INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 255

The Existence of Nash Equilibria in Two–Person, Infinitely Repeated Undiscounted Games of Incomplete Information: A Survey

by

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June 1996

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Abstract

This article concerns infinitely repeated and undiscounted two-person games of incomplete information. It surveys what is known about sufficient conditions to guarantee the existence of a Nash equilibrium, and presents some open problems. Journal of Economic Literature classification numbers C62, C72

This research was supported by the German Society for the Encouragement of Scientific Research (Deutsche Forschungsgemeinschaft.)

1 Introduction

We concern ourselves with games in which there are at least two states of nature and at least one player does know the state of nature at the start of the repeated game. Unlike stochastic games, the state of nature can not change during the course of the repeated game, though the knowledge of the players concerning the state of nature may change. The payoffs to the players at each stage are not progressively discounted, so that the limit properties of the averages of the payoffs up to finite stages must be considered.

In the model presented here all knowledge learned about the state of nature or about what the other players have done will be deterministic. There are important aspects of learning from signals that are not considered here; this author's goal is to present the topic in its maximal simplicity. Repeated games with more than two players also will not be considered.

This author has chosen to concentrate on the pioneer work in questions of equilibrium existence, including some contemporary open questions. In this article only a tiny fraction of all the superb results in the field of repeated games are mentioned.

The interested reader could consult the earlier survey article by Aumann (1985), the new book *Repeated Games* by Mertens, Sorin, and Zamir (1994), *The Handbook of Game Theory*, (Aumann and Hart, Eds., 1992), or the new book *Repeated Games of Incomplete Information* by Aumann and Maschler (1995), with the colaboration of R. Stearns.

2 The model

Let $K = \{1, 2, \ldots, n\}$ and let $p = (p_1, \cdots, p_n) \in \Delta(K)$ be a probability distribution on the set K. Let I and J be finite index sets for the pure actions of players one and two, of cardinality m_1 and m_2 respectively. Let $\{A^k \mid k \in K\}$ and $\{B^k \mid k \in K\}$ be m_1 x m_2 real matrices representing the payoffs to the players in all stages of play, with $A^k(i,j)$ and $B^k(i,j)$ being the payoffs to Player One and Player Two, respectively, when Player One chooses $i \in I$ and Player Two chooses $j \in J$.

The information structure of the repeated game is given by two families of partitions $\{\mathcal{P}^i \mid i \in I\}$ of $J \times K$ and $\{\mathcal{Q}^j \mid j \in J\}$ of $I \times K$ with an additional pair of partitions $\mathcal{P}^0$ of K and $\mathcal{Q}^0$ of K. The partitions $\mathcal{P}^0$ and

$\mathcal{Q}^0$ represent the initial knowledge of Players One and Two, respectively, on the states of nature $k \in K$, and the partitions $\{\mathcal{P}^i \mid i \in I\}$ and $\{\mathcal{Q}^j \mid j \in J\}$ represent what knowledge the players learn from their actions about the state of nature and the actions of their opponents.

If $\mathcal{P}$ is a partition of a set S and $x \in S$ then $\mathcal{P}(x)$ will be the member of $\mathcal{P}$ containing x. If S is a finite set, $s \in S$, and $p \in \Delta(S)$, then p_s is the probability p assigns to the singleton $\{s\}$.

A behavior strategy of player one is an infinite sequence $\sigma = \{\sigma^1, \sigma^2, \ldots\}$ such that for each l σ^l is a mapping from all tuples of the form

$$\left(\mathcal{P}^0(k)\ ,\ i_1\ ,\ \mathcal{P}^{i_1}(j_1, k)\ ,\ \ldots\ ,\ i_{l-1}\ ,\ \mathcal{P}^{i_{l-1}}(j_{l-1}, k)\right)$$

to $\Delta(I)$. Behavior strategies for Player Two are defined symmetrically. Let $\mathcal{I}$ and $\mathcal{J}$ be the set of behavior strategies of Players One and Two, respectively.

From the definition of the information structure and behavior strategies, the players have perfect recall. (See Kuhn, 1953.)

3 Histories and the definition of Nash equilibrium

For every $l < \infty$ define the set of finite histories of length l to be $\mathcal{H}^l := K \times (I \times J)^l$.

