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Consistency and its Convers
An Approach for Economies

by

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Consistency and its Converse. An Approach for Economies

Abstract

The question how to define consisteny for economies is not easy to answer. There have been different approaches to solve this problem. In this paper we will extend the class of economies in consideration to a class of so-called generalized economies like in Thomson [12], Dagan [1] or van den Nouweland, Peleg, Tijs [9]). Here, the idea of consistency, i.e. agreeing on some outcome of an economic situation, paying agents who want to leave the economy according to this outcome and then reconsidering the reduced economic situation, requires the introduction of some net trade vector.

We will consider solution concepts – especially generalized Walrasian equilibrium concepts – for generalized economies. Extending the Walras concept, two points should not be neglected. First, one should concentrate on everywhere non-empty solutions. Second, the extension should be consistent. It turns out that there may possibly be a huge variety of non-empty consistent extensions of the concept of Walras equilibrium. However, we will be able to show, that the most prominent extensions, the porportional solution and the equal sharing solution, are minimal in the family of non-empty consistent extensions on some classes of generalized economies.

Finally, we will consider converse consistency, which will provide the class of solution concepts with additional structure from a bottom-up point of view.

1 Introduction

Recently, there have been efforts to apply the game-theoretical consistency property to economies. Papers which have to be mentioned in this context are Thomson ([10], [12]), Dagan [1] and van den Nouweland, Peleg, Tijs [9]. A survey on general aspects of consistency was given by Thomson [11]. In analogy to reduced games in game theory, reduced economies have to be developed to define consistency properties for economies.

Following this line - and not the rather exotic alternative of Korthues [4] who extends the notion of Walras equilibrium by exactly specifying who trades what with whom - one necessarily has to extend the class of economies in consideration to a class of so-called generalized economies (see Thomson [12], Dagan [1] and van den Nouweland, Peleg, Tijs [9]). The latter is inevitable since the idea of consistency, i.e. agreeing on some outcome of an economic situation, paying agents who want to leave the economy according to this outcome and then reconsidering the reduced economic situation, requires the introduction of some net trade vector - a not necessarily positive bundle of goods.

After enlarging the class of economies one has to extend the considered solution concepts - in our case the Walras equilibrium concept - as well if one does not want to deal with concepts which are empty almost everywhere. Extending the Walras concept, two points should not be neglected. First, one should concentrate on everywhere non-empty solutions. Second, the extension should be consistent. It turns out that there may possibly be a huge variety of non-empty consistent extensions of the concept of Walras equilibrium. However, we will be able to show, that one of most prominent extensions, the porportional solution, is minimal in the class of non-empty consistent extensions on a standard class of economies. A direct analogon for another very prominent solution, the equal sharing solution, holds only on a smaller class of economies. This is due to a less general existence result for the equal sharing equilibrium. The property of minimal consistency helps to understand the structure of the family of all consistent solution concepts, especially of those which are extensions of the Walras equilibrium concept.

In the last part of the paper we will consider the idea of converse consistency, which will give us further insight in the structure of the family of solution concepts on the domain of generalized economies. This notion can be roughly described by "choosing an outcome for a generalized economies if it was chosen in all proper reduced economies". It will turn out that the main solution concepts all satisfy converse consistency on certain classes of economies. For Walrasian concepts and the Pareto concept it will be necessary to restrict considerations to classses of economies in which supporting prices are unique (smoothness assumption). To provide more structure minimal conversely consistent extensions and maximal conversely consistent subsolutions are introduced.

2 Generalized Economies and Solution Concepts

A generalized economy E is a tuple $((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$. Here, $N = \{1, \ldots, n\}$ is the set of the agents of the economy, who are represented by their initial endowments $\omega_i \in \mathbf{R}^l_{++}$ and their preferences $\succeq_i \subset \mathbf{R}^l_+ \times \mathbf{R}^l_+$. In addition, $T \in \mathbf{R}^l$ with $\sum_{j=1}^n \omega_j + T \in \mathbf{R}^l_+$ represents the net trade vector of this economy. Its components can be positive (indicating imports of the good in question) as well as negative (indicating exports of the good in question). Imports can be distributed among the economy's agents; exports have to be brought up by them. In this wider context a usual economy can be seen as a generalized economy E' with net trade vector $T' = 0$. An **allocation** of a generalized economy (for short: economy) $E = ((\omega_i)_{i\in N}, (\succeq_i)_{i\in N}, T)$ is a vector $z = (z_1, \ldots, z_n) \in \mathcal{A}(E) := \{\zeta \in (\mathbf{R}^l_+)^n | \sum_{j=1}^n \zeta_j = \sum_{j=1}^n \omega_j + T\}$. $\mathcal{A}(E)$ is called the set of allocations. A price system is a vector P in the $(l-1)$-dimensional unity simplex Δ^l. Very often boundary prices will be excluded from consideration. Then we will use the notation $P \in \overset{\circ}{\Delta}{}^l$, where $\overset{\circ}{\Delta}{}^l$ is the interior of the price simplex Δ^l.

As is known from cooperative game theory, varying the notion of reduced games has a great impact on what solutions turn out to be consistent. The specific way of definition of reduced economies seems to be important, too. The easiest and most adhoc way of doing it is described in

Definition 2.1 *For every economy E, every subset of its agents $S \subset N$ and* *every allocation* $x \in \mathcal{A}(E)$ *the* **reduced economy** $E^{S,x}$ *is given by*

$$E^{S,x} := ((\omega_i)_{i\in S}, (\succeq_i)_{i\in S}, T')$$

with $T' := T + \sum_{j\in N\setminus S}(\omega_j - x_j)$.

