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An Improvement on the Existence Proof of Joint Plan Equilibria

by

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Abstract

This article concerns infinitely repeated and undiscounted two-person non-zero-sum games of incomplete information on one side. Following the spirit of the Folk Theorem it establishes a sufficient condition for the existence of Nash equilibria with payoffs superior to what the players would receive from observable deviation. Examples are presented that show both the difficulty and the desirability for stronger results than those presented here.

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1 Introduction

In SIMON, SPIEZ and TORUNCZYK (1995) the existence of a "independent and two safe" joint plan equilibrium was proven; (see SORIN, 1983.) In the non-trivial case of the proof, where there exists no *non-revealing* independent and two-safe joint plan equilibrium, the equilibria whose existence was proven give to the informed player for at least two states of nature the same payoffs that she would receive if she deviated from the equilibrium play in an observable way and were punished by the un-informed player – in general this is likely to be true for all states of nature. In the opinion of this author, one cannot speak reasonably about an equilibrium being the result of a negotiating process when for some player there is no payoff distinction between the equilibrium and the threat, carried out by the other player, provoked by her observable deviation from the equilibrium. Therefore the first efforts toward developing a theory of equilibrium selection or negotiation in such games must be a better understanding of sufficient conditions for the existence of an equilibrium with positive motivations for the players not to deviate in an observable way.

To illustrate the above problem, consider the following Example 1. The states I and II are chosen with even probability. The informed player has two moves, "x" and "y", and the un-informed player has three moves, "a", "b", and "c." As with all examples, the row player is the informed player, the first entry is the payoff for the informed player and the second entry is the payoff for the un-informed player.

See the diagram labelled Example 1 on the next page:

Example 1

State I

	a	b	c
x	-1 , 1	1 , -9	2 , 0
y	-1 , 1	1 , -9	-2 , 0

State II

	a	b	c
x	1 , -9	-1 , 1	-2 , 0
y	1 , -9	-1 , 1	2 , 0

Let us assume the informed player does not reveal anything about the state of nature and an agreement is made that the un-informed player will play "c" always and regardless of her knowledge of the state of nature the informed player will play some sequence of "x" and "y" with a distribution in the limit of α for "x" and $1-\alpha$ for "y" with $1/4 \leq \alpha \leq 3/4$. This gives a payoff of 0 to the un-informed player and a payoff to the informed player of $4\alpha - 2$ and $2 - 4\alpha$ for the states I and II, respectively. If Player One cheats on this agreement and plays either "x" and "y" with greater frequency, Player Two will punish her by playing (forever or for some increasingly long sequences) the appropriate combination of "a" and "b" to duplicate the payoffs of $4\alpha - 2$ and $2 - 4\alpha$ for the states I and II, respectively. (Any greater frequency for "a" or "b" would give the informed player motivation for one or the other state of nature to provoke punishment.) This describes an equilibrium, but why would the informed player wish to play this way? She doesn't suffer by being punished at all, and by deviating observably and being punished she inflicts great damage upon the other player!

2 Background

Let $K = \{1, \ldots, n\}$ be the set of states of nature. Let I and J be the pure actions (or moves) of Players One and Two, respectively, with $|I| = m_1$ and $|J| = m_2$. Let A^k and B^k for all $k \in K$ be the corresponding $m_1 \times m_2$ payoff matrices for Players One and Two, with $A^k(i,j)$ and $B^k(i,j)$ being the payoffs to Player One and Player Two, respectively, when Player One chooses $i \in I$, Player Two chooses $j \in J$ and the state of nature is k. For every $p \in \Delta(K)$ we define the matrix $A(p)$ by $A(p) := \sum_{k \in K} p_k A^k$; and likewise we define $B(p)$. The function $a^* : \Delta(K) \to \mathbf{R}$ is defined so that $a^*(p)$ is the value of the matrix $A(p)$ with the first player as maximizer in the corresponding zero-sum game, and likewise the function b^* is defined so that $b^*(p)$ is the value of the matrix $B(p)^t$, (with the second player as the maximizer; t stands for the transpose.) If C is a convex set and f is a bounded real valued function on C then cav(f) is defined to be the minimal concave function greater than or equal to f and vex(f) is defined to be the maximal convex function less than or equal to f.