For every $h \in \mathcal{H}^l$ with $h = (k, i_1, j_1, \ldots, i_l, j_l)$ define

$$f_l(h) = 1/l \sum_{m=1}^{l} A^k(i_m, j_m) \text{ and}$$

$$g_l(h) = 1/l \sum_{m=1}^{l} B^k(i_m, j_m).$$

Every pair of behavior strategies $\sigma \in \mathcal{I}$ and $\tau \in \mathcal{J}$ induces a probability measure $\mu^l_{\sigma,\tau}$ on the set of finite histories of length l, so that if $h = (k, i_1, j_1, \cdots i_l, j_l) \in \mathcal{H}_l$ then

$$\mu^l_{\sigma,\tau}(\{h\}) = p_k \cdot \sigma^1\left(\mathcal{P}^0(k)\right)_{i_1} \cdot \tau^1\left(\mathcal{Q}^0(k)\right)_{j_1} \cdots \sigma^l\left(\mathcal{P}^0(k), i_1, \mathcal{P}^{i_1}(j_1, k), \ldots,\right.$$

$$\left.i_{l-1}, \mathcal{P}^{i_{l-1}}(j_{l-1}, k)\right)_{i_l} \cdot \tau^l\left(\mathcal{Q}^0(k), j_1, \mathcal{Q}^{j_1}(i_1, k), \ldots, j_{l-1}, \mathcal{Q}^{j_{l-1}}(i_{l-1}, k)\right)_{j_l}.$$

A Nash equilibrium is a pair of behavior strategies $\sigma \in \mathcal{I}$ and $\tau \in \mathcal{J}$ such that for every pair $\sigma^* \in \mathcal{I}$ and $\tau^* \in \mathcal{J}$

$$\lim_{m\to\infty} \sup \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma^*,\tau} \leq \lim_{m\to\infty} \inf \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau}$$

and

$$\lim_{m\to\infty} \sup \int_{\mathcal{H}_m} g_m(h) d\mu^m_{\sigma,\tau^*} \leq \lim_{m\to\infty} \inf \int_{\mathcal{H}_m} g_m(h) d\mu^m_{\sigma,\tau} \quad .$$

(By replacing σ^* by σ and τ^* by τ notice that if $\sigma \in \mathcal{I}$ and $\tau \in \mathcal{J}$ are a Nash equilibrium then both $\lim_{m\to\infty} \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau}$ and $\lim_{m\to\infty} \int_{\mathcal{H}_m} g_m(h) d\mu^m_{\sigma,\tau}$ exist. An alternative formulation of the definition of Nash equilibria exists; see Aumann and Maschler, (1995): page 140)

An "ϵ-Nash equilibrium" is a pair of behavior strategies $\sigma \in \mathcal{I}$ and $\tau \in \mathcal{J}$ such that for every pair $\sigma^* \in \mathcal{I}$ and $\tau^* \in \mathcal{J}$ the same inequalities as above hold with the quantity ϵ added to the right side of the inequalities. A game has "epsilon-Nash equilibria" when it has an ϵ-Nash equilibrium for every $\epsilon > 0$. When a repeated game has a Nash equilibria it has epsilon-Nash equilibria, but the converse does not hold.

If a zero-sum repeated game has epsilon-Nash equilibria, by exchanging the pairs of ϵ-Nash equilibria for different ϵ one can show through Cauchy convergence that there is a unique quantity such that the distance supremum between either $\lim_{m\to\infty} \sup \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma^\epsilon,\tau^\epsilon}$ or $\lim_{m\to\infty} \inf \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma^\epsilon,\tau^\epsilon}$ of any ϵ-Nash equilibrium $\sigma^\epsilon, \tau^\epsilon$ and this quantity converges to zero as ϵ goes to zero. In this case this quantity is called the value of the zero-sum game, and conversely if a zero-sum game has a value it means that it has epsilon-Nash equilibria. If a zero-sum game, repeated or not repeated, has an equilibrium, all strategies (or behavior strategies) that are half of an equilibrium pair are called "optimal" for the player in question.

4 Special conditions

Special conditions on the information structure are to be considered. Although these conditions are defined explicitely only for Player One, the definitions for Player Two will be symmetric.

– Perfect monitoring for Player One: for every $i \in I$ and $A \in \mathcal{P}^i$ there is some subset $K' \subseteq K$ and a $j \in J$ such that $A = \{j\} \times K'$.

- State independent learning for Player One: For every $A \in \mathcal{P}^0$ and $i \in I$ there exists a partition $\mathcal{P}^i_A$ of J such that $B \in \mathcal{P}^i$ implies that $B = C \times A$ for some $A \in \mathcal{P}^0$ and $C \in \mathcal{P}^i_A$. (The information structure satisfies "strong" state independent learning for Player One if $\mathcal{P}^i_A$ is the same for every $A \in \mathcal{P}^0$.)