The definition follows the idea that every agents who leaves the economy is paid in goods according to the outcome x and leaves his initial endowments in the economy. This automatically leads to the definition of the net trade vector T' of the reduced economy. Initial endowments of the remaining agents are kept fixed to give them the same starting position for the discussion of redistribution in the reduced economy.

Throughout the paper, preferences are assumed to be reflexive, transitive, complete, continuous, monotonic and strictly convex. Sometimes consumers' tastes will be described by means of utility functions instead of preferences. Furthermore, generalized economies will be denoted by E or E_T. Economies with same initial endowments and preferences but with net trade vector 0 will be called **corresponding usual economies** and will be denoted by E_0.

The most important aspects of the Walras equilibrium are the market clearing condition and the preference maximization of the agents as regards their budget constraints. The following definition is made to emphasize these aspects, based on which several generalizations of the Walras correspondence can be obtained by varying only the amount of the budget constraints.

Definition 2.2 $(z, P) \in \mathcal{A}(E) \times \Delta^l$ *is called an* **equilibrium** *of* E **relative to the budget constraints** $v_i(P)$, *if and only if*

1. $\sum_{j=1}^n z_j = \sum_{j=1}^n \omega_j + T$ *(market clearing condition)*
2. $z_i \in B_i(P) := \{x \in \mathbf{R}_+^l | \langle P, x\rangle \leq v_i(P)\} \quad \forall i$
3. $\forall x_i \in B_i(P): \quad z_i \succeq_i x_i \quad \forall i$.

That is, agent i chooses his consumption bundle z_i within his budget $B_i(P) := \{x \in \mathbf{R}_+^l | \langle P, x\rangle \leq v_i(P)\}$ such that his preferences are maximized. Given monotonicity of preferences $\langle P, z_i\rangle = v_i(P)$ is satisfied for all $i \in N$.[1] Thus

$$\sum_{j=1}^n v_j(P) = \langle P, \sum_{j=1}^n \omega_j + T\rangle = \langle P, \sum_{j=1}^n \omega_j\rangle + \langle P, T\rangle$$

[1]Independent of monotonicity of preferences this equality has to be fulfilled in equilibrium, because if anyone does not choose z_i in the boundary of his budget set, someone else has to exceed his budget set, which is not allowed in equilibrium.

$$= \sum_{j=1}^{n} \langle P, \omega_j \rangle + \langle P, T \rangle =: \sum_{j=1}^{n} w_j(P) + \langle P, T \rangle \qquad (1)$$

where $w_j(P) := \langle P, \omega_j \rangle$ are the budget constraints of the corresponding usual economy. Since the value $\langle P, T \rangle$ of the net trade vector does not have to be zero – think for example of $T \in \mathbf{R}^l_{++}$ –, one cannot expect that v_j and w_j are always equal. To give a starting point for our discussion we state

Definition 2.3 *(z, P) is called* **simple equilibrium**, *if it is an equilibrium relative to the budget constraints $v_i(P) := w_i(P)$.*

In the case of $T = 0$ this coincides with the original Walras equilibrium. Obviously, for almost every net trade vector there will be no simple equilibrium.

2.1 The Proportional Equilibrium

We are now looking for new concepts which generalize the concept of Walras equilibrium. As it will, in general, not be possible to choose budget constraints $v_i = w_i$ for all $i \in N$, one has to think about how to deviate from equality without causing too much damage (and without violating equation (1), of course). One way to solve the problem is to do it proportionally, i.e. to choose budget constraints v_i such that v_i/w_i is independent of i. This ensures equality $v_i(P) = w_i(P)$ in the case that $\langle P, T \rangle = 0$, which leads to the concept of simple equilibrium.

Definition 2.4 *(z, P) is called* **proportional equilibrium**, *if it is an equilibrium relative to the budget constraints $v_i(P) := \lambda_i \langle P, \sum_{j=1}^{n} \omega_j + T \rangle$ with $\lambda_i := \langle P, \omega_i \rangle / \langle P, \sum_{j=1}^{n} \omega_j \rangle$.*

Since monotonicity of preferences is assumed, j's share of the value of total endowments is the same in $E_T := ((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, T)$ and $E_0 := ((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, 0)$, i.e.

$$\frac{v_i(P)}{\sum_{j=1}^{n} v_i(P)} = \frac{v_i(P)}{\langle P, \sum_{j=1}^{n} \omega_j + T \rangle} = \lambda_i = \frac{w_i(P)}{\langle P, \sum_{j=1}^{n} \omega_j \rangle} = \frac{w_i(P)}{\sum_{j=1}^{n} w_j(P)}.$$

The foregoing concept is the same as the one defined by the budget constraints

$$\overline{v_i}(P) := \langle P, \omega_i \rangle + \lambda_i \langle P, T \rangle \text{ with } \lambda_i := \frac{\langle P, \omega_i \rangle}{\langle P, \sum_{j=1}^{n} \omega_j \rangle} .$$

Here, agents get their budget constraints w_i plus a share λ_i of the value of the net trade vector, where λ_i is proportional to w_i.[2] Both ways lead to the

[2]This concept is due to Thomson, see [12] and [13].

same concept because the budget constraints are equal as can be seen from the following chain of equations.

$$\begin{aligned}\frac{\overline{v_i}(P)}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} &= \frac{\langle P, \omega_i\rangle + \lambda_i \langle P, T\rangle}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} \\ &= \frac{\langle P, \omega_i\rangle}{\langle P, \sum_{j=1}^n \omega_j\rangle} \frac{\langle P, \sum_{j=1}^n \omega_j\rangle + \langle P, T\rangle}{\langle P, \sum_{j=1}^n \omega_j + T\rangle} \\ &= \lambda_i = \frac{v_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T\rangle}\end{aligned}$$

2.2 The Equal Sharing Equilibrium

Another idea of sharing $\langle P, T\rangle$ is, of course, to distribute it equally among the agents. This may sometimes lead to problems since not every agent is able to bear the nth part of the net trade vector and has to declare bankruptcy, i.e. is assigned the goods bundle 0. But in this context, we will also speak of an equilibrium, if for one agent i the bundle $\omega_i + T/n$ is not in the strictly positive orthant but equilibrium trades lead him to a strictly positive goods bundle. Since for agent i the necessity to declare bankruptcy heavily depends on prices, definition of equal sharing equilibrium is necessarily a bit blown up.

