_T(\mathbf{y}_M) =< 1, 1; \min[-(x_2-1)^2, -(y_1^1-1)^2], \min[-(x_2-1)^2, -(y_1^2-1)^2] >$$

Clearly, $(1, 1; 0)$ is a c.e. of $E_T(\mathbf{y}_M)$. Thus, (2.11) is satisfied. As the reader may easily verify (2.12) is violated.

References

[1] Arrow, K.J. and Hahn, F. (1971) *General Competitive Analysis,* Holden Day, San Francisco, CA.

[2] Lipsey, R.G. and Steiner, P.O. (1972) *Economics*, third edition, Harper & Row, New York, NY.

[3] Rockafellar, R.T. (1970), *Convex Analysis,* Princeton University Press, Princeton, NJ.

[4] Shapley, L.S. and Shubik, M. (1969), "On Market Games", *Journal of Economic Theory* 1, 9-25.