- One-sided information for Player One: $\mathcal{P}^0$ is the discrete partition of K and for every $i \in I$ $\mathcal{P}^i$ is the discrete partition of $J \times K$.

- Symmetric information: Both players have perfect monitoring, $\mathcal{P}^0 = \mathcal{Q}^0$, and for every $i \in I$ and $j \in J$ the information received by both players about the states of nature is the same; in other words for every $i \in I$ and $j \in J$ there is a subset $K^{i,j} \subseteq K$ such that $\mathcal{P}^i = \{\{j\} \times K^{i,j} \mid j \in J\}$ and $\mathcal{Q}^j = \{\{i\} \times K^{i,j} \mid i \in I\}$.

- Independence of initial information : There exist non-negative real numbers $\{r_A \mid A \in \mathcal{P}_0\}$ and $\{s_B \mid B \in \mathcal{Q}_0\}$ such that $\sum_{x \in A \cap B} p_x = r_A s_B$.

Additionally, there are conditions on the information structure that pertain to knowledge of the payoff matrices.

- Player One knows her own payoff: $k, k' \in S \in \mathcal{P}^0$ implies that $A^k = A^{k'}$.

- Player One knows the other's (Two's) payoff: $k, k' \in S \in \mathcal{P}^0$ implies that $B^k = B^{k'}$.

In the non-zero-sum case, there are 16 different possible combinations of the above conditions on the knowledge of the payoffs; however there are only 10 combinations after accounting for symmetries generated by switching the players.

And finally there is the zero-sum condition: $B^k = -A^k$ for every $k \in K$.

Although the class of repeated games is rather restricted compared to that of stochastic games with incomplete information, this author does not want to imply that *all* of the interesting special cases are obtained through combinations of the above conditions.

When does there necessarily exist a Nash equilibrium or epsilon-Nash equilibria? Not all combinations of the above mentioned special conditions will be considered, since some combinations answer the question for others. Stronger assumptions will be considered only when the answer is "no" and weaker assumptions will be considered only when the answer is "yes." If a case is labeled "no" it means that there exist examples where epsilon-Nash equilibria don't exist. If the case is labeled yes, then there always exists a Nash equilibrium; and if the case is labeled "yes-ϵ" then there always exist epsilon-Nash equilibria. An exclamation point means that a published result answered exactly this question; a "yes" without an exclamation point means that the question has already been answered in the affirmative in a more general case; and a "no" without an exclamation point means that the question has already been answered in the negative in a more specific case.

5 One-sided information

For every $p \in \Delta(K)$ define the matrices $A(p) := \sum_{k=1}^{n} p_k A^k$ and $B(p) = \sum_{k=1}^{n} p_k B^k$. The value of a finite real matrix A, "val (A)" for short, will be the value of the corresponding zero-sum matrix game with the row player as the maximizer.

If C is a convex set and f is a bounded real valued function on C then $\text{cav}(f)$ is defined to be the minimal concave function greater than or equal to f and $\text{vex}(f)$ is defined to be the maximal convex function less than or equal to f. If f is a bounded real valued function on $A \times B$, both convex sets, then $\text{cav}_A(f)(a,b) = \text{cav}(f(\cdot,b))(a)$ where $f(\cdot,b)$ is a function on A. Likewise we define $\text{cav}_B(f)(a,b)$, $\text{vex}_A(f)(a,b)$, and $\text{vex}_B(f)(a,b)$.

Table 1

One-Sided Information for Player One

	General	Player Two Knows His Payoff	Zero-sum
General	?	Yes-$\mathcal{E}$	Yes !
Perfect Monitoring	?	Yes	Yes
State Independent Learning	?	Yes-$\mathcal{E}$	Yes
Perfect Monitoring and State Independent Learning	Yes !	Yes !	Yes !

The fundamental result in this topic is from the 1966-68 work of R. Aumann and M. Maschler (1995), with the colaboration of R. Stearns. For the zero-sum repeated game with perfect monitoring and state independent learning they proved that a Nash equilibrium exists and that the value for the informed player, Player One, is $\text{cav}(\text{val}(A(\cdot)))(p)$. The determination of the value came in two parts. First they proved that the informed player can guarantee herself on the average a function on $\Delta(K)$ that is both concave and greater than or equal to $\text{val}(A(\cdot))$. Second, with the help of Blackwell's generalization of the min-max theorem to vector valued payoffs (Blackwell, 1956), they showed that the uninformed player can hold down the payoffs of the informed player to any vector $v \in \mathbf{R}^K$ such that $v \cdot p \geq \text{val}(A(p))$ for all $p \in \Delta(K)$.