Definition 2.5 *(z, P) is called* **equal sharing equilibrium,** *if a permutation*

$$\Pi := \Pi_P : N \to N$$

exists with $0 \le \langle P, \omega_{\Pi(1)}\rangle \le \ldots \le \langle P, \omega_{\Pi(n)}\rangle$ *and*

$$m := m(P) := \max\{\ i \ |\langle P, \omega_{\Pi(i)}\rangle \ge -\frac{1}{n-i+1}\langle P, T + \sum_{j=1}^{i-1} \omega_{\Pi(j)}\rangle\} \quad ,$$

such that (z, P) is an equilibrium relative to the budget constraints

$$v_i(P) := \begin{cases} 0 & \text{if} \quad \Pi(i) < m(P) \\ \langle P, \omega_i\rangle + \frac{1}{n-m+1}\langle P, T + \sum_{j=1}^{m-1} \omega_{\Pi(j)}\rangle\} & \text{if} \quad \Pi(i) \ge m(P) \end{cases}$$

$m(P)$ is well defined since for $i = n$ the inequality only states the condition $\sum_{j=1}^n \omega_j + T \in \mathbf{R}_+^l$ evaluated at prices P. If the inequality holds for $i = 1$ nobody declares bankruptcy because of

$$\langle P, \omega_j\rangle \ge \langle P, \omega_{\Pi(1)}\rangle \ge -\frac{1}{n}\langle P, T\rangle \quad .$$

More precisely, exactly those agents i with $\Pi(i) < m(P)$ declare bankruptcy at prices P.

The existence of an equal sharing equilibrium is difficult to show because varying prices may change the set of bankruptcy declaring agents and may therefore create discontinuities.

A similar equilibrium notion which follows the idea of splitting up $\langle P, T\rangle$ equally among agents was given by Thomson [12] and [13]: He also divides agents into two groups. The bankruptcy declaring agents receive a 0-budget and the non-bankruptcy declaring agents split up $\langle P, T\rangle$ equally among themselves, i.e. their budget constraints are $\langle P, \omega_i\rangle + \frac{1}{\nu}\langle P, T\rangle$ where ν is the cardinality of non-bankruptcy declaring agents. This notion, of course, leads to the same outcome as our notion of equal sharing equilibrium as long as there are no bankruptcy declaring agents. But as soon as one agent declares bankruptcy, the budget constraints (0 or $\langle P, \omega_i + \frac{1}{\nu}T\rangle$) of the agents do not sum up to $\langle P, \sum_{j=1}^{n} \omega_j + T\rangle$. But this is a necessary condition for existence of an equilibrium relative to some budget constraints. Thus, Thomson's equal sharing equilibrium notion will lead to inexistence results for economies with "poor" agents and implicit deficit sharing problems (i.e. $\langle P, T\rangle < 0$ for equilibrium price vector candidates).

Moreover, there is a non-technical reason for choosing our notion instead of Thomson's: In every day life a person is called bankrupt, if claims of other persons on the property of the agent (in our case $\frac{1}{n}\langle P, T\rangle$) exceed what the agent owns (here $\langle P, \omega_i\rangle$). Then the possessions of the agent are used to compensate a part of the claims. The creditors only have to take care of the rest of the claims

$$\langle P, \omega_i + \frac{T}{n}\rangle \in (\langle P, \frac{T}{n}\rangle, 0) \quad .$$

This is an exact description of what happens when our equilibrium notion is applied. Thomson's notion does not use ω_i to compensate the creditors. This is like letting the bankrupt live on some sunny caribbean island without making him pay for his debts. Of course, this is almost more realistic, but can be described by an equilibrium notion only if one allows for free disposal.

2.3 Solution Concepts

We will get more insight into the nature of the equilibrium notions we defined up to now, if we compare them in different economies. This will be done by considering solution concepts. A **solution concept** (or just **solution**) Φ on $\mathcal{F}$ is a correspondence that assigns to each economy $E \in \mathcal{F}$ a (possibly empty) set of allocations $\Phi(E) \subset \mathcal{A}(E)$. For the largest possible class of generalized economies, $\mathcal{E}$, we are now able to introduce the following solution concepts.

Definition 2.6

- **Empty solution** $\emptyset$ *with* $\emptyset(E) := \emptyset$,
- **Solution of all allocations** $\mathcal{A}$ *with* $\mathcal{A}(E)$ *including all allocations of economy* E,
- **Pareto optimal solution** PO *with* $PO(E)$ *including all Pareto optimal allocations* $x \in \mathcal{A}(E)$,
- **Simple Walras solution** W_0 *with* $W_0(E)$ *including all simple equilibrium allocations of* E,
- **Usual Walras solution** W *with* $W(E) = W_0(E)$ *if* $T(E) = 0$ *and* $W(E) = \emptyset$ *else*,
- **Proportional solution** W_P *with* $W_P(E)$ *including all proportional equilibrium allocations of* E,
- **Equal sharing solution** W_E *with* $W_E(E)$ *including all equal sharing equilibrium allocations of* E..