We define G to be the subset of $\Delta(K) \times \mathbf{R}^K \times \mathbf{R}$ such that $(p, x, y) \in G$ satisfies the following properties:

(1) $x \cdot q \geq a^*(q)$ for all $q \in \Delta(K)$,
there exists some $\gamma \in \Delta(I \times J)$ such that
(2) $y = \sum_{(i,j) \in I \times J} \gamma_{(i,j)} B(p)(i,j) \geq \text{vex}(b^*(\cdot))(p)$,
(3) $\sum_{(i,j) \in I \times J} \gamma_{(i,j)} A^k(i,j) = x_k$ if $p_k > 0$,
(4) $\sum_{(i,j) \in I \times J} \gamma_{(i,j)} A^k(i,j) \leq x_k$ if $p_k = 0$.
Recall that an equilibrium payoff for the players at the initial probability distribution $p \in \Delta(K)$ is an $(x, y) \in \mathbf{R}^K \times \mathbf{R}$ such that there is a martingale in $\Delta(K) \times \mathbf{R}^K \times \mathbf{R}$ starting at (p, x, y) and converging a.e. to elements in G with the property that at each stage of the martingale either the $\Delta(K)$ coordinate is held constant or the $\mathbf{R}^K$ is held constant, (otherwise known as a bi-martingale.) Conversely, such a martingale generates an equilibrium payoff of the repeated game (HART, 1985.) Such a martingale we call an "equilibrium bi-martingale." A joint plan equilibrium is an equilibrium generated by an equilibrium bi-martingale with only one stage for which the x-coordinate is held constant; (see AUMANN and MASCHLER, 1995 and SORIN, 1983.)

Call an element (p, x, y) of $\Delta(K) \times \mathbf{R}^K \times \mathbf{R}$ "properly separated" from a^* if there is a vector $x' \in \mathbf{R}^K$ such that $q \cdot x' \geq a^*(q)$ for every $q \in \Delta(K)$, $x'_k \leq x_k$ for every $k \in K$, and $x'_k < x_k$ for some $k \in K$ with $p_k > 0$. Call an element (p, x, y) of $\Delta(K) \times \mathbf{R}^K \times \mathbf{R}$ "completely separated" from a^* if there is a vector x' such that $q \cdot x' \geq a^*(q)$ for every $q \in \Delta(K)$, $x'_k \leq x_k$ for every $k \in K$ and $x'_k < x_k$ for every $k \in K$ with $p_k > 0$.

An equilibrium bi-martingale is called "good" for the informed player if with positive probability the bi-martingale converges to elements of G properly separated from a^*. An equilibrium bi-martingale is called "very good" for the informed player if with probability one the bi-martingale converges to elements of G completely separated from a^*.

An equilibrium bi-martingale is called "initially good" for the informed player if the (p, x, y) at the start is properly separated from a^*. An equilibrium bi-martingale is called "initially very good" for the informed player if the (p, x, y) at the start is completely separated from a^*.

Initially good and initially very good for the informed player pertain to the initial expected payoff for the informed player, while good and very good for the informed player pertain to a lack of motivation to deviate in an observable way. Good and very good for the informed player imply initially good and initially very good for the informed player, respectively, but in general the converses do not hold. In the case of joint plan equilibria, however, (very) good and initially (very) good for the informed player are equivalent, (respectively.)

An equilibrium bi-martingale is called "good" for the un-informed player if with positive probability the bi-martingale converges to elements (p, x, y) of G such that $y >$ vex $(b^*(\cdot))(p)$. An equilibrium bi-martingale is called "very good" for the un-informed player if with probability one the bi-martingale converges to elements (p, x, y) of G such that $y >$ vex $(b^*(\cdot))(p)$. Notice for a fixed initial probability that an equilibrium bi-martingale that is not good for a player could give a higher average expected payoff for this player than some other equilibrium bi-martingale that is good or very good for this player.

3 Some Examples

We recall from the "Folk Theorem" concerning infinitely repeated un-discounted games of *complete* information defined by a single pair of real matrices A and B of the same dimensions that any payoff from a correlated strategy in $\Delta(I \times J)$ giving at least the zero-sum values of A and B^t to Players One and Two, respectively, is an equilibrium payoff of the infinitely repeated game. We attempt to follow the spirit of the Folk Theorem. We look for weak conditions on a game of incomplete information on one side that make it both different from a zero-sum game and insure the existence of equilibria that are good or very good for either or both players.

First, for the existence of an initially good payoff for the informed player, it does not suffice that for every $k \in K$ the pair of payoff matrices A^k and B^k have a correlated strategy giving to Players One and Two strictly more than the values of A^k and $(B^k)^t$, respectively. Consider the following Example 2.