There is a curious asymmetry to these two claims concerning the value: the informed player can guarantee $\text{cav}(\text{val}(A(\cdot)))(p)$ only on the average determined by the distribution p; she can have extremely bad luck. As an example of S. Zamir's demonstrates (Zamir, 1992: page 112) the informed player may choose an optimal behavior strategy such that for some behavior strategy choice of the uninformed player, some state of nature, and some finite history starting with this state of nature and reached with positive probability, the conditional expectation of average payoff for the informed player at any stage following this finite history are lower than what the informed player can guarantee for herself at this state of nature. (Needless to say, such a pair of behavior strategies is not an equilibrium and gives the informed player very good average payoffs elsewhere!) Is there an example where the informed player can have such bad luck with every optimal behavior strategy? As far as this author is aware, this is an open question. On the other hand, for every vector v with the above property the uninformed player has a behavior strategy such that, with probability one (a.e.) the payoff to the informed player will be held down to no more than v_k for the $k \in K$ that happens to be the true state of nature (Aumann and Maschler, 1995; Blackwell, 1956). $\text{cav}(\text{val}(A(\cdot))) : \Delta(K) \rightarrow \mathbf{R}$ is also the function limit of the values of the corresponding finitely repeated games; the differences of these values from their limit is also an important topic (Aumann and Maschler, 1995; Zamir, 1972).

R. Aumann and M. Maschler (1995) showed that the zero-sum repeated game of one-sided information still has a value when the uninformed player may not have perfect monitoring or may not have state independent learning. They showed also that the informed player has an "optimal" behavior strategy that obtains for her on the average at least this repeated game value. Later, E. Kohlberg (1975) showed that the uninformed player also has an "optimal" behavior strategy, hence that the repeated game has an Nash equilibrium.

With regard to the non-zero-sum one-sided information case in which the uninformed player has perfect monitoring and state independent learning, the existence of an Nash equilibrium has been established. S. Sorin showed that there always exists an Nash equilibrium when there are two states of nature (Sorin, 1983). The principles at work behind Sorin's proof, and also behind the previous consideration of this class of games by R. Aumann, M. Maschler, and R. Stearns (Aumann and Maschler, 1995), are the concepts of "non-revealing" Nash equilibria and "joint plans." With perfect monitoring and state independent learning a non-revealing Nash equilibrium is a Nash equilibrium for which the informed player's behavior strategy is independent of her knowledge of the state of nature – in other words she never makes use of her information. (If perfect monitoring or state independent learning were not assumed, a more sophisticated definition of "non-revealing" would be necessary.) A joint plan is a set of signals S, for each $k \in K$ a probability distribution $q^k \in \Delta(S)$, and for each $s \in S$ an infinite sequence $(i_1^s, j_1^s, i_2^s, j_2^s, \ldots) \in (I \times J)^\infty$ such that for every $k \in K$ the distribution on $I \times J$, in the limit, converges. This implies that $\lim_{m\to\infty} f_m(k, i_1^s, j_1^s, \ldots, i_m^s, j_m^s)$ and $\lim_{m\to\infty} g_m(k, i_1^s, j_1^s, \ldots, i_m^s, j_m^s)$ exist. By allowing $s \in S$ to be chosen according to q^k if k is the true state of nature and communicating the result s to the uninformed player, the informed player can alter the uninformed player's subjective conditional probability distribution on the state of nature K, which we label $P(\cdot \mid s)$, satisfying $P(k \mid s) = q_s^k / (\sum_{k' \in K} q_s^{k'})$. (We assume without loss of generality that every $s \in S$ has a positive value q_s^k for some $k \in K$. Furthermore the informed player is allowed to use a finite number of stages to communicate the signals before they can play according to the corresponding sequence of moves.) The joint plan delivers a Nash equilibrium if

1) for every $s \in S$ the sequence of pure actions associated with s is the non-deviating behavior of a non-revealing Nash equilibrium of the repeated game with initial probability distribution $P(\cdot \mid s)$, and

2) for every $k \in K$ there is no payoff incentive for the informed player, given that the true state of nature is k, to choose $s \in S$ is any way other than by q^k.