The definitions of these concepts were straightforward. The starting point for our discussion will be the usual Walras concept W since this is the trivial extension (by $\emptyset$s) of Walras' concept for usual exchange economies. A less trivial extension is W_0 which was introduced by Dagan [1] and van den Nouweland, Peleg, Tijs [9] to deal with consistency in the framework of generalized economies. However, W_0 is empty almost everywhere (with respect to initial endowments and net trade vector). In the remainder of the paper we will concentrate on the solutions W_P and W_E since they both are extensions of W (and even of W_0) and non-empty on non-trivial classes of generalized economies.

3 Consistency

Consistency, i.e. consistent treatment of economic situations by solution concepts, is, of course, a question of the considered class of economic situations, too. To start with, we shall first define some classes of generalized economies we will discuss later on.

Definition 3.1 *The class of all generalized economies is denoted by $\mathcal{E}$. The class of all economies with agents whose preferences are strictly convex and monotone and are represented by n times continuously differentiable utility functions is denoted by $\mathcal{E}^n$.*

Note, that for $n \geq 2$ a proportional equilibrium exists for all economies $E \in \mathcal{E}^n$ as was shown in Korthues [3]. To assure existence of equal sharing equilibrium one, unfortunately, has to restrict considerations to subclasses of $\mathcal{E}$ in which every agent can bear T/n on his own (i.e. $\omega_i + T/n \in \mathbf{R}^l_+$). As is easy to imagine, this subclass is not closed with respect to formation of reduced economies. If an "extremist" – someone who wants to sell almost everything for one specific good – leaves an economy, he will in general leave open a possibly very great gap of this specific good. Not all remaining agents will then in general be able to cover their part of this gap by their initial endowments in this specific good.

Definition 3.2 *By $\mathcal{E}_E$ we denote the class of all generalized economies E which satisfy*

$$\forall i \in N(E): \quad \omega_i + T(E)/n \in \mathbf{R}^l_+$$

$\mathcal{E}^n_E$ is defined analogously.

Before we go on, let us first define the property of consistency in the setting of generalized economies.

Definition 3.3 *(Consistency) A solution concept Φ is called* **consistent** *on $\mathcal{F}$ if for all $E \in \mathcal{F}$ and for all $x \in \Phi(E)$ we get*

$$\forall S \neq \emptyset, S \subset N: \quad x^S \in \Phi(E^{S,x}) \text{ whenever } E^{S,x} \in \mathcal{F} \quad .$$

If one considered $E^{S,x} \in \mathcal{F}$ as a consequence and not as a condition, the consistency notion would be much stronger. It would then imply closedness of $\mathcal{F}$ with respect to formation of reduced economies. Since we will consider consistency only on closed classes of economies, this would not change anything.

Proposition 3.4 *The solution concepts $\emptyset$, $\mathcal{A}$, PO, W_0, W_E and W_P are consistent on the class $\mathcal{E}$ of generalized economies.*

Proof: Consistency for the empty and the all allocation solution are obvious. In case of the PO solution the proof can be seen as follows: Let $x \in PO(E)$ for some generalized economy E and let $E^{S,x}$ be a reduced economy for some proper subset S of agents. Assume that there exists an agent $i \in S$ who can improve upon x_i^S without making the other agents of S worse off. Let y^S be the new choice of S. Then $(y^S, x_{N\setminus S})$ is a feasible allocation of economy E and a Pareto improvement of x. But this yields a contradiction to Pareto optimality of x. Thus x^S is also Pareto optimal.
Consistency holds for the Walrasian concepts W_0, W_E and W_P because the shape of budget sets does not change if one goes to reduced economies and keeps prices fixed. Then, maximizing preferences within agents' budget sets yields the same outcome in the reduced economies. Following this line the proof for the proportional concept can be seen very easily whereas it turns out to be very technical for the case of the equal sharing solution. Therefore, we shall give a sketch of a proof for the equal sharing solution.
Let x be Pareto optimal. Then, there is a unique supporting price system P. Let x^S be an equal sharing outcome of the reduced economy $E^{S,x}$ for all two-agent groups $S \subset N$. Then, x can only be equal sharing equilibrium with respect to the price system P. Let therefore be $v_i := \langle P, x_i \rangle$, $w_i := \langle P, \omega_i \rangle$ and $t := \langle P, T \rangle$. All we have to show is that $v_i = v_i(P)$ where $v_i(P)$ is taken from the definition of equal sharing equilibrium. Without loss of generality we will assume

$$0 \leq \langle P, \omega_1 \rangle \leq \cdots \leq \langle P, \omega_n \rangle \quad .$$

Moreover let Ω be defined to be

$$\Omega(l) := \frac{\sum_{j \leq l-1} w_j + t}{n - l + 1} \quad .$$

The proof is shown once the following two lemmas have been proved.

Lemma 3.5 *There is an $m \in N$ such that*

$$\forall i < m \ : \ v_i = 0$$
$$\forall i \geq m \ : \ v_i = w_i + \Omega(m) > 0 \quad .$$

Lemma 3.6 $m = m(P)$, *where* $m(P)$ *is taken from the definition of equal sharing equilibrium.*

The proof for the first lemma can be seen by considering the right reduced economy $E^{S,x}$ for $S = \{i, j\}$. For the proof of the second lemma we proceed indirectly:

- Suppose $m < m(P)$:
  Then $v_m = w_m + \Omega(m) \leq 0$ by the definition of $m(P)$, which is a contradiction to the construction in the first lemma.

