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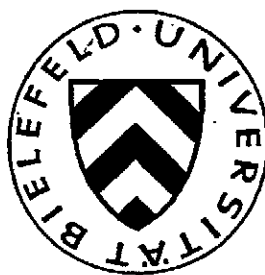
On the Value of Discounted Stochastic Games

by

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Abstract

Without using the Tarski-Seidenberg Principle, we show that the value of a finite discounted stochastic game is an algebraic function of the discounted factor in the neighborhood of zero.[1]

1   Introduction

The idea of zero-sum two-person stochastic games goes back to Shapley [7]. Consider finitely many states, each of them being endowed with two finite sets of actions available for the players. In a given state, the simultaneous choice by each player of an action induces a payoff and a probability to select a subsequent state. From some initial state onward, a play proceeds as described previously. The discounted average of the sequence of payoffs with a discount factor $0 < \lambda < 1$, provides an overall payoff for which any stochastic game has a value v_λ as Shapley proved it. This value can be achieved by the players using stationnary strategies i.e. there is a prescribed mixed action for each state to play independently of the past.

Bewley-Kohlberg [1] established the existence of Puiseux series expansions of v_λ in λ, when it is close enough to zero. The idea of their proof is to use a

consequence of the Tarski-Seidenberg Principle on real closed fields [4]. A real closed field is an ordered field i.e. a field endowed with a total order relation $\leq$ compatible with the operations such that any algebraic extension cannot be ordered.

A first order formula (for short a formula) is a boolean expression of finitely many polynomial equalities or inequalities between variables of the real closed field with possibly quantifiers attached to them. It is assumed that the coefficients of these polynomials are positive or negative integers. A free variable has no quantifier attached to it.

The Tarski-Seidenberg Principle asserts that given a formula with at least one quantifier, it is possible to find a logically equivalent formula with one quantifier less and without any additionnal free variable. In other words it is possible to eliminate any variable which is not free.

This implies that a true sentence on a given real closed field , i.e. any true formula without free variables, is also true on any real closed field since it is equivalent to a true formula dealing with integers.

Bewley-Kohlberg remark that the Shapley Theorem can be written as a true sentence on the field of real numbers, a real closed fields. As seen above, this sentence is true on the real closed field of Puiseux series in X with real coefficients. By substituting the discount factor λ for X in the 'formal value', the series that we obtain is convergent and it gives indeed an expansion of v_λ.

The expansion of the value in Puiseux series of λ shows immediately the convergence of v_λ toward some limit v when λ goes to zero. This limit turns out to be the value of the infinite average payoff game [5]. See [2],[9] for further applications.

Infortunately such a proof does not give any insight of what is exactly going

on. It is difficult to understand why such expansions arise.

There is a more intuitive approach [6]. Consider the set of tuples figuring the tuples of values for each initial state, the components of the corresponding optimal mixed action and λ. This set can be seen as the set of tuples satisfying finitely many polynomial equalities and inequalities i.e. it is a semi-algebraic set. The coefficients of these polynomials are obtained from the parameters of the game. Using the Tarski-Seidenberg Principle, it is easy to see that the projection of this set on the set of couples made of the value for any given initial state and the discount factor λ is also semi-algebraic, since this amounts to eliminate quantifiers in a formula. Therefore we have shown that the value for any initial state is a semi-algebraic function of the discount factor. As it is also a continuous function of the discount factor, it can be considered as algebraic function in the vicinity of zero and therefore it can be expanded in Puiseux Series [3] application of Puiseux Theorem.

The local algebraicity of the value around zero seen as function of the discount factor is precisely what we intend to prove in this paper but without invoking the Tarski-Seidenberg Principle. Bewley-Kohlberg [1] write:

‘At first sight, the problem we study might appear to require the use of the methods of real analysis. However our approach is algebraic rather than analytic.’

Our method uses elementary methods of real analysis even if it is difficult to avoid dealing with polynomials (of algebraic nature). The first interest of the paper is to provide a very clear proof from the viewpoint of intuition. Second, we offer a new way to deal with the tuple of values (one for each initial state). We show that it is indeed possible to work separately with each one of them. The third interest is that we might have a tool to prove algebraic properties in

cases where maybe we cannot apply the Tarski-Seidenberg Principle.

To conclude this introduction, let us mention a completely different method to establish stronger properties on the values [8]. The idea is to write the fixed point equation satisfied by the tuple of values as a system of algebraic equations. Then we see this system as defining a complex variety i.e. we extend our point of view. This allows us to describe the behavior of the values with respect to the discount factor.

The paper is organized as follows. In section 2 we give the basic definitions and properties. In section 3, the crucial Proposition 3 is presented. It will be proved recursively on the number of moves available to player I. In section 4, all the useful properties on algebraic functions and polynomials are collected, especially Proposition 4 on the resultant of two polynomials and Proposition 5 which deals about the composition of algebraic functions. Section 5 presents the proof of Proposition 7 which is the cornerstone of the paper. Finally in section 6 we proceed with the proof of Proposition 3 and Theorem 3 and section 7 delivers the last comments.

2   Definitions

In this section, $\Delta(X)$ is the set of probability distributions on the (finite) set X. See [6] for an extended presentation of the topic of this section.

A (finite) set of states $k \in K$ is given and in each one of them finitely many actions $i \in I(k)$ (resp. $j \in J(k)$) are available for player I (resp. II). Let $p(k,.,.) : I(k) \times J(k) \to \Delta(K)$ be the transitions probabilities and $g(k,.,.) : I(k) \times J(k) \to [-1,1]$ the payoff function for $k \in K$. Remark that player I (resp. II) is seen as a maximizer (resp. minimizer).

A stochastic game is a tuple $\Gamma = (I(k), J(k), p(k,.,.), g(k,.,.))_{k \in K}$.