Define $p^s \in \Delta(K)$ by $p_k^s := P(k \mid s)$. In general, finding a joint plan that satisfies both conditions 1 and 2 is not easy. Condition 2 means that there is a vector $x \in \mathbf{R}^{|K|}$ such that $\lim_{m\to\infty} f_m(k, i_1^s, j_1^s, \ldots, i_m^s, j_m^s) = x_k$ when $p_k^s = P(k \mid s) > 0$ and otherwise $\lim_{m\to\infty} f_m(k, i_1^s, j_1^s, \ldots, i_m^s, j_m^s) \leq x_k$. Condition 1 implies that for every $s \in S$ $\lim_{m\to\infty} \sum_{k\in K} p_k^s g_m(k, i_1^s, j_1^s, \ldots, i_m^s, j_m^s) \geq$ vex $(\text{val}(B(\cdot)^t))(p^s)$ and that $q \cdot x \geq \text{val}(A(q))$ for every $q \in \Delta$. For two states of nature, Sorin demonstrated the existence of such an Nash equilibrium delivered by a joint plan of a special kind he called "independent and two-safe." The "independent" property is that for every $s \in S$ the sequence of pairs of moves i^s_m, j^s_m has, in the limit, an independent distribution determined by a mixed strategy pair $(\sigma^s, \tau^s) \in \Delta(I) \times \Delta(J)$. The "two-safe" property is that for every $s \in S$ the τ^s is an optimal (or "safe") strategy for the uninformed player in the "one-shot" zero-sum game determined by the matrix $B(p^s)$. For arbitrarily many states of nature the existence of an independent and two-safe joint plan Nash equilibria was proven with the help of a new theorem of algebraic topology. (See Simon, Spież, and Toruńczyk, 1995. For general information on this case, see Forges, 1992.)

In the non-zero-sum case where the uninformed player knows his own payoffs, even if perfect monitoring and state independent learning are not assumed, an epsilon-Nash equilibria must exist. (No matter what happens, the uninformed player plays on every stage according to an optimal strategy $s \in \Delta(J)$ in the "one-shot" zero-sum game defined by his known payoff matrix. The informed player choices any pure strategy that maximizes her payoff for the true state of nature in response to $s \in \Delta(J)$. Statistically significant deviation by the uninformed player can be detected and punished; deviation by the informed player would be senseless. For the existence of an Nash equilibrium, perfect monitoring suffices, since the players can agree to play in a deterministic way that mimics this solution.) Adding the perfect monitoring and state independent learning conditions, J. Shalev (1994) did much more than show that an Nash equilibrium must exist; using the characterization of all Nash equilibria developed by S. Hart (1985), he showed that with all Nash equilibria the informed player completely reveals, through her observable behavior, her knowledge of her payoff to the uninformed player.

One can consider what happens in the non-zero-sum case when the uninformed player may not have perfect monitoring or may not have state independent learning. Whether or not epsilon-Nash equilibria exist always is still an open question, and a difficult one. This author thinks that with state independent learning epsilon-Nash equilibria should always exist, but he is very uncertain about the general case!

One can consider the condition that the uninformed player knows the payoff of the informed player (but not his own,) and perfect monitoring and state independent learning are not assumed. In the opinion of this author, this case is more interesting in the more general context of independent information structures, something discussed in the 7th section.

Other variations of the one-sided information condition can be studied. One can assume that Player One knows the state of nature ($\mathcal{P}^0$ is discrete) but does not have perfect monitoring. Such an information structure could no longer be called strictly "one-sided;" yet this author suspects that with state independent learning for Player Two there would be epsilon-Nash equilibria.

6 Existence results: general

Let us consider the cases, both zero-sum and non-zero-sum, where both players have perfect monitoring and we don't necessarily assume state independent learning.