- Suppose $m \geq m(P)$:
  Then, $m - 1 \geq m(P)$ and therefore

  $$w_{m-1} + \Omega(m-1) > 0 \quad .$$

  But take $S := \{m-1, m\}$. Then, we get $v_{m-1} = 0$ and $v_m = w_m + \Omega(m) > 0$ and $t_{m-1,m} := \langle P, T^{S,x} \rangle = -w_{m-1} + \Omega(m)$.

  $$\begin{aligned} \Rightarrow 0 = v_{m-1} &\geq w_{m-1} + \frac{t_{m-1,m}}{2} = \frac{w_{m-1} + \Omega(m)}{2} \\ &\geq \frac{w_{m-1} + \Omega(m-1)}{2} \ (*) \\ &> 0 \text{ which is a contradiction.} \end{aligned}$$

  $(*)$ holds since for all $k \geq m(P) + 1$ the function Ω is increasing, i.e. $\Omega(k) \geq \Omega(k-1)$, as can easily be demonstrated. □

Proposition 3.7 *Intersections and unions of consistent solution concepts are consistent as well.*

The proof is straightforward and can be omitted.
This proposition enables us to consider minimal consistent extensions (MCE) and maximal consistent subsolutions (MCS) of solution concepts. Furthermore, we can define a minimal non-empty consistent extension (MNCE) of some solution concept Φ, being a minimal extension among the non-empty and consistent extensions of Φ. A solution concept may possibly have several MNCEs, whereas it has a unique MCE

$$\overline{\Phi} := \bigcap_{\Psi \supset \Phi,\, \Psi \text{cons.}} \Psi$$

and a unique MCS

$$\underline{\Phi} := \bigcup_{\Psi \subset \Phi, \Psi \text{cons.}} \Psi \quad .$$

Corollary 3.8 *$W_E \cap W_P$ and $W_E \cup W_P$ are consistent.*

For the next result we will use an argument similar to the Debreu-Sonnenschein-Mantel Theorem. Therefore, it will be necessary to restrict the considerations to the class of twice continuously differentiable economies.

Theorem 3.9 *W_P is a MNCE (minimal non-empty consistent extension) of W on $\mathcal{E}^2$.*

Proof: Suppose Φ is a non-empty consistent extension W with $W \subset \Phi \subset W_P$ with $\Phi \neq W_P$. Then

$$\Phi \neq W_P \quad \iff \quad \exists E \in \mathcal{E}^2 : \Phi(E) \neq W_P(E) \wedge \Phi(E) \subset W_P(E) \quad .$$

Let E be an economy such that the foregoing assumption is fulfilled and x be an allocation in $W_P(E) \setminus \Phi(E)$. Now, x is as well Pareto optimal and has a unique supporting price system as an (usual) equilibrium. Together with this price system the tuple $(x, \bar{P})$ is a proportional equilibrium.

Now, the idea is to construct a larger economy $\bar{E}$ such that $W_P(\bar{E})$ is a singleton $\{y\}$ and the reduced economy $\bar{E}^{N,y}$ coincides with E.

This will be done by constructing an excess demand function which has only one zero and can be represented as the excess demand function of the desired economy $\bar{E}$. Note that in this case we have to consider proportional excess demand functions since we want to state a result on proportional equilibrium.

Proportional excess demand functions are defined as follows: Given a price system $P \in \overset{\circ}{\Delta}{}^l$ proportional budget sets of the agents are well defined and compact, so that under our assumptions on preferences agents have unique demand. It can be shown that in our context this demand varies continuously in prices. The sum of individual proportional demand functions minus initial endowments forms the desired excess demand function. A price system is a proportional equilibrium price system if and only if it is a zero of the proportional excess demand function.

Let z^P be the proportional excess demand function of economy E. Then we need to construct an economy E' with proportional excess demand function

$z^{P'}$ such that $\bar{z}^P := z^P + z^{P'}$ has only one zero at some $\bar{P}$ so that $\bar{E} := E \cup E'$ is the desired economy. This looks like a standard application of the Debreu-Sonnenschein-Mantel Theorem.[3]

However, I will choose a different approach by constructing a comparably simple one-agent economy where the agent has Cobb-Douglas preferences.

Let this agent be given by initial endowments $\omega' := \alpha \sum_{j=1}^n \omega_j$ and utility function $u(x) := \prod_{k=1}^l x_k^{\alpha_k}$ with

$$\alpha_k := \frac{(\sum_{j=1}^n \omega_j + T)_k}{\langle \bar{P}, \sum_{j=1}^n \omega_j + T\rangle} \bar{P}_k \quad .$$

α will be a positive constant and dertermined later on. Define the net trade vector T' of the economy E' in construction to be $T' := \alpha T$.

Now, proportional demand of this agent – and thus of the whole economy E' – is given by

$$d^P(P) := \langle \bar{P}, \omega' + T'\rangle (\frac{\alpha_1}{P_1}, \ldots, \frac{\alpha_l}{P_l}) = \alpha \langle \bar{P}, \sum_{j=1}^n \omega_j + T\rangle (\frac{\alpha_1}{P_1}, \ldots, \frac{\alpha_l}{P_l}) \quad .$$

and proportional excess demand by

$$z^{P'}(P) := \alpha \left[\langle \bar{P}, \sum_{j=1}^n \omega_j + T\rangle (\frac{\alpha_1}{P_1}, \ldots, \frac{\alpha_l}{P_l}) - (\sum_{j=1}^n \omega_j + T) \right] \quad .$$

By definition of $(\alpha_1, \ldots, \alpha_n)$ the proportional excess demand function has the unique zero $\bar{P}$.