See the diagram labelled Example 2 on the next page:

Example 2

State I

	a	b	c
x	1 , -9	1+ε , -9	0 , 0
y	1 , 1	1+ε , -9	0 , 0

State II

	a	b	c
x	1+ε , -9	1 , -9	0 , 0
y	1+ε , -9	1 , 1	0 , 0

The functions a^* and b^* are uniformly 0. If $\epsilon = 0$ then there is an equilibrium giving 1 to Player One and 1 to Player Two for all initial probabilities and both states of nature. But if $\epsilon > 0$ and the initial probability gives strictly more than a 1/10 probability for both states I and II, then there is no equilibrium payoff other than that of 0 for both players.

Why? Let the variable p stand for the probability of the state of nature II. If $1/10 < p < 9/10$ then $(p, x, y) \in G$ implies that $x = (0, 0)$. If $0 \leq p \leq 1/10$ and $(p, x, y) \in G$ with $x \neq (0, 0)$ then the $\gamma \in \Delta(I \times J)$ generating the payoff $x = (x_I, x_{II})$ for Player One must give a frequency for "a" at least nine times that of "b." This implies that $(1 + .1\epsilon)x_{II} \geq (1 + .9\epsilon)x_I$.

Now suppose for the sake of contradiction that an equilibrium bi-martingale starting at $p < 9/10$ also starts with $(x_I, x_{II}) \in \mathbf{R}^2$ satisfying $(1 + .1\epsilon)x_{II} < (1 + .9\epsilon)x_I$. Consider the set $C := [0, (10p + 9)/20]$ and the closed half plane in $\mathbf{R}^2$ defined by $D := \{(r, s) \in \mathbf{R}^2 \mid (1 + .9\epsilon)r - (1 + .1\epsilon)s \geq 1/2\ ((1 + .9\epsilon)x_I - (1 + .1\epsilon)x_{II})\}$. Since p and (x_I, x_{II}) are in the interior of C and D, respectively, any equilibrium bi-martingale starting with p and (x_I, x_{II}) must converge with positive probability to elements of G with the $\Delta(K) \times \mathbf{R}^2$ coordinate in $C \times D$ (Theorem 4.7, AUMANN and HART, 1986) – however there are no such elements of G! Therefore any equilibrium bi-martingale starting at $p < 9/10$ must also start with the $\mathbf{R}^2$ coordinate (x_I, x_{II}) satisfying $(1 + .1\epsilon)x_{II} \geq (1 + .9\epsilon)x_I$. The symmetrical result for $p > 1/10$ allows only for $x_I = x_{II} = 0$.

One shortcoming with the above condition is that it does not consider all bi-matrix games determined by the pairs $A(p)$ and $B(p)$ for all $p \in \Delta(K)$. However, this author suspects that there exists an example such that for every $p \in \Delta(K)$ the pair of payoff matrices $A(p)$ and $B(p)$ have correlated strategies giving to the players strictly more than val $(A(p))$ and val $(B(p)^t)$, respectively, yet there is no initial probability $p \in$ interior $(\Delta(K))$ with an equilibrium bi-martingale initially good for the informed player. To illustrate this opinion, consider the following Example 3. There are two states of nature, I and II, Player One has four moves, "x", "x^*", "y", and "y^*", and Player Two has seven moves, "a" through "g". Again, we will let the variable p stand for the probability of the state of nature II.

See the diagram labelled Example 3 on the next page:

Example 3 State I t=1000

	a	b	c	d	e	f	g
x	0 , 1	4 , -9	2 , 1/2	2 , -9/2	2 , -t	1 , -t	1 , -t
x*	-t , 1	-t , -9	-t , 1/2	-t , -9/2	$2+\varepsilon$, -1/2	$.8+\varepsilon$, $1+\varepsilon$	$.8+\varepsilon$, $-6-3\varepsilon$
y	0 , 1	4 , -9	-2 , 1/2	-2 , -9/2	-2 , -t	1 , -t	1 , -t
y*	-t , 1	-t , -9	-t , 1/2	-t , -9/2	$-2+\varepsilon$, -1/2	$.8+\varepsilon$, $1+\varepsilon$	$.8+\varepsilon$, $-6-3\varepsilon$

State II

	a	b	c	d	e	f	g
x	4 , -9	0 , 1	-2 , -9/2	-2 , 1/2	-2 , -t	1 , -t	1 , -t
x*	-t , -9	-t , 1	-t , -9/2	-t , 1/2	$-2+\varepsilon$, -1/2	$.8+\varepsilon$, $-6-3\varepsilon$	$.8+\varepsilon$, $1+\varepsilon$
y	4 , -9	0 , 1	2 , -9/2	2 , 1/2	2 , -t	1 , -t	1 , -t
y*	-t , -9	-t , 1	-t , -9/2	-t , 1/2	$2+\varepsilon$, -1/2	$.8+\varepsilon$, $-6-3\varepsilon$	$.8+\varepsilon$, $1+\varepsilon$