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Denote by $\mathcal{H}_t$ the set of histories $h_t = k_1, i_1, j_1, \ldots, k_{t+1}$ of length $t \geq 0$, with for $s = 1, \ldots, t$, $(i_s, j_s) \in I(k_s) \times J(k_s)$. Assume that k_1 is fixed and let us call it the initial state.

A behavior strategy σ (resp. τ) for player I (resp. II) is a sequence of mappings

$$\sigma_t : \mathcal{H}_{t-1} \to \Delta(I(k_t))$$

(resp. $\tau_t : \mathcal{H}_{t-1} \to \Delta(J(k_t))$) for $t \geq 1$. A stationnary strategy of player I (resp. player II) ignores the full history i.e. $\sigma_t(h_{t-1})$ ($\tau_t(h_{t-1})$ depends only on k_t for all $t \geq 1$.

The Kolmogorov extension Theorem allows us to define a family $(P_{\sigma,\tau})_{\sigma,\tau}$ of probability distributions on the set $\mathcal{H}$ of histories of infinite length such that:

$$P_{\sigma,\tau}(h_t) = P_{\sigma,\tau}(h_{t-1})p(k_t, i_t, j_t)[k_{t+1}]\sigma_t(h_{t-1})[i_t]\tau_t(h_{t-1})[j_t].$$

The problem consists of studying the discrete process $g_t = g(k_t, i_t, j_t)$ defined on $(\mathcal{H}, P_{\sigma,\tau})$ depending on σ and τ.

Definition 1 *Let the discounted payoff be:*

$$g_\lambda = \lambda \sum_{t=1}^{\infty} (1-\lambda)^{t-1} g_t$$

for $0 < \lambda < 1$.

The game with initial state k and discount factor λ is denoted by $\Gamma_\lambda(k)$. The payoff associated to (σ, τ) is $g_\lambda(\sigma, \tau) = E_{\sigma,\tau}(g_\lambda)$. The Shapley Theorem (1953) can be expressed as follows:

Theorem 1 $\Gamma_\lambda(k)$ *has a value $v_\lambda(k)$ achieved by stationnary strategies i.e. there exists a couple of stationnary strategies $(\sigma_\lambda, \tau_\lambda)$ such that for all (σ, τ):*

$$g_\lambda(\sigma, \tau_\lambda) \leq g_\lambda(\sigma_\lambda, \tau_\lambda) \leq g_\lambda(\sigma_\lambda, \tau).$$

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For any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_K \end{pmatrix}$$

denote by $\mathbf{T}_\lambda(\mathbf{v})$ the vector

$$\begin{pmatrix} T_\lambda^1(\mathbf{v}) \\ \vdots \\ T_\lambda^K(\mathbf{v}) \end{pmatrix}$$

where $T_\lambda^k(\mathbf{v})$ stands for the value (in the sense of Von Neumann) of the finite $I(k) \times J(k)$-game

$$\left[\lambda g(k,i,j) + (1-\lambda) \sum_{h \in K} p(k,i,j)[h] v_h \right]_{i,j}.$$

Remark that $\mathbf{T}_\lambda$ is a contracting map. In order to show Theorem 1, it is necessary to see that:

Proposition 1 *The unique fixed point of* $\mathbf{T}_\lambda$ *is precisely*

$$\mathbf{v}_\lambda = \begin{pmatrix} v_\lambda(1) \\ \vdots \\ v_\lambda(K) \end{pmatrix}.$$

Remark 1 Any family of optimal strategies of the finite games

$$\left[\lambda g(k,i,j) + (1-\lambda) \sum_{h \in K} p(k,i,j)[h] v_\lambda(h) \right]_{i,j}$$

induce an optimal stationnary strategy.

On one side, the vector $\mathbf{v}_\lambda(\Gamma)$ can be seen as a function of $\lambda > 0$ when the stochastic game Γ is fixed. On the other side, it can be considered as a function of stochastic games when λ is fixed.

Proposition 2 *The vector* $\mathbf{v}_\lambda(\Gamma)$ *is separately continuous with respect to* $(\lambda, \Gamma) \in \,]0,1] \times \{\Gamma\}$.

The fundamental Bewley-Kohlberg Theorem (1976) means that $\mathbf{v}_\lambda$ can be expanded in Puiseux Series i.e.

Theorem 2 *The vector $\mathbf{v}_\lambda$ can be expanded in fractionnal powers of λ i.e.*

$$\mathbf{v}_\lambda = \mathbf{v} + \sum_{t=1}^{\infty} \mathbf{c}_t \lambda^{t/T}$$

where $T > 0$ is a well chosen integer, $\mathbf{v}$ and the $\mathbf{c}_t$'s are constant vectors. Remark that:

$$\lim_{\lambda \to 0} \mathbf{v}_\lambda = \mathbf{v}.$$

3 Main Results

Denote by $\mathcal{P}(X)$ the set of polynomials in X with real coefficients. Write $\mathcal{P}^\star(X)$ for $\mathcal{P}(X) \setminus \{0\}$ and denote by $\deg P$ the degree of P.

Similarly denote by $\mathcal{P}(X, Y)$ the set of polynomials in X and Y with real coefficients. Define $\mathcal{P}^\star(X, Y)$ as $\mathcal{P}(X, Y) \setminus \{0\}$.

Given $P \in \mathcal{P}(X, Y)$, $\deg_X P$ (resp. $\deg_Y P$) represents the degree of P with respect to X (resp. Y).