Then, $\bar{P}$ is also a zero of $\bar{z}^P := z^P + z^{P'}$. But by addition other zeros may appear which have not been there before. To avoid this, we show that outside a sufficiently small ϵ-boundary (i.e. for $P \in \overset{\circ}{\Delta}{}^l$ with $P_k \geq \epsilon$ for all components $k = 1, \ldots, l$), $z^{P'}$ stricty dominates z^P (i.e. $\|z^{P'}(P)\| > \|z^P(P)\|$ for all $P \in \Delta_\epsilon^l \setminus \{\bar{P}\}$).

To see domination show that the Jacobian of $z^{P'}$ has full rank everywhere.

$$(Dz^{P'}(P))_{ki} = \frac{\partial \left[\langle P, \sum_{j=1}^n \omega_j + T\rangle \frac{\alpha_k}{P_k} - (\sum_{j=1}^n \omega_j + T)_k \right]}{\partial P_i}$$

[3]The problem could in principle be solved by deriving $z^{P'}$ from some function $z^{W'}$ which can be shown to be the usual excess demand function of some economy with differentiable agents. The latter could be done by a differentiable version of the Sonnenschein-Debreu-Mantel Theorem as, for example, stated in Mas-Colell's book "The Theory of General Economic Equilibrium, A Differentiable Approach". But transforming $z^{W'}$ into $z^{P'}$ turns out to be very difficult if one wants to preserve certain properties.

$$= \begin{cases} (\sum_{j=1}^n \omega_j + T)_i \frac{\alpha_k}{P_k} & : \quad i \neq k \\ (\sum_{j=1}^n \omega_j + T)_i \frac{\alpha_k}{P_k} - \langle P, \sum_{j=1}^n \omega_j + T \rangle \frac{\alpha_k}{P_k^2} & : \quad i = k \end{cases} .$$

By dividing every row by $\frac{\alpha_k}{P_k}$ and every column by $(\sum_{j=1}^n \omega_j + T)_j$ one sees that $Dz^{P'}(P)$ has full rank if and only if $\det A \neq 0$ for

$$A = (A_{ki}) \text{ with } A_{ki} = \begin{cases} 1 & : \quad i \neq k \\ 1 - \frac{\langle P, \sum_{j=1}^n \omega_j + T \rangle}{(\sum_{j=1}^n \omega_j + T)_i P_i} & : \quad i = k \end{cases} .$$

But this is obviously true.

In the next step we will show that $\|z^P\|/\|z^{P'}\|$ is bounded on Δ_ϵ^l:

- $\|z^P\|/\|z^{P'}\|$ is continuous on $\Delta_\epsilon^l \setminus N(z^{P'}) = \Delta_\epsilon^l \setminus \{\bar{P}\}$
- Sufficiently close to $\bar{P}$ we have

$$\begin{aligned} \|z^P(P)\| &\leq 2\|Dz^P(P)(P-\bar{P})\| \leq \bar{m}\|P-\bar{P}\| \text{ and} \\ \|z^{P'}(P)\| &\geq \frac{1}{2}\|Dz^{P'}(P)(P-\bar{P})\| \geq \underline{m}\|P-\bar{P}\| \end{aligned}$$

with $\bar{m} < \infty$ and $\underline{m} > 0$ so that

$$\frac{\|z^P(P)\|}{\|z^{P'}(P)\|} \leq \bar{m} - \underline{m} \quad .$$

Both points together yield the boundedness of $\frac{\|z^P(P)\|}{\|z^{P'}(P)\|}$ on Δ_ϵ^l by some $M > 0$.

Without loss of generality assume M to be 1 (choose α appropiatly), so that for all $P \neq \bar{P}$ we get $\|z^P(P)\| < \|z^{P'}(P)\|$. Then $\bar{z}^P := z^P + z^{P'}$ has only one zero (at $\bar{P}$) because

- $\bar{z}^P$ does not have a zero on $\Delta_\epsilon^l \setminus \{\bar{P}\}$ because of the triangle inequality,
- $\bar{z}^P$ does not have a zero outside Δ_ϵ^l because of the boundary conditions both z^P and $z^{P'}$ have to satisfy.

Last but not least we have to verify that $\bar{z}^P$ is really the proportional excess demand function of the economy $\bar{E} := E \cup E'$. To see this check that

$$\frac{\langle P, \omega' + T' \rangle}{\langle P, \omega' \rangle} = \frac{\alpha \langle P, \sum_{j=1}^n \omega_j + T \rangle}{\alpha \langle P, \sum_{j=1}^n \omega_j \rangle} = \frac{\langle P, \sum_{j=1}^n \omega_j + T \rangle}{\langle P, \sum_{j=1}^n \omega_j \rangle} =: \lambda(P)$$

and that every agent (especially the one of economy E') maximizes preferences within his proportional budget set

$$B_i^P(P) := \{ y \in \mathbf{R}_+^l \quad | \quad \langle P, y \rangle \leq \lambda(P) \langle P, \omega_i \rangle \} \quad .$$