The function a^* is piece-wise linear, whose graph is defined by the line segments connecting the points $(0,0)$, $(1/4,1)$, $(1/2,0)$, $(3/4,1)$, and $(1,0)$, and determined by the moves "x" and "y", and "a", "b", "c" and "d". The moves "x^*" and "y^*" paired with the moves "e", "f", and "g" are the optional cooperative moves for the two players, with appropriate payoffs to discourage the alternative mismatches. The values of the convex function b^* are obtained for Player Two by the move "a" in the closed interval $[0,1/10]$, by "b" in the closed interval $[9/10,1]$, by "c" in the closed interval $[1/10,1/2]$, and by "d" in the closed interval $[1/2,9/10]$. In combination with the moves "x^*" and "y^*" the value of b^* or more is delivered by the move "e" in the closed interval $[1/5,4/5]$, by the move "f" in the closed interval $[0,1/4]$, and by the move "g" in the closed interval $[3/4,1]$. Because of the overlap between the above intervals associated with "e", "f", and "g" and the payoff of $.8+\epsilon > a^*(1/5) = a^*(4/5) = .8$ for the informed player in the event of a cooperative combination with the moves "f" or "g", for every p there is a pair of moves that delivers more than $a^*(p)$ and $b^*(p)$ for Player One and Two, respectively.

With regard to Example 3, this author suspects that if ϵ is equal to .01 then the only members of G come from the moves "x" and "y" combined with "a", "b", "c", and "d", namely that there is no member of G for p strictly between 1/10 and 9/10, for $p = 1/10$ all members of G have for the $\mathbf{R}^2$ coordinate a convex combination of $(0,4)$ and $(1,1)$, (and symmetrically so for $p = 9/10$,) and for p strictly less than 1/10 all members of G have $(0,4)$ for the $\mathbf{R}^2$ coordinate, (and symmetrically so for p strictly more than 9/10.) If this suspicion is correct, that would imply that there is no equilibrium bi-martingale good for either player. The reason for this suspicion is the large separation between the intervals where "a" or "b" deliver the function value of "b^*" for Player Two and the interval where cooperative combination with "e" delivers this function value.

4 A Folk Theorem

For every positive integer $d \geq 1$, a convex and bounded set $D \subseteq \mathbf{R}^d$ and an $s \in S^{d-1}$ define the interval $I(D,s)$ in $\mathbf{R}$ by

$$I(D,s) := \{r \mid v \cdot s = r \quad \text{for some } v \in D\}$$

and define a real value $r(D,s) := \sup(I(D,s))$. For every such D and a subset $S \subseteq S^{d-1}$ define the set $\overline{C}(D,S)$ by

$$\overline{C}(D,S) := \bigcap_{s\in S}\{v \mid v\cdot s \le r(D,s)\}.$$

Lemma 1: For every positive integer $d \ge 1$ and $\epsilon > 0$ there exists a finite subset S of S^{d-1} with the property that for every non-empty convex and compact set D in $\mathbf{R}^d$ every point in $\overline{C}(D,S)$ has a distance from the set D no greater than ϵ times the diameter of D.

Proof: Without loss of generality, we can assume that $\epsilon < 1$. Since S^{d-1} is a compact set, we can choose S so that for every point t in S^{d-1} there is a point s in S such that the angle between t and s is less than arctan (ϵ).

Let w be a point in $\overline{C}(D,S)\backslash D$, and let w' be the nearest point in D to w. Consider any member s of S close enough to $w-w'/||w-w'|| \in S^{d-1}$ so that the angle between s and $w-w'/||w-w'||$ is less than arctan (ϵ). Since tan $(\pi/4) = 1 > \epsilon$, the hyperplane $\{v \mid v\cdot s = r(D,s)\}$ intersects the ray starting at w' and passing through w; without loss of generality we can assume that w is the intersection of the hyperplane $\{v \mid v\cdot s = r(D,s)\}$ with this ray. Let u be any point in the intersection of $\{v \mid v\cdot s = r(D,s)\}$ and D, let H be the hyperplane containing u and perpendicular to the line passing through w and w', and let u' be the intersection of H with this line. Because w' is the nearest point in D to w, no point of D is strictly on the w-side of the hyperplane parallel to H passing through w' (perpendicular to the line passing through w and w';) therefore u' is either equal to w' or w' is strictly between u' and w, so we have $||w-w'|| \le ||w-u'||$. The angle at u formed by the rays toward w and u' can be no more than the angle between s and $w-w'/||w-w'||$, and therefore $||w-u'||$ can be no more than $||u-u'||$ times the tangent of the angle between s and $w-w'/||w-w'||$. Lastly, we notice that $||u-u'||$ is no greater than $||u-w'||$, a distance between two points of D. q.e.d.