Definition 2 *A continuous function $f(.)$ on a domain $\mathcal{D} \subset \mathbb{R}$ is algebraic if there exists $P(X, Y) \in \mathcal{P}^\star(X, Y)$ such that:*

$$\forall x \in \mathcal{D}, \ P(x, f(x)) = 0.$$

Remark 2 There exist $(p_m, \ldots, p_0) \in (\mathcal{P}(X))^{m+1} \setminus \{(0, \ldots, 0)\}$ such that

$$P(X, Y) = p_m(X)Y^m + \ldots + p_1(X)Y + p_0(X)$$

i.e.

$$\forall x \in \mathcal{D}, \ p_m(x)f^m(x) + \ldots + p_1(x)f(x) + p_0(x) = 0.$$

What we call the 'non-trivial condition' means that for all $x \in \mathcal{D}$:

$$P(x, Y) \neq 0.$$

Here is the main result of the paper:

Theorem 3 *Given a stochastic game, there exists $\bar{\lambda} > 0$ such that the function $\lambda \mapsto v_\lambda(k)$ is algebraic on the domain $]0, \bar{\lambda}[$ for $k = 1, \ldots, K$.*

From now on, we assume that 1 is the initial state. Write $v_\lambda(1) = v_\lambda$ and simply call it 'the value'.

Theorem 3 is consequence a consequence of Proposition 3 which is shown recursively on the cardinal of the $I(k)$'s. Call $\mathcal{G}(n_1, \ldots, n_K)$ the set of stochastic games with $\#I(k) = n_k$ for $k = 1, \ldots, K$ whereas the $\#J(k)$'s are given constants independent of $n_1, \ldots, n_K$.

The transition probabilities and the payoffs are seen as parameters of stochastic games in $\mathcal{G}(n_1, \ldots, n_K)$.

Proposition 3 *There are finitely many mappings $\phi_i : \mathcal{G}(n_1, \ldots, n_K) \rightarrow \mathcal{P}(X, Y)$ such that each coefficient of $\phi_i(\Gamma) = P_i^\Gamma(X, Y)$ is a polynomial function (independent of Γ) of the parameters of Γ and:*

$$\forall \Gamma, \ \forall \lambda \in]0, 1], \ \exists i, \ \phi_i(\Gamma) \in \mathcal{P}^\star(X, Y) \wedge P_i^\Gamma(\lambda, v_\lambda) = 0.$$

4 Basic Properties

This section investigates some properties of polynomials and algebraic functions. See [3] for instance.

Definition 3 $P \in \mathcal{P}^\star(X, Y)$ *is irreducible if it is not possible to decompose it in the product of two polynomials of strictly positive and strictly lower degree.*

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Definition 4 *Let $f(.)$ be an algebraic function with $P(x', f(x')) = 0$ for all $x' \in \mathcal{D}$. Any pair $(x, f(x))$ such that:*

$$P_Y(x, f(x)) \neq 0,$$

where P_Y denotes the formal partial derivative of P with respect to Y, is called an ordinary point.

Remark 3 If P is irreducible, then there are finitely many such points.

Lemma 1 *If $(x, f(x))$ is an ordinary point, then the derivative $f'(x)$ of $f(.)$ at x exists. Moreover:*

$$f'(x) = -\frac{P_X(x, f(x))}{P_Y(x, f(x))},$$

where P_X is the formal partial derivative with respect to X.

The following Definition 5, Proposition 4 and Corollary 1 are the basic tools for this paper.

Definition 5 *Let $P \in \mathcal{P}^*(Y)$ (resp. $Q \in \mathcal{P}^*(Y)$) be of degree $n > 0$ (resp. $m > 0$).*

For $k = 0, \ldots, n-1$ (resp. $h = 0, \ldots, m-1$), call $(Y^k P)$ (resp. $(Y^h Q)$), the row of length $n + m$ representing the coefficients of $Y^k P$ (resp. $Y^h Q$) ordered by the increasing powers of Y.

The determinant:

$$R = \begin{vmatrix} (Y^{n-1}P) \\ \ldots \\ (P) \\ (Y^{m-1}Q) \\ \ldots \\ (Q) \end{vmatrix}$$

is called the resultant of P and Q. Remark that it is of the same nature as the coefficients of these polynomials.

Proposition 4 *There exists $N \in \mathcal{P}^*(Y)$ (resp. $M \in \mathcal{P}^*(Y)$) with $\deg M \leq m - 1$ (resp. $\deg M \leq n - 1$) such that:*

$$R = N(Y)P(Y) + M(Y)Q(Y).$$

Sketch of the proof:

Case 1: $R \neq 0$. We are looking for two polynomials with unknown coefficients

$$N(Y) = b_{m-1}Y^{m-1} + \ldots + b_1 Y + b_0,$$

$$M(Y) = c_{n-1}Y^{n-1} + \ldots + c_1 Y + c_0,$$

such that the following system is true:

$$b_{m-1}(Y^{m-1}P) + \ldots + b_1(YP) + b_0(P) +$$

$$c_{n-1}(Y^{n-1}Q) + \ldots + c_1(YQ) + c_0(Q) = (0, \ldots, R).$$

This system has one and only one solution since its determinant is $R \neq 0$.

Case 2: $R = 0$. Instead of the previous sytem, we solve the following one:

$$b_{m-1}(Y^{m-1}P) + \ldots + b_1(YP) + b_0(P) +$$

$$c_{n-1}(Y^{n-1}Q) + \ldots + c_1(YQ) + c_0(Q) = (0, \ldots, 0).$$

Since it is homogeneous, it is easy to find a non trivial solution. For instance fix as many coefficients as possible to 1. Up to some multiplication, one can insure that the coefficients of N and M are polynomial functions of the coefficients of P and Q. Remark that there are finitely many such functions. $\square$

Corollary 1 *Let $R \in \mathcal{P}(X)$ be the resultant of $P \in \mathcal{P}^*(X,Y)$ and $Q \in \mathcal{P}^*(X,Y)$ with respect to Y.*