Finally, we are done with the proof of the theorem: Due to the non-emptyness of our solution concept Φ we get $\Phi(\bar{E}) = W_P(\bar{E}) = \{y\}$ where $y = (x, \omega' + T')$. Then $\bar{E}^{N,y} = E$, so that

$$x = y^N \in \Phi(\bar{E}^{N,y}) = \Phi(E) \quad ,$$

which contradicts the assumption $x \notin \Phi(E)$. □

A similar result can be shown for the equal sharing solution W_E. Unfortunately, a genereal existence result for W_E does not hold on $\mathcal{E}$ or one of its subclasses $\mathcal{E}^n$, because – as already mentioned – the set of bankruptcy declaring agents may change with varying prices so that discontinuities cannot be excluded. Restricting our considerations to $\mathcal{E}_E$ or one of its subclasses $\mathcal{E}_E^n$ solves the existence problem. Nethertheless, $\mathcal{E}_E$ is not closed with respect to formation of reduced economies under W_E as can be seen in the following

Example 3.10 *Let E be an economy with three agents with initial endowments $\omega_1 = \omega_2 = (5,5)$ and $\omega_3 = (1,1)$, net trade vector $T = (0,0)$ and preferences $\succeq_i$ such that at prices $P = 1/2(1,1)$ agents want to consume $x_1 = (1,9)$, $x_2 = (9,1)$ and $x_3 = (1,1)$ respectively. For subset $S = \{2,3\}$ of agents we now get*

$$T^{S,x} = T + \omega_1 - x_1 = (0,0) + (4,-4) = (4,-4) \quad .$$

Then $\omega_3 + T^{S,x}/2 = (1,1) + (2,-2) = (3,-1) \notin \mathbf{R}^2_{++}$.

But consistency of a solution concept does not require the class of economies to be closed with respect to formation of reduced economies. So, we can state

Theorem 3.11 *W_E is a MNCE (minimal non-empty consistent extension) of W on $\mathcal{E}^2_E$.*

The proof is very similar to the one of the foregoing theorem. It will turn out that it is even simpler since we do not have to deal with proportional (excess) demand functions: For economies in $\mathcal{E}_E$ equal sharing demand functions of the agents are just translated usual demand functions. One can think of agents having initial endowments $\omega_i + T/n \in \mathbf{R}^l_{++}$.

Let Φ be a consistent solution concept with $\emptyset \neq \Phi(E') \subset W_E(E')$ for all $E' \in \mathcal{E}^2_E$. Assume that there is an economy $E \in \mathcal{E}^2_E$ such that $x \in W_E(E) \setminus \Phi(E) \neq \emptyset$. Let $\bar{P}$ be the price system of x as a equal sharing equilibrium. The equal sharing excess demand function d of E has a zero in $\bar{P}$, but possibly more then one zero. Construct an agent $n+1$ with initial endowments ω_{n+1} and preferences $\succeq_{n+1}$ such that the economy $E' := (\omega_{n+1}, \succeq_{n+1}, T/n) \in \mathcal{E}_E^2$ and such that the excess demand function d' of E' has a zero in $\bar{P}$ and d' dominates d (like in the proof of the last theorem).

Then $E \cup E' := ((\omega_i)_{i=1,\dots,n+1}, (\succeq_i)_{i=1,\dots,n+1}, \frac{n+1}{n}T) \in \mathcal{E}_E^2$. Thus, $\{y\} := \{(x, \omega_{n+1})\} = W_E(E) = \Phi(E)$ and

$$x = y^N \in \Phi(E^{N,y}) = \Phi(E) \quad ,$$

which is a contradiction to our assumption. □

4 Converse Consistency

Following the line of van den Nouweland, Peleg, Tijs [9] one can also consider consistency the other way round: Let x be an allocation of some generalized economy E, such that all reduced economies $E^{S,x}$ with $S \subset N, S \neq N$ agree upon x^S as one (but not necessarily the only) reasonable outcome. Why then not choosing x as an outcome of economy E? More formally

Definition 4.1 *A solution concept Φ is called* **conversely consistent** *(CO-CONS) on $\mathcal{F}$, if for every economy $E \in \mathcal{F}$ with at least 3 agents and $x \in \mathcal{A}(E)$ we get*

$$\left[\forall S \subset N, |S| \leq 2 : E^{S,x} \in \mathcal{F} \text{ and } x^S \in \Phi(E^{S,x}) \Rightarrow x \in \Phi(E)\right].$$

The restriction to $|E| \geq 3$ is very important. Dropping this assumption and considering all proper subsets S of the set of agents N leads to the fact that only extensions of PO could be conversely consistent **and** everywhere non-empty.

Van den Nouweland, Peleg and Tijs [9] use a weaker version of converse consistency, since their definition only requires considering allocations $x \in PO(E)$ rather than general allocations. They characterize the simple Walras solution by means of converse consistency and are thus interested in a version of converse consistency which is as weak as possible. Their definition makes PO conversely consistent by definition.

Proposition 4.2 *The solution concepts $\emptyset$ and $\mathcal{A}$ are conversely consistent on $\mathcal{E}$. On the contrary PO, W_0, W_E and W_P are conversely consistent only on the class $\mathcal{E}^2$.*

Proof: The concepts $\emptyset$ and $\mathcal{A}$ are obviously conversely consistent on $\mathcal{E}$. To see that Walrasian solutions and the Pareto solution are not COCONS on $\mathcal{E}$, construct an allocation $x := (x_1, x_2, x_3)$ of an economy $E \in \mathcal{E}$ with at least 3 goods such that the cones of individual marginal valuations ("potential equilibrium price vectors") have pairwise non-empty intersection and that the intersection of all 3 cones is empty. Then there is no common supporting price vector for x_1, x_2 and x_3, so that x can be neither a Pareto optimum nor some Walrasian outcome. □

For a proof of COCONS for PO on $\mathcal{E}^2$ see Goldman and Starr [2].[4] A proof for converse consistency of W_0 can be found in van den Nouweland, Peleg, Tijs [9]. The cases of W_E and W_P can be treated similarly. In addition, one has to verify, that equal resp. proportional splitting of $\langle P, T^{S,x}\rangle$ in the budgets of the reduced economies $E^{S,x}$ for all $S \subset N$ implies equal resp. proportional splitting of $\langle P, T\rangle$ in E. □