Lemma 2: Let X and Y be compact and convex subsets of Euclidean spaces, let Y also be a polytope, and let $F: X \rightarrow 2^Y$ be an upper-hemi-continuous convex valued non-empty correspondence. For every $\epsilon > 0$ there is a continuous correspondence (upper and lower hemi-continuous) $\overline{F}: X \rightarrow 2^Y$ such that for every $x \in X$ $\overline{F}(x)$ is a polytope, $F(x) \subseteq \overline{F}(x)$, and if $y^* \in \overline{F}(x^*)$

then the Euclidean distance between (x^*, y^*) and $\{(x, y) \mid y \in F(x)\}$, the graph of F, is less than ϵ.

Proof: First, we prove that when Y is a subset of $\mathbf{R}$ then there exists a continuous function $f : X \to Y \subseteq \mathbf{R}$ such that $f(x) \geq y$ for all $x \in X$ and $y \in F(x)$ and every point in the graph of f is less than a distance of ϵ from the graph of F. Consider any simplicial subdivision of the Euclidean space containing X such that the diameter of every simplex is less than ϵ. For any simplex σ in this subdivision define the function $g_\sigma : \sigma \cap X \to \mathbf{R}$ by $g_\sigma(x) = \sup_{y \in F(x)} y$. cav $(g_\sigma) : \sigma \cap X \to \mathbf{R}$ is continuous and bounded because Y is bounded and $g_\sigma(x)$ is an upper-semi-continuous function. Since cav (g_σ) and cav $(g_{\sigma'})$ agree on any $x \in X$ shared by two different simplexes σ and σ', the functions cav (g_σ) define a continuous function on all of X. It remains to show that if cav $(g_\sigma)(x) = y$ then there is some $x' \in \sigma \cap X$ with $y \in F(x')$. cav $(g_\sigma)(x) = y$ means that there is at least one $x_1 \in \sigma \cap X$ and $y_1 \in F(x_1)$ with $y_1 \leq y$ and at least one $x_2 \in \sigma \cap X$ and $y_2 \in F(x_2)$ with $y_2 \geq y$. The existence of such an x' follows from the fact that the image of g_σ must be connected, (from the Vietoris mapping theorem, for example.)

Let $\mathbf{R}^d$ be the Euclidean space containing Y. Without loss of generality we assume that $\epsilon < 1$ and that the diameter of Y is equal to 1. By Lemma 1, we choose $S \subseteq S^{d-1}$ so that for every non-empty compact and convex subset $D \subseteq \mathbf{R}^d$ every point of $\overline{C}(D, S)$ is no more than a distance of $\epsilon/8$ times the diameter of D from the set D.

Since F is upper-hemi-continuous, for every $x \in X$ there exists a $0 < \delta_x < \epsilon/4$ such that $x' \in B(x, 2\delta_x)$, (the open ball of radius $2\delta_x$ about x,) implies that every point of $F(x')$ is less than a distance of $\epsilon/4$ from the set $F(x)$. Cover X with $\cup_{x \in X} B(x, \delta_x)$; since X is compact we can choose a finite subcover, $\cup_{t \in T} B(t, \delta_t)$, for some finite subset $T \subseteq X$. Let $\delta = \min_{t \in T} \delta_t$.

For every $s \in S$ define the correspondence $\phi_s : X \to 2^{\mathbf{R}}$ by $\phi_s(x) := I(F(x), s)$. Since F is upper-hemi-continuous and convex valued, so is ϕ_s. By the first paragraph, for every $s \in S$ there exists a continuous function $\overline{\phi}_s : X \to \mathbf{R}$ such that for every $x \in X$ $\overline{\phi}_s(x) \geq \sup(\phi_s(x)) = r(F(x), s)$ and $(x, \overline{\phi}_s(x))$ is less than a distance of δ from the graph of ϕ_s. Define $\overline{F} : X \to 2^Y$ by

$$\overline{F}(x) := \cap_{s \in S} \{v \in Y \mid v \cdot s \leq \overline{\phi}_s(x)\}.$$

We have $\overline{F}(x) \supseteq \cap_{s \in S} \{v \in Y \mid v \cdot s \leq r(F(x), s)\} \supseteq F(x)$ for every $x \in X$.