If $R = 0$, then there exists $N \in \mathcal{P}^*(X,Y)$ (resp. $M \in \mathcal{P}^*(X,Y)$) with $\deg_Y N \leq \deg_Y Q - 1$ (resp. $\deg_Y M \leq \deg_Y P - 1$) such that:

$$0 = N(X,Y)P(X,Y) + M(X,Y)Q(X,Y).$$

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<u>Proof:</u> For each x such that $P(x,Y) \in \mathcal{P}^*(Y)$ (resp. $Q(x,Y) \in \mathcal{P}^*(Y)$), apply Proposition 4 to find $N_x \in \mathcal{P}^*(Y)$ (resp. $M_x \in \mathcal{P}^*(Y)$) with $\deg N_x \le \deg_Y Q - 1$ (resp. $\deg_Y M_x \le \deg_Y P - 1$) such that:

$$0 = N_x(Y)P(x,Y) + M_x(Y)Q(x,Y).$$

Remark that the coefficients of N_x and M_x are polynomial functions of x. Since there are finitely many possible such functions, they are the same for infinitely many x. Hence, it is possible to find $N \in \mathcal{P}^*(X,Y)$ (resp. $M \in \mathcal{P}^*(X,Y)$) such that:

$$0 = N(X,Y)P(X,Y) + M(X,Y)Q(X,Y).$$

<div align="right">□</div>

The next Lemma 2 and Lemma 3 which are independent, are used to show the final Proposition 5, the second goal of the section.

Lemma 2 *For any integer $n > 0$ there are finitely many mappings*

$$\psi_i : \ \{(Q,P) \in (\mathcal{P}^*(X))^2 \mid \deg Q \le n; \deg P \le n\}$$

$$\rightarrow \{(H,H') \in (\mathcal{P}(X))^2 \mid \deg H < n; \deg H' < n\} \times \mathbb{R}^2$$

such that, with $\psi_i(Q,P) = (H_i, H'_i, f_i, f'_i)$, each coefficient of H_i (resp. H'_i), f_i (resp. f'_i) are polynomial functions (independent of Q and P) of the coefficients of Q and P, and $\deg H_i < \deg Q$ (resp. $\deg H'_i < \deg Q$).

Moreover, for any $x \in \mathbb{R}$ one has:

$$Q(x) = 0 \ \wedge P(x) \ne 0 \Rightarrow \exists i, \ f_i P(x) = H_i(x) \wedge \frac{f'_i}{P(x)} = H'_i(x) \wedge f_i \ne 0 \wedge f'_i \ne 0.$$

<u>Proof:</u> Let us make the construction of the H'_i (resp. f'_i) recursively on $\deg Q$.

1. $\deg Q = 1$:

   It is possible to solve $Q(x) = 0$. It is clear that $1/P(x)$ (resp. $P(x)$) can be expressed as a rationnal functions of the coefficients of P and Q.

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2. Assume that we have finitely many mappings working for $\deg Q = p < n$. Let us make the construction when $\deg Q = p + 1$.

Take the resultant $R \in \mathbb{R}$ of Q and P. If $R \neq 0$ then there exist $N \in \mathcal{P}^\star(X)$ and $M \in \mathcal{P}^\star(X)$ with $\deg M < \deg Q$ such that each coefficient of M (resp. N) is a fixed polynomial function of those of P and Q and

$$R = N(X)Q(X) + M(X)P(X).$$

In particular, if $Q(x) = 0$, then $P(x) \neq 0$ and

$$\frac{R}{P(x)} = M(x).$$

Choose $M(X)$ as one of the H_i''s and R as one of the f_i''s.

On the other side, when $R = 0$, apply Proposition 4 to find $N \in \mathcal{P}^\star(X)$ and $M \in \mathcal{P}^\star(X)$ with $\deg M < \deg Q$ such that each coefficient of M (resp. N) is a fixed polynomial function of those of Q and P with:

$$0 = N(X)Q(X) + M(X)P(X).$$

Recall that there are finitely many such polynomial functions. Remark that for x such that $Q(x) = 0$ and $P(x) \neq 0$, one has:

$$M(x) = 0.$$

Apply the recursive assumption, bearing in mind that Q is replaced by M. This ends the construction of the H_i''s and the f_i''s.

To conclude, let us make the construction for the H_i's and the f_i's. Take any pair (Q, P) such that $Q(x) = 0$ and $P(x) \neq 0$. Let i be such that:

$$\frac{f_i'}{P(x)} = H_i'(x).$$

Considering the new pair (Q, H_i') let i' be such that:

$$\frac{f_{i'}'}{H_i'(x)} = H_{i'}'(x).$$

Remark that $f_{i'}'P(x) = f_i'H_{i'}'(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3 *There exist finitely many mappings $\mu_j = (\mu_j^1, \ldots, \mu_j^n) : (\mathbb{R}^n)^{n+1} \to$*
$\mathbb{R}^n$ such that each μ_j^i is a polynomial function and for any $(\mathbf{u_i})_{i=1,\ldots,n+1} =$
$((u_i^1, \ldots, u_i^n))_{i=1,\ldots,n+1} \in (\mathbb{R}^n \setminus (0,\ldots,0))^{n+1}$ there exists j such that:

$$\mu_j \neq (0, \ldots, 0)$$

and

$$\sum_{i=1}^{n+1} \mu_j^i \mathbf{u_i} = (0, \ldots, 0).$$

<u>Proof:</u> Remark that up to some permutation it is possible for some $j \in [2, n+1]$,
to express in a unique way (by solving some system) $\mathbf{u_j}$ as a linear combination
of the $\mathbf{u_i}$'s with $i \in [1, j-1]$. $\qquad\qquad\qquad\qquad\qquad\square$

Proposition 5 means roughly that if $x \in \mathbb{R}$ is any algebraic number and
$y \in \mathbb{R}$ an algebraic function of x, then y is also algebraic. We require additional
stronger properties.