Proposition 4.3 *Intersections of conversely consistent solution concepts are conversely consistent as well.*

Proposition 4.4 *Unions of conversely consistent solution concepts are not necessarily conversely consistent.*

For the proof we will use the following two solution concepts:

- **Equal Division Solution** *ED*
  $x \in ED(E) :\Leftrightarrow \forall i \in N \quad x_i = (\sum_{j \in N(E)} \omega_j + T)/|N(E)|$
- **No Envy Solution** *NEnv*
  $x \in NEnv(E) :\Leftrightarrow \forall i,j \in N \quad \omega_i + (x_j - \omega_j) \in \mathbf{R}^l_+ \wedge x_i \succeq_i \omega_i + (x_j - \omega_j)$.

[4] They use the notion of t-wise optimality and show for the class of usual economies with agents whose preferences can be represented by quasi-concave C^2-utility functions that pairwise optimality implies Pareto optimality. Since for the concept of Pareto optimality initial endowments do not matter and only total endowments are important, we can easily adopt this approach to our setting.

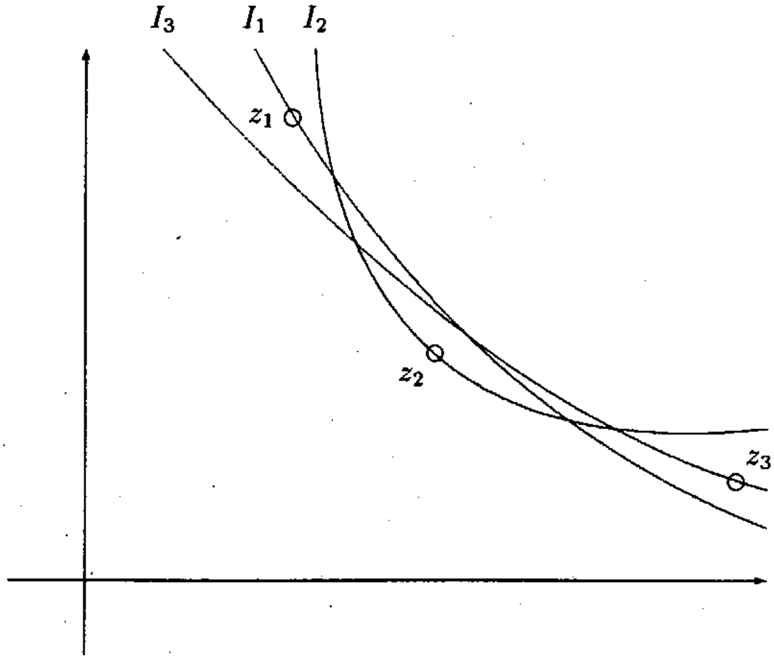


Figure 1: Counterexample for Proposition 4.4

Both solutions are obviously conversely consistent. Let us now show that the union $ED \cup NEnv$ is not conversely consistent. Construct an economy E with three agents and an allocation x such that $x_1 = x_3 \neq x_2$ and $\omega_1 \neq \omega_3$. And such that for the net trades $z_i := x_i - \omega_i$ we get

$$\begin{aligned} z_1 \succeq_1 z_2 \quad \wedge \quad z_2 \succeq_2 z_1 \quad \wedge \\ z_2 \succeq_2 z_3 \quad \wedge \quad z_3 \succeq_3 z_2 \quad . \end{aligned}$$

This constellation, which is also illustrated in Figure 4, yields

$$\begin{aligned} (x_1, x_2) &\in NEnv(E^{\{1,2\},x}) \text{ and} \\ (x_2, x_3) &\in NEnv(E^{\{2,3\},x}) \text{ and} \\ (x_1, x_3) &\in ED(E^{\{1,3\},x}) \text{ but} \\ (x_1, x_2, x_3) &\notin NEnv(E) \text{ since } z_1 \not\succeq_1 z_3 \text{ and} \\ (x_1, x_2, x_3) &\notin ED(E) \text{ since } x_1 \neq x_2 \square \end{aligned}$$

It is not immediate to find a non-trivial counterexample since unions of typical representatives of Walrasian solutions are conversely consistent as well. This is due to the following fact: If equilibrium prices are unique (under smoothness, for example) and (x_1, x_2) and (x_2, x_3) form equilibria of a special type, x_1, x_2 and x_3 are supported by the same price vector, so that in general (x_1, x_3) also forms an equilibrium of that special type. Apart from many Walrasian solutions a lot of other solutions show this transitivity property.

Definition 4.5 *A solution concept Φ on $\mathcal{F}$ is called* **transitive** *if for every $E \in \mathcal{F}$ and for every $x \in \mathcal{A}(E)$ the relation $R_{\Phi,E,x}$ defined by*

$$R_{\Phi,E,x} := \{(i,j) \in N(E)^2 | i \neq j \wedge (x_i, x_j) \in \Phi(E^{\{i,j\},x})\}$$

on $N(E)$ is transitive.

Note that $R_{\Phi,E,x}$ is by definition irreflexive and symmetric.

Lemma 4.6 *Let Φ_1 and Φ_2 be two transitive solutions. For $\Phi := \Phi_1 \cup \Phi_2$ let there be $E \in \mathcal{F}$ and $x \in \mathcal{A}(E)$ such that for all two-agent sets S we get $x^S \in \Phi(E^{S,x})$. Then*