For the rest of the proof, consider any $x \in X$. For every $s \in S$ there exists an $x_s \in X$ and $y_s \in F(x_s)$ such that $|y_s \cdot s - \overline{\phi}_s(x)| < \delta$ and $||x_s - x|| < \delta$. Let $t \in T$ satisfy $||t-x|| < \delta$, which means for every $s \in S$ that $||x_s - t|| < 2\delta$ and y_s is less than a distance of $\epsilon/4$ from the set $F(t)$. Let Q be $\{v \in \mathbf{R}^d \mid$ distance $(v, F(t)) \leq 1/2\ \epsilon\}$, a convex and compact set. Since $(y_s + \epsilon/4\ s) \cdot s > \overline{\phi}_s(x)$, we have that $r(Q,s) \geq \overline{\phi}_s(x)$ for every s and therefore $\overline{C}(Q,S)$ contains $\overline{F}(x)$. By the choice of S, since the diameter of Q is less than 2, every point in $\overline{C}(Q,S)$ is less than a distance of $\epsilon/4$ from Q. Therefore if $y \in \overline{F}(x) \subseteq \overline{C}(Q,S)$ then y is less than a distance of $3/4\ \epsilon$ from $F(t)$. It follows that the distance of (x,y) from the set $\{t\} \times F(t)$ is less than ϵ. q.e.d.

Theorem 1: Let $a : \Delta(K) \to \mathbf{R}$ be a real valued continuous function on $\Delta(K)$. If $F : \Delta(K) \to 2^{\Delta(I \times J)}$ is an upper-hemi-continuous non-empty convex valued correspondence such that for every pair $p \in \Delta(K)$ and $q \in \Delta(K)$

$$\max_{\gamma \in F(p)} \sum_{(i,j)} \gamma_{(i,j)} A(q)(i,j) \geq a(q),$$

then for every $p_0 \in \Delta(K)$ there exists an $x \in \mathbf{R}^K$, a set $T \subseteq \Delta(K)$, and for every $t \in T$ a $\gamma^t \in F(t)$ such that
1) $x \cdot q \geq a(q)$ for all $q \in \Delta(K)$,
2) $p_0 \in$ convex hull (T),
and for every $t \in T$
3) $\sum_{(i,j)} \gamma^t_{(i,j)} A^k(i,j) \leq x_k$ and
4) $\sum_{(i,j)} \gamma^t_{(i,j)} A^k(i,j) = x_k$ if $t_k > 0$.

Proof: We fix $\epsilon > 0$. By Lemma 2 there is a continuous polytope valued correspondence $\overline{F} : \Delta(K) \to 2^{\Delta(I \times J)}$ such that $\overline{F}(p) \supseteq F(p)$ for every $p \in \Delta(K)$ and if $y^* \in \overline{F}(p^*)$ then the Euclidean distance between (p^*, y^*) and $\{(p,y) \mid y \in F(p)\}$ is less than ϵ. Define $[\Delta(K)] := \{v \in \mathbf{R}^K \mid \sum_{k \in K} v_k = 1\}$. Let $r : [\Delta(K)] \to \Delta(K)$ be the canonical retraction $r(v)_k := \max(v_k, 0)/\sum_{v_l \geq 0} v_l$ and let $u : [\Delta(K)] \to \mathbf{R}^K_+ = \{v \mid v_k \geq 0\ \forall k \in K\}$ be defined by $u(v)_k := |min(v_k, 0)|$, also so defined in the Appendix of Section II of SIMON, SPIEZ, and TORUNCZYK (1995). For every $\gamma \in \Delta(I \times J)$ and $u \in \mathbf{R}^K_+$ let $h(\gamma, u) \in \mathbf{R}^K$ be defined by $h(\gamma, u)_k := \sum_{(i,j)} \gamma_{(i,j)} A^k(i,j) + u_k$. For every $v \in [\Delta(K)]$ we define the convex function b_v on $[\Delta(K)]$ by