Proposition 5 *For any integer $n > 0$, there exist finitely many mappings*

$$\xi_i : \{(Q, P) \in \mathcal{P}^\star(X) \times \mathcal{P}^\star(X, Y) \mid \deg Q \leq n; \deg_X P \leq n; \deg_Y P \leq n\}$$

$$\to \mathcal{P}(Y)$$

such that each coefficient of $\xi_i(Q, P) = L_i$ is a polynomial function (independent
of P and Q) of the coefficients of P, Q. Moreover, for all $(x, y) \in \mathbb{R}^2$:

$$Q(x) = 0 \wedge P(x, Y) \in \mathcal{P}^\star(Y) \wedge P(x, y) = 0 \Rightarrow L_i \in \mathcal{P}^\star(Y) \wedge L_i(y) = 0.$$

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<u>Proof:</u>

Assume that $Q(x) = 0$ with $\deg Q = m'$. One starts from:

$$P(x, y) = p_m(x)y^m + \ldots + p_1(x)y + p_0(x) = 0$$

with $p_m(x) \neq 0$. We are going to make finitely times, operations on polynomials in such a way that we always obtain polynomials with coefficients being polynomial functions of the previous ones. Depending on them, there are only finitely many possible such functions.

Using Lemma 2, find $p_{1,k} \in \mathcal{P}(X)$ with $\deg p_{1,k} < \deg Q$ for $k = 1, m-1$ and $\ell_1 \in \mathbb{R}^*$ such that:

$$\ell_1 y^m = p_{1,m-1}(x)y^{m-1} + \ldots + p_{1,1}y + p_{1,0}.$$

Recall that ℓ_1 (resp. the coefficients of $p_{i,j}$) are polynomials functions of the coefficients of Q and P.

In other words:

$$\ell_1 y^m = \sum_{i=0}^{m'-1} \sum_{j=0}^{m-1} u_{i,j}^0 x^i y^j.$$

For $h = 1, mm'$, by multiplying the previous expression by y^h and by using Lemma 2, write similarly:

$$\ell_{h+1} y^{m+h} = \sum_{i=0}^{m'-1} \sum_{j=0}^{m-1} u_{i,j}^h x^i y^j.$$

Two cases are possible:

- $\underline{y = 0}$: Take $Y \in \mathcal{P}^*(Y)$ among the possible L_i's.

- $\underline{y \neq 0}$: Remark that $(\mathbf{u}^h)_{h \in [0, mm']} \in (\mathbb{R}^{mm'})^{mm'+1}$ with $\mathbf{u}^h = (u_{i,j}^h)_{i,j} \in \mathbb{R}^{mm'} \setminus \{(0, \ldots, 0)\}$ satisfies the assumptions of Lemma 3. Find a tuple $(a_0, \ldots, a_{mm'}) \in \mathbb{R}^{mm'+1} \setminus \{(0, \ldots, 0)\}$ of polynomials functions of

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$(\mathbf{u}^h)_{h \in [0, mm']} \in (I\!\!R^{mm'})^{mm'+1}$. such that :

$$\sum_{t=0}^{mm'} a_t y^t = 0.$$

□

5 Preliminaries

For $P \in \mathcal{P}(X, Y)$ (resp. $Q \in \mathcal{P}(X, Y)$): 'P divides Q' is denoted by $P \mid Q$. If $P \mid Q$ and $Q \mid P$ then one writes $P \sim Q$ and this means that they differ only by some real constant factor.

Lemma 4 *Let $P \in \mathcal{P}^*(X, Y)$ (resp. $Q \in \mathcal{P}^*(X, Y)$) be irreducible of degree n (resp. m). One has:*

$$\#\{x \mid \exists y, \ P(x, y) = 0 \land Q(x, y) = 0\} < \infty$$

or $P \sim Q$.

<u>Proof:</u> Let $R(X)$ be the resultant of $P(X, Y)$ and $Q(X, Y)$ with respect to Y.

- $R \neq 0$. Remark that

$$\{x \mid \exists y, \ P(x, y) = 0 \land Q(x, y) = 0\} \subset \{R(x) = 0\}$$

and $\#\{R(x) = 0\} < \infty$.

- $R = 0$. Apply Corollary 1 and choose $N \in \mathcal{P}^*(X, Y)$ (resp. $M \in \mathcal{P}^*(X, Y)$) with $\deg_Y N \leq m - 1$ (resp. $\deg_Y M \leq n - 1$) such that:

$$N(X, Y)P(X, Y) + M(X, Y)Q(X, Y) = 0.$$

Since P (resp. Q) is irreducible and $P \mid M$ (resp. $Q \mid N$) is not possible, one has $P \mid Q$ (resp. $Q \mid P$). □

Lemma 4 has a very useful consequence:

Proposition 6 *Let $(P_1, \ldots, P_r) \in (\mathcal{P}^*(X, Y))^r$ and $f(.)$ be a continuous function defined on the neighborhood $\mathcal{D} \subset \mathbb{R}$ of x such that:*

$$\forall x' \in \mathcal{D}, \exists i, \ P_i(x', f(x')) = 0.$$

There exists $\epsilon > 0$ such that:

- $\exists i, \forall x' \in V^+(\epsilon), \ P_i(x', f(x')) = 0.$

- $\exists j, \forall x' \in V^-(\epsilon), \ P_j(x', f(x')) = 0.$

with $V^+(\epsilon) =]x, x + \epsilon[$ and $V^-(\epsilon) =]x - \epsilon, x[$.