Table 2

Perfect Monitoring for Both Players

	General	Zero-sum	Zero-sum, Player One Knows the Payoffs
Symmetric	Yes-ε !	Yes-ε !	Reduces to Complete Information
State Independent Learning : General	No	No	No !
State Independent Learning and Independent Information	No	No !	Reduces to One-sided Information

We assume that the game is zero-sum, both players have perfect monitoring and state independent learning, and the initial information is independent. The 1966-68 work of R. Aumann, M. Maschler, and R. Stearns shows that there may not be a value to the undiscounted repeated game (Aumann and Maschler, 1995). We assume without loss of generality that the join $\mathcal{P}^0 \wedge \mathcal{Q}^0$ is the discrete partition of K and consider the set of all independent distributions on K represented by the set $\Delta(\mathcal{P}^0) \times \Delta(\mathcal{Q}^0)$. For any $(r,s) \in \Delta(\mathcal{P}^0) \times \Delta(\mathcal{Q}^0)$ define $p(r,s) \in \Delta(K)$ to be the corresponding independent probability distribution on K. Aumann, Maschler and Stearns proved that

$$\sup_{\sigma\in\mathcal{I}} \inf_{\tau\in\mathcal{J}} \lim_{m\to\infty} \inf \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau} = \sup_{\sigma\in\mathcal{I}} \inf_{\tau\in\mathcal{J}} \lim_{m\to\infty} \sup \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau} =$$

$$\mathrm{cav}_{\Delta(\mathcal{P}^0)}\Big(\mathrm{vex}_{\Delta(\mathcal{Q}^0)}\big(\mathrm{val}(A(p(\cdot,\cdot)))\big)\Big)(r,s)$$

and

$$\inf_{\tau\in\mathcal{J}} \sup_{\sigma\in\mathcal{I}} \lim_{m\to\infty} \sup \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau} = \inf_{\tau\in\mathcal{J}} \sup_{\sigma\in\mathcal{I}} \lim_{m\to\infty} \inf \int_{\mathcal{H}_m} f_m(h) d\mu^m_{\sigma,\tau} =$$

$$\mathrm{vex}_{\Delta(\mathcal{Q}^0)}\Big(\mathrm{cav}_{\Delta(\mathcal{P}^0)}\big(\mathrm{val}(A(p(\cdot,\cdot)))\big)\Big)(r,s).$$

The infinitely repeated game has a value if and only if the two above quantities are equal. An application of the Stone-Weierstrass theorem implies that there exists a plethora of examples of non-inequality for some $(r,s) \in \Delta(\mathcal{P}^0) \times \Delta(\mathcal{Q}^0)$ (Mertens, Sorin, and Zamir, 1994; pages 357-8); explicit examples of such have also been found (Aumann and Maschler, 1995; Mertens and Zamir, 1972). (The four expressions on the top of the two equations are not to be confused with what results when one switches the "sup inf" or "inf sup" with the limits inferior and superior; in this case there would be equality always of these four expressions and the resulting quantity would be the limit of the values of the finite stage repeated games (See Mertens and Zamir, 1972).)

If one drops the initial information independence condition but adds the condition that one of the players know the payoffs, even the zero-sum context is not sufficient to guarantee the existence of epsilon-Nash equilibria. Such a game is called a game of incomplete information on one and a half sides.

An explicit zero-sum example of Sorin and Zamir (1985) without a repeated game value answers the existence question in the negative for many cases, and allows us to restrict ourselves to only a few open questions. (Also see Mertens and Zamir, 1972; Mertens and Zamir, 1977.)

If one assumes that the information structure is symmetric, there will always be epsilon-Nash equilibria. The proof, however, comes historically in two parts, following the proofs of the necessary existence of epsilon-Nash equilibria in stochastic games with absorbing states, first done in the zero-sum context (Kohlberg, 1974; Kolhberg and Zamir, 1974) and later in the non-zero-sum context (Thuijsman and Vriez, 1989; Neyman and Sorin, 1995a.) (An absorbing state is a states of nature that the players cannot leave once this states is reached. A stochastic game with absorbing states is a stochastic game in which all states of nature but one are absorbing.) One can perceive the gaining of knowledge by both players concerning the state of nature as a kind of transition to a new state in a corresponding stochastic game. A special quality of this transition in this context is that the attaining of more information is not reversible, which introduces an directed tree property to these transitions. This allows, through finite induction, an application of the existence of epsilon-Nash equilibria for stochastic games with absorbing states. These results have also been generalized to learning structures more complex than that presented by the model in this paper (Neyman and Sorin, 1996b; Forges, 1982.)

Two logical variations of the symmetric condition would be that of one-sided information with perfect monitoring but without the state independent learning condition for the uninformed player and that of information structures that maintain the independence of information, the latter a topic discussed below. Other variations of the symmetric condition are perhaps of interest.