$$b_v(q) := \max_{\gamma \in \overline{F}(r(v))} h(\gamma, u(v)) \cdot q.$$

As in SIMON, SPIEZ, and TORUNCZYK (1995), we say that $\varphi \in \mathbf{R}^K$

"separates" a b_v from a continuous function $f : \Delta(K) \to \mathbf{R}$ if $b_v(q) \geq \varphi \cdot q$ for every $q \in [\Delta(K)]$, $\varphi \cdot q \geq f(q)$ for every $q \in \Delta(K)$ and there is some $q \in \Delta(K)$ such that $b_v(q) = \varphi \cdot q$; and we say that φ "tightly separates" b_v from f if φ separates b_v from f and there is some $q \in \Delta(K)$ such that $\psi \cdot q = b_v(q)$ for all $\psi \in \mathbf{R}^K$ separating b_v from f. Because the correspondence $\overline{F}$ is continuous, the function $b_v(q)$ is continuous on $(v, q) \in [\Delta] \times \Delta$. Since the correspondence $\overline{F}$ contains the correspondence F we have $b_v(q) \geq a(q)$ for every $v \in [\Delta(K)]$ and $q \in \Delta(K)$. Because there is a real number $W < \infty$ with $|\max_{i,j,k} A^k(i,j)| < W$, condition (1) to Theorem 2 of SIMON, SPIEZ, and TORUNCZYK (1995) is satisfied; therefore by the theorem: for every $p_0 \in \Delta(K)$ either there is an $h \in \mathbf{R}^K$ that separates b_{p_0} from a or there exists a set $V \subset [\Delta(K)]$ and an $h \in \mathbf{R}^K$ such that $p_0 \in$ convex hull $(r(V))$ and for every $v \in V$ h tightly separates b_v from a. In either case, the same argument for (15) in the proof of Proposition 2 in SIMON, SPIEZ, and TORUNCZYK (1995) implies that for every $v \in V$ (or $p_0 = v$ in the former case) $h = h(\gamma^v, u(v))$ for some $\gamma^v \in \overline{F}(r(v))$. (The original argument by Sorin used the Min-Max Theorem; see SORIN, 1983.) After removing the auxiliary values $u(v)$ we have Conditions 1) through 4) with $T = r(V)$ but with $\gamma^t \in \overline{F}(t)$ instead of $\gamma^t \in F(t)$ for every $t \in T$. Since F is upper hemi-continuous and by Catheodory's Theorem we can always choose T to be a finite set of cardinality no more than $|K|$, taking the limit as ϵ goes to 0 of a convergent subsequence of such solutions gives the desired conclusion. q.e.d.

By letting $F(p)$ equal $\{\gamma \in \Delta(I \times J) \mid \sum_{(i,j)} \gamma_{(i,j)} B(p)(i,j) \geq \text{ vex}(b^*(\cdot))(p)\}$ and $a = a^*$ we have a slightly alternative proof of the existence of joint plan equilibria (SIMON, SPIEZ, and TORUNCZYK, 1995.)

Corollary: If for every p and q in $\Delta(K)$ there exists a correlated strategy $\gamma \in \Delta(I \times J)$ such that

$$\sum_{(i,j)} \gamma_{(i,j)} B(p)(i,j) > \text{ vex}(b^*(\cdot))(p)$$

and

$$\sum_{(i,j)} \gamma_{(i,j)} A(q)(i,j) > a^*(q)$$

then for every initial probability there exists a joint plan equilibrium that is very good for both players.

Proof: Let $g : \Delta(K) \times \Delta(K) \to \mathbf{R}$ be defined by $g(p,q) :=$

$$\max_{\gamma \in \Delta(I \times J)} \min \Big(\sum_{(i,j)} \gamma_{(i,j)} A(q)(i,j) - a^*(q) \ , \ \sum_{(i,j)} \gamma_{(i,j)} B(p)(i,j) - \ \mathrm{vex}(b^*(\cdot))(p) \Big).$$

Because $\Delta(K) \times \Delta(K)$ and $\Delta(I \times J)$ are compact and $\min \Big(\sum_{(i,j)} \gamma_{(i,j)} A(q)(i,j) - a^*(q) \ , \ \sum_{(i,j)} \gamma_{(i,j)} B(p)(i,j) - \ \mathrm{vex}(b^*(\cdot))(p) \Big)$ is a continuous function on $(p,q,\gamma) \in \Delta(K) \times \Delta(K) \times \Delta(I \times J)$, the function g is continuous. Because $\Delta(K) \times \Delta(K)$ is compact, the function g attains a minimum value; therefore by the hypothesis there is a $w > 0$ with $g \geq w$. That means the convex valued correspondence $F : \Delta(K) \to 2^{\Delta(I \times J)}$ defined by $F(p) := \{\gamma \in \Delta(I \times J) \mid \sum_{(i,j)} \gamma_{(i,j)} i B(p) j \geq \ \mathrm{vex}(b^*(\cdot))(p) + w\}$ satisfies $\max_{\gamma \in F(p)} \sum_{(i,j)} \gamma_{(i,j)} A(q)(i,j) \geq a^*(q) + w$ for all $q \in \Delta(K)$. Since $vex(b^*)$ is continuous, the correspondence F is upper hemi-continuous. The rest follows from Theorem 1. q.e.d.