Proof: WLOG one can assume that the P_i's are non equivalent irreducible polynomials. Indeed: write P_i as product of irreducible polynomials $P_{1,i} \ldots P_{t(i),i}$. Remark that $P_i(x', f(x')) = 0$ implies

$$P_{1,i}(x', f(x')) = 0 \vee \ldots \vee P_{t(i),i}(x', f(x')) = 0.$$

Remark:

Sublemma 1 *There exists $\epsilon > 0$ such that for all $x' \in V^+(\epsilon)$ (resp. $x' \in V^-(\epsilon)$) there exists a unique i (resp. j) with $P_i(x', f(x')) = 0$ (resp. $P_j(x', f(x')) = 0$).*

Proof of the Sublemma: Use the previous lemma to remark that for all $i \neq j$, $\{x' \mid P_j(x', f(x')) = 0 \wedge P_i(x', f(x')) = 0\}$ is a finite set. $\square$

Let us continue the proof of Proposition 6. Fix an integer $n > 0$ large enough. Assume that for any integer large enough $m > 0$, it is possible to find a pair $(x'_m, x''_m) \in]x - \epsilon - 1/n, x - 1/n[^2$ with $|x'_m - x''_m| \leq 1/m$ so that the unique polynomials associated to each one of them by Sublemma 1 are different. Take a convergent subsequence to $\tilde{x} \in [x - \epsilon - 1/n, x - 1/n]$.

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By continuity of $f(.)$ at $\tilde{x}$, we obtain a contradiction to Sublemma 1.

Therefore, there exists $m > 0$, such that if $|x' - x''| \leq 1/m$ then the unique polynomials associated to x' and x'' are the same. This means that it is the same on all $]x - \epsilon, x - 1/n[$. Let n goes to infinity to obtain the result. Proceed similarly for V^+. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2 *Let $f(.)$ be an algebraic function on the domain $\mathcal{D} \subset \mathbb{R}$ characterized by $P(X,Y) \in \mathcal{P}^\star(X,Y)$. For a given $x \in \mathcal{D} \cup \partial\mathcal{D}$ assume that there is $Q(X,Y) \in \mathcal{P}^\star(X,Y)$ and a sequence of distinct points $x_n \to x$ as $n \to \infty$ such that:*

$$Q(x_n, f(x_n)) = 0.$$

Then, there exists $\epsilon > 0$ such that two cases are possible:

- *Either for all $x' \in V^-(\epsilon)$ or for all $x' \in V^+(\epsilon)$, $Q(x', f(x')) = 0$ is true.*

- *For all $x' \in V^-(\epsilon) \cup V^+(\epsilon)$, $Q(x', f(x')) = 0$ is true.*

<u>Proof:</u> Decompose P (resp. Q) in a product $P_1 \ldots P_r$ (resp. $Q_1 \ldots Q_s$) of irreducible polynomials. The assumptions of Proposition 6 are satisfied with the P_i's. Hence, there exists $\epsilon > 0$, such that:

- $\exists i, \forall x' \in V^+(\epsilon),\ P_i(x', f(x')) = 0.$

- $\exists i', \forall x' \in V^-(\epsilon),\ P_{i'}(x', f(x')) = 0.$

This means that for at least a pair (i,j) or (i',j') (possibly both) infinitely many $x' \in V^+(\epsilon)$ or $V^-(\epsilon)$ satisfy:

$$P_i(x', f(x')) = 0 \wedge Q_j(x', f(x')) = 0$$

or

$$P_{i'}(x', f(x')) = 0 \wedge Q_{j'}(x', f(x')) = 0$$

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By using Lemma 4, remark that $P_i \sim Q_j$ or $P_{i'} \sim Q_{j'}$. $\qquad\square$

Proposition 7 is the main result of this section.

Proposition 7 *For any integers $n > 0$, $m > 0$ and $p > -1$ there exist finitely many mappings*

$$\theta_i : \{(P, Q) \in (\mathcal{P}^\star(X, Y))^2 \mid \deg_X P \le p; \deg_X Q \le p; \deg_Y P \le n; \deg_Y Q \le m\}$$

$$\to \mathcal{P}(X)$$

(resp.

$$\theta_j' : \{(P, Q) \in (\mathcal{P}^\star(X, Y))^2 \mid \deg_X P \le p; \deg_X Q \le p; \deg_Y P \le n; \deg_Y Q \le m\}$$

$$\to \mathcal{P}(Y).)$$

such that each coefficient of $\theta_i(P, Q) = T_i$ (resp. $\theta_j'(P, Q) = R_j$) is a polynomial function (independent of P and Q) of the coefficients of P and Q. Moreover for any local maximum $x \in \mathbb{R}$ of a continuous function $f(.)$ defined on a domain $\mathcal{D} \subset \mathbb{R}$ with:

- *For all $x' < x$ close enough to x, $P(x', f(x')) = 0$.*

- *For all $x' > x$ close enough to x, $Q(x', f(x')) = 0$.*

the following holds:

$$\exists i, \ T_i \in \mathcal{P}^\star(Y) \wedge T_i(f(x)) = 0$$

or

$$\exists j, \ R_j \in \mathcal{P}^\star(X) \wedge R_j(x) = 0.$$

<u>Proof:</u> Denote by $\mathcal{T}(n, m, p)$ (resp. $\mathcal{T}(n, m, \infty)$) : 'Proposition 7 holds for (n, m, p) (resp. $\mathcal{T}(n, m, 0) \wedge \mathcal{T}(n, m, 1) \wedge \ldots$).

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The proposition will be shown recursively according to the following schemes:

$$T(1,1,p-1) \Rightarrow T(1,1,p),$$

which added to $T(1,1,0)$ implies $T(1,1,\infty)$, and

$$T(n,m,p-1) \wedge T(n,m-1,\infty) \wedge T(n-1,m,\infty) \wedge T(n-1,m-1,\infty) \Rightarrow T(n,m,p),$$

wich added to $T(n,m,0)$ implies that:

$$T(n,m-1,\infty) \wedge T(n-1,m,\infty) \wedge T(n-1,m-1,\infty) \Rightarrow T(n,m,\infty).$$

Initial step:

If there is only one polynomial of degree 0 with respect to X, for instance $P(X,Y) = a_n Y^n + \ldots + a_1 Y + a_0$, then we are done since:

$$a_n f^n(x) + \ldots + a_1 f(x) + a_0 = 0$$

can be taken among the T_i's.