7 Non-zero-sum: many open questions

Let us consider now the cases which are not necessarily zero-sum where both players have perfect monitoring and state independent learning. We order the cases in two dimensions, whether or not the initial information is independent, and by the knowledge of the two players concerning the payoffs. Although the latter dimension contains ten cases, we do not need to consider some of these cases, due to the two previously mentioned non-existence results on zero-sum repeated games. One is left with eleven interesting cases (out of an original 20), six of which are generate open questions on the existence of epsilon-Nash equilibria and two of which belong to the one-sided information topic, discussed above.

Table 3

Perfect Monitoring and State Independent Learning for Both Players

	General	Independent Information
Player One Knows Two's Payoff	No	?
Both Know One's Payoff	?	?
Both Know Own Payoffs	No	No !
Both Know Others' Payoffs	?	Yes
Both Know Others' Payoffs, Player One Knows Own Payoff	?	Reduces to One-Sided Information
Both Know Own Payoffs, Player One Knows Two's Payoff	?	Reduces to One-Sided Information

Assuming the independent initial information condition, (along with perfect monitoring and state independent learning,) interesting is the difference between both players knowing their own payoffs and both players knowing the payoffs of the other player.

If both players know the payoffs of the other player, then one has an easy proof of the existence of a Nash equilibrium. Both players play indefinately according to a Nash equilibrium of the "one-shot" bi-matrix game determined by the matrices $A(p)$ and $B(p)$, and continue to play in this manner no matter what the opponent does. Since both of the above behavior strategies are independent of the knowledge of the players, neither of the players can learn anything from the actions of the other player about her payoff. Any deviation from the above non-revealing behavior strategies could not bring any higher payoff expectation than that from the "one-shot" bi-matrix Nash equilibrium. (A comparison of the Nash equilibria of this repeated game with the "Folk Theorem" Nash equilibria of the corresponding infinitely repeated bi-matrix game determined by $A(p)$ and $B(p)$ may prove to be very interesting.) The above argument for the existence of a Nash equilibrium would remain valid if the players still had state independent learning but didn't have perfect monitoring.

Given intitial information partitions $\mathcal{P}^0$ and $\mathcal{Q}^0$, define a rectangle to be a subset $A \cap B$ of K for an A that is the union of some members of $\mathcal{P}^0$ and a B that is the union of some members of $\mathcal{Q}^0$. We keep the perfect monitoring and initial independence of information conditions, drop the state independent learning condition, and add the condition that for every pair $i \in I, j \in J$ there is a partition $\mathcal{R}^{i,j}$ of K all of whose members are rectangles such that $\mathcal{P}^i = \{\{j\} \times R \mid j \in J \ , \ R \in \mathcal{R}^{i,j}\}$ and $\mathcal{Q}^j = \{\{i\} \times R \mid i \in I \ , \ R \in \mathcal{R}^{i,j}\}$ In this case, learning always retains a structure of information independence. Whether there must exist epsilon-Nash equilibria is an open question. Can one establish a similar relationship between these repeated games and stochastic games to prove the necessary existence of epsilon-Nash equilibria as one could with symmetric repeated games? The problem is that, although a player has no private information on her payoff matrices, knowledge of the other's payoff matrix could give a player private information on the subgame equilibria that result from a transition.