5 The Desirability of a Stronger Theorem

There is a major shortcoming of Theorem 1 and the corollary: For some initial probabilities it is possible to have no joint plan equilibrium that is initially good for either player but have an equilibrium bi-martingale that is good and initially very good for the informed player and delivers for both players average payoffs much better than from any joint plan equilibrium.

Consider the following Example 4. The moves of the first player are duplicated so that she can send signals.

See the diagram labelled Example 4 on the next page:

Example 4

State I

a	b	c	d	e	f	g
1 , 10	1 , 1	4 , 1/2	1 , 0	0 , -3/2	1 , -3	3 , -354
1 , 10	1 , 1	4 , 1/2	1 , 0	0 , -3/2	1 , -3	3 , -354

State II

a	b	c	d	e	f	g
3 , -354	1 , -3	0 , -3/2	1 , 0	4 , 1/2	1 , 1	1 , 10
3 , -354	1 , -3	0 , -3/2	1 , 0	4 , 1/2	1 , 1	1 , 10

Again, letting p represent the probability of state II, the graph of the function a^* is created by the line segments connecting the points $(0,0)$, $(1/4,1)$, $(3/4,1)$, and $(1,0)$. Exactly the values of the convex function b^* are obtained for Player Two by the move "a" in the closed interval $[0,.025]$, by the move "b" in the closed interval $[.025,1/4]$, by the move "d" in the closed interval $[1/4,3/4]$, by the move "f" in the closed interval $[3/4,.975]$, by the move "g" in the closed interval $[.975,1]$, and by the moves "c" and "e" at the points $p=1/4$ and $p=3/4$, respectively. Due to the separation between the points where "a" "c", "e", and "g" are optimal for the second player, for all initial probabilities other than $p \in [0,.025]$, $p=1/4$, $p=3/4$, or $p \in [.975,1]$ all joint plan equilibria deliver $(1,1)$ to the first player.

On the other hand, for every value of $p \notin \{0,1\}$ there is an equilibrium bi-martingale that is good and initially very good for the first player and delivers for the second player an average payoff better than from any joint plan equilibrium. Consider the following process: For $p=3/4$, with $2/3$ probability the joint plan equilibrium associated with the strategy "e" is played, for a payoff of $(0,4)$ for the first player. With $1/3$ probability there follows a lottery performed by the first player between revealing the posterior probability $p=1/4$ (with $1/3$ chance) and the posterior probability $p=1$ (with $2/3$ chance.) If $p=1$ is revealed, then the equilibrium associated with the move "g" is played, with a payoff of $(3,1)$ for the first player. Otherwise, if $p=1/4$ is revealed then the process is repeated, but symmetrically. Letting (x,y) be the value of the above process to the first player and (y,x) the value of the symmetric process starting at $t=1/4$, we notice that $x=1$ and $y=3$ satisfy $(x,y)=2/3\ (0,4)+1/3\ (3,1)$. Therefore there is a equilibrium payoff of $(1,3)$ for the first player for all values of p in $[0,3/4]$ associated with a payoff of $5/2$ for the second player at the initial probability of $3/4$ (and likewise for all values of p in $[1/4,1]$.) For the initial probability of $p=1/2$, the most Player Two could get on the average from a joint plan equilibrium is .9, while the use of either of these equilibrium bi-martingales gives him an average of $2/3 \cdot 5/2\ + 1/3 \cdot 10 = 5$.

This author suspects that for a slight variation of Example 4 and some initial probability there would be no joint plan equilibrium initially good for either player but an equilibrium bi-martingale very good for both players.

Examples 4 was based largely on the example in FORGES (1990); however in the Forges example only the equilibrium payoffs of the *informed* player are improved in comparison to all payoffs from joint plan equilibria; furthermore in the Forges example for every initial probability there exists a joint plan equilibrium which is very good for the informed player.

Examples 4 demonstrates the desirability of a stronger theorem than Theorem 1 that makes use of the greater generality of equilibrium bi-martingales. Whether the possible improvements in equilibrium payoffs for both players using equilibrium bi-martingales over joint plans are contingent on the examples studied, as suggested by Example 2, or are supported by a theorem more advanced than that of Theorem 1, is something that this author has not been able, as yet, to ascertain.

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7 References

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