Recursive Step:

Let us take $P(X,Y) = p_n(X)Y^n + \ldots + p_1(X)Y + p_0(X)$ and $Q(X,Y) = q_m(X)Y^m + \ldots + q_1(X)Y + q_0(X)$ with $p_n \in \mathcal{P}^\star(X)$ (resp. $q_m \in \mathcal{P}^\star(X)$) and $0 < \max_{i,j}(\deg p_i, \deg q_j) = p$.

Case I: $P \neq Q$

Take the resultant $R(X)$ of $P(X,Y)$ and $Q(X,Y)$ with respect to the variable Y.

- $R \neq 0$. Take R as one of the R_j's.

- $R = 0$. Apply Corollary 1 and select $N \in \mathcal{P}^\star(X,Y)$ (resp. $M \in \mathcal{P}^\star(X,Y)$) with $\deg_Y N \leq m - 1$ (resp. $\deg_Y M \leq n - 1$) and

$$0 = N(X,Y)P(X,Y) + M(X,Y)Q(X,Y).$$

Clearly, this implies that for all $x' < x$ (resp. $x' > x$) close enough to x,

$$M(x', f(x'))Q(x', f(x')) = 0 \text{ (resp. } N(x', f(x'))P(x', f(x')) = 0).$$

In view of Corollary 2, this implies that around x the same polynomial of degree $\min(n, m)$ can be chosen and we are led to <u>Case II</u> or instead of P or Q, M and N can be considered. In the last case remark that the degree with respect to Y is strictly lower for at least one of the polynomials.

<u>Case II</u>: $P = Q$

This case means that $f(.)$ is algebraic on some neighborhood $V(\epsilon) =]x - \epsilon, x + \epsilon[$ of x.

<u>Subcase A</u>: $(x, f(x))$ is an ordinary point. Since the derivative $f'(x)$ exists, if x is a local maximum of $f(.)$, then $P_X(x, f(x)) = 0$. Consider two different cases:

1. $P_X(x, Y) = 0$. Since there exists i such that $p'_i \neq 0$, take p'_i among the R_j's.

2. $P_X(x, Y) \neq 0$. Clearly $P_X(X, Y) \neq 0$, so let $R(X)$ be the resultant of $P(X, Y)$ and $P_X(X, Y)$ with respect to Y. Remark that $R(x) = 0$ when for Y, we substitute $f(x)$.

   - $R \neq 0$. Take R as one of the R_j's.

   - $R = 0$. In view of Corollary 1, there exists $N \in \mathcal{P}^\star(X, Y)$ (resp. $M \in \mathcal{P}^\star(X, Y)$) with $\deg_Y M \leq n - 1$ such that:

     $$0 = N(X, Y)P(X, Y) + M(X, Y)P_X(X, Y).$$

     This implies that for any $x' \in V$ one has:

     $$0 = N(x', f(x'))P(x', f(x')) + M(x', f(x'))P_X(x', f(x')),$$

i.e.

$$0 \doteq M(x', f(x')) P_X(x', f(x')).$$

Invoke Corollary 2 to conclude that there is a neighborhood $V(\epsilon') =]x - \epsilon', x + \epsilon'[$ of x such that at least one of the following items is true:

$\forall x' \in V(\epsilon'), \ 0 = M(x', f(x'))$,

$\forall x' \in V(\epsilon'), \ 0 = P_X(x', f(x'))$,

$\forall x' \in V^+(\epsilon'), \ 0 = P_X(x', f(x')) \ \wedge \forall x' \in V^-(\epsilon'), \ 0 = M(x', f(x'))$,

$\forall x' \in V^-(\epsilon'), \ 0 = P_X(x', f(x')) \ \wedge \forall x' \in V^+(\epsilon'), \ 0 = M(x', f(x'))$.

We are led to polynomials with a lower degree in Y or in X.

Subcase B: The point $(x, f(x))$ is not an ordinary point i.e. $P_Y(x, f(x)) = 0$. Mimick what we have done with P_Y instead P_X.

□

6   Main Proofs

Proof of Proposition 3:

The assertion 'Proposition 3 is true for $(n_1, \ldots, n_K)$' is denoted by $\mathcal{P}(n_1, \ldots, n_K)$.

Proposition 3 will be shown recursively according to the following scheme:

$$\mathcal{P}(n_1, \ldots, n_K) \Rightarrow \mathcal{P}(n_1, \ldots, n_k + 1, \ldots, n_K)$$

for any $k \in [1, \ldots, K]$.

(i) Step $\mathcal{P}(1, \ldots, 1)$

This means, that for all states $k = 1, \ldots, K$ player I has only one move available $i = 1$. Therefore:

$$v_\lambda(k) = \max_j [\lambda g(k, 1, j) + (1 - \lambda) \sum_{h \in K} p(k, 1, j)[h] v_\lambda(h)],$$

i.e. for some j_k

$$v_\lambda(k) = [\lambda g(k, 1, j_k) + (1 - \lambda) \sum_{h \in K} p(k, 1, j_k)[h] v_\lambda(h)].$$

Therefore $(v_\lambda(1), \ldots, v_\lambda(K))$ is the unique solution of the following linear system:

$$\begin{cases} \lambda g(1, 1, j_1) & = & (1 - (1 - \lambda) p(1, 1, j_1)[1]) v_\lambda(1) - (1 - \lambda) \sum_{h>1} p(1, 1, j_1)[h] v_\lambda(h) \\ \vdots & & \vdots \\ \lambda g(K, 1, j_K) & = & -(1 - \lambda) \sum_{h<K} p(1, 1, j_K)[h] v_\lambda(h) + (1 - (1 - \lambda) p(K, 1, j_K)[K]) v_\lambda(K). \end{cases}$$