A special case of the above paragraph is generated by the additional assumption that for every i and j there is a pair of partitions $\mathcal{A}^{i,j}$ and $\mathcal{B}^{i,j}$ of K coarser than $\mathcal{P}^0$ and $\mathcal{Q}^0$, respectively, such that $\mathcal{R}^{i,j} = \mathcal{A}^{i,j} \wedge \mathcal{B}^{i,j}$. (This includes the case of one-sided information in which the uninformed player has perfect monitoring and knows the payoff matrix of the informed player but does not have state independent learning.) In this case, there does exist a Nash equilibrium. Consider the join partitions $\mathcal{A} := \wedge_{i\in I, j\in J}\mathcal{A}^{i,j}$ and $\mathcal{B} := \wedge_{i\in I, j\in J}\mathcal{B}^{i,j}$. For every subset $S \subseteq K$ let p^S be the distribution on S determined conditionally by the initial probability distribution p. (Without loss of generality we can assume that p assigns positive probability to every member of K.) The non-deviational behavior of a Nash equilibrium is constructed in the following way: Player One and Player Two play all pairs in $I \times J$ in the first $|I||J|$ moves, so that subsequently Player One knows in which member of $\mathcal{B}$ lies the true state of nature, and the same is true for Player Two and $\mathcal{A}$. After all pairs of $I \times J$ have been played, the repeated game is reduced to a subgame of state independent learning for which the set of possible states of nature is some member of $\mathcal{A} \wedge \mathcal{B}$. Given that the state of nature is in $S \in \mathcal{A} \wedge \mathcal{B}$, both players play according to a Nash equilibrium patterned after a Nash equilibrium of the "one shot" bi-matrix game determined by the matrices $A(p^S)$ and $B(p^S)$, as described above for the corresponding repeated game of state independent learning. If one of the players, say Player One, refuses to cooperate in the process of playing all pairs in $I \times J$, Player Two will minimize Player One's expected payoff according to the zero-sum matrix game determined by the matrix $A(p^T)$ where T is the member of $\mathcal{B}$ that contains the true state of nature. Due to information independence, p^T induces the same probability distribution on Player One's payoff matrices as that of p^S and $p^{S'}$ for all $S \in \mathcal{A} \wedge \mathcal{B}$ and $S' \in \mathcal{P}^0 \wedge \mathcal{B}$ such that $S \subseteq T$ and $S' \subseteq T$. Since any Nash equilibrium of the bi-matrix game determined by $A(p^S)$ and $B(p^S)$ for $S \in \mathcal{A} \wedge \mathcal{B}$ delivers for Player One at least the value of $A(p^S)$, Player One has no motivation not to cooperate in playing all pairs in $I \times J$. As with the corresponding state independent learning case, studying the set of equilibrium payoffs of such a repeated game may be very interesting, as well as considering subgame perfect properties.

We assume again, for the rest of this section, that both players have state independent learning and perfect monitoring.

Let us assume that the players know the payoffs of their opponents and remove the independence of initial information condition. This generates what this author considers to be a good open question concerning the existence of Nash equilibria. One can also strengthen the assumptions by adding the condition that one of the players knows her own payoff. (One should settle these questions first before investigating what happens if the perfect monitoring or state independent learning conditions are dropped.)

Furthermore, one can consider three other similar cases for which the Nash equilibrium existence question has not been answered: the case where one of the players knows the payoff of the other player and independence is assumed, and the two cases where independence is either assumed or not assumed and both players know the payoff of one of the players.

On the other hand, if both players know their own payoffs and independence of initial information is assumed, there may not exist an epsilon-Nash equilibrium. This was proved by G. Koren (1988), who also proved that with all Nash equilibria that do exist both players completely reveal to the other player through their observable behavior what they know about their payoffs.

One can strengthen the "both players know their own payoff" condition and weaken the "both players know the others' payoffs" condition by assuming that both players know their own payoff and one of the players knows the payoff of the other (and of course dropping the independence of initial information assumption.) As far as this author is aware, for this case the epsilon-Nash equilibria existence question is open.

8 Conclusion

This author concludes with a comment on the case he knows best, that of non-zero-sum, one-sided information, with perfect monitoring and state independent learning for the uninformed player.

In the event that there is no non-revealing Nash equilibrium, the existence result of Simon, Spież, and Toruńczyk (1995) shows only the necessary existence of a very strange kind of Nash equilibrium, one in which the informed player always receives in general the same payoff as she would receive when she deviates and is punished by the uninformed player. Needless to say, it is difficult to understand why the informed player would want to play in this way, except of course in situations that are in some way similar to that of a zero-sum game. The Nash equilibria for which necessary existence is proved involve very little of the beauty of S. Hart's complete characterization of the set of all Nash equilibria using bi-martingales and signaling moving in both directions (Hart, 1985; Aumann and Hart, 1986). In games where the existence of an Nash equilibrium is obvious, some fine work has been done on determining the set of Nash equilibria payoffs and how they may be obtained; (see Forges, 1990). So far there have been few bridges between these two directions of research. For example, one could try to prove that for some weak conditions implying common interests between the players there must exist a Nash equilibrium that both players would be motivated to play by a positive difference in expected payoffs between the non-deviating behavior and all detectably deviating behavior. It would be interesting to see if one could obtain a stronger result in this direction using the set of all equilibrium payoffs instead of just those from joint plan equilibria. This direction of research, in the opinion of this author, should be both difficult and rewarding.

Acknowledgements: Writing this article required a lot of assistance; this author thanks many people, and especially Robert Aumann, Abraham Neyman, and Sylvain Sorin for answering many questions concerning what has been done in the field of repeated games. I thank also the Deutsche Forschungsgemeinschaft, not only for supporting this work but also for supporting the last few months of corrections of this author's joint publication with Spiez and Torunczyk.

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