Remark that

$$\begin{vmatrix} (1 - (1 - \lambda) p(1, 1, j_1)[1]) & \cdots & -(1 - \lambda) p(1, 1, j_1)[K] \\ \vdots & \vdots & \vdots \\ -(1 - \lambda) p(1, 1, j_K)[1] & \cdots & (1 - (1 - \lambda) p(K, 1, j_K)[K]) v_\lambda(K). \end{vmatrix} \neq 0.$$

The system being a Cramer System, $v_\lambda(1)$ is expressed as the quotient of two polynomial functions of all the parameters of the game. The polynomial of degree 1 which describes $v_\lambda(1)$ satisfies obviously 'the non trivial condition'.

(ii) Step $\mathcal{P}(n_1, \ldots, n_K) \Rightarrow \mathcal{P}(n_1, \ldots, n_k + 1, \ldots, n_K)$.

Assume now that $\lambda \in]0, 1]$ is fixed. Therefore, the polynomials that will be considered do not mention it.

Let Γ be a game of dimensions $(n_1, \ldots, n_k + 1, \ldots, n_K)$. Call v its value. In state k, choose the two rows $n_k + 1$ and n_k for instance. Define a new one n'_k as follows:

$$\forall j, \forall h, \ p(k, n'_k, j)[h] = \alpha p(k, n_k, j)[h] + (1 - \alpha) p(1, n_k + 1, j)[h],$$

$$\forall j, \ g(k, n'_k, j) = \alpha g(k, n_k, j) + (1 - \alpha) g(k, n_k + 1, j).$$

where $\alpha \in [0, 1]$ is some new parameter. The game obtained by substituting the new row n'_k for $n_k + 1$ and n_k is denoted by $\Gamma(\alpha)$. The value of this game is written $v(\alpha)$. According to Proposition 2, the function $v(.)$ is continuous with respect to α.

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We have this lemma:

Lemma 5 $v = \max_\alpha v(\alpha)$.

Proof: If player I restricts himself to play the last two rows of state with relative weight α (resp. $1 - \alpha$), then he shall receive a lower equilibrium payoff. Hence:

$$\forall \alpha, \ v(\alpha) \leq v.$$

The value v is achieved by playing stationnary strategies by Theorem 1. It prescribes to play the rows n_k (resp. $n_k + 1$) with the relative weight α_λ (resp. $1 - \alpha_\lambda$). Therefore:

$$v(\alpha_\lambda) = v.$$

□

The problem is simply to describe the set of parameters α where the maximum is achieved. Remark that the game $\Gamma(\alpha)$ satisfies the conditions of $\mathcal{P}(n_1, \ldots, n_K)$.

There are two cases:

1. The maximum is achieved for $\alpha = 0$ (resp. $\alpha = 1$). Since $\mathcal{P}(n_1, \ldots, n_K)$ holds, one can find a polynomial P among a finite set of possible ones, such that:

   $$P(0, v(0)) = P(0, v) = 0 \ (resp. \ P(1, v(1)) = P(1, v) = 0),$$

   satisfying the 'non trivial condition' with respect to v.

2. The maximum is achieved only for some $\alpha \in]0, 1[$. Apply Proposition 6 to select $\epsilon > 0$ such that there exists two polynomials P and Q selected among a finite set of possible ones, such that:

   $$\forall \alpha' \in]\alpha - \epsilon, \alpha[, \ P(\alpha', v(\alpha')) = 0,$$

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and

$$\forall \alpha' \in]\alpha, \alpha + \epsilon[, \ Q(\alpha', v(\alpha')) = 0.$$

Now, apply Proposition 7, to deduce that for some $T_1, \ldots, T_s$ and $R_1, \ldots, R_r$, one have:

$$T_1(v) = 0 \vee \ldots \vee T_s(v) = 0$$

or

$$R_1(\alpha) = 0 \vee \ldots \vee R_r(\alpha) = 0.$$

In the first case, then we are done. In the second case, we will succeed to reach the conclusion after application of Proposition 5. Remember indeed that there exists, among a finite set of possible choices, $P(X, Y) \in \mathcal{P}^*(X, Y)$ such that $P(\alpha, Y) \in \mathcal{P}^*(Y)$ and

$$P(\alpha, v(\alpha)) = 0.$$

□

Proof of Theorem 3: Apply Proposition 6 to Proposition 3 on the interval $]0, 1[$. □

7 Discussion

We have also proved the following property:

Proposition 8 *Let f be a function defined on the set of stochastic games $\{\Gamma\}$ such that:*

(i) Assuming that in each state player I has only one action, and given a fixed number of actions for player II, there exists $P(X) \in \mathcal{P}^*(X)$ chosen among a finite set of possible ones each coefficient being a polynomial function of

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the parameters of the game, such that:

$$P(f(\Gamma)) = 0.$$

(ii) $\sup_{0 \leq \alpha \leq 1} f(\Gamma(\alpha)) = f(\Gamma).$

(iii) The function $f(\Gamma(.))$ is continuous on $]0, 1[$.

Then, for any stochastic game Γ there exists $P(X) \in \mathcal{P}^\star(X)$ with each coefficient being a polynomial function of the parameters of the game such that:

$$P(f(\Gamma)) = 0.$$

Remark 4 Our proof is not much simpler than Tarski's theorem. However, in view of Proposition 8, it depends much more on the specific properties of the games.

Remark 5 It should be interesting to find a non trivial example (at least different from a dicounted value) of a function satisfying the assumptions of the previous Proposition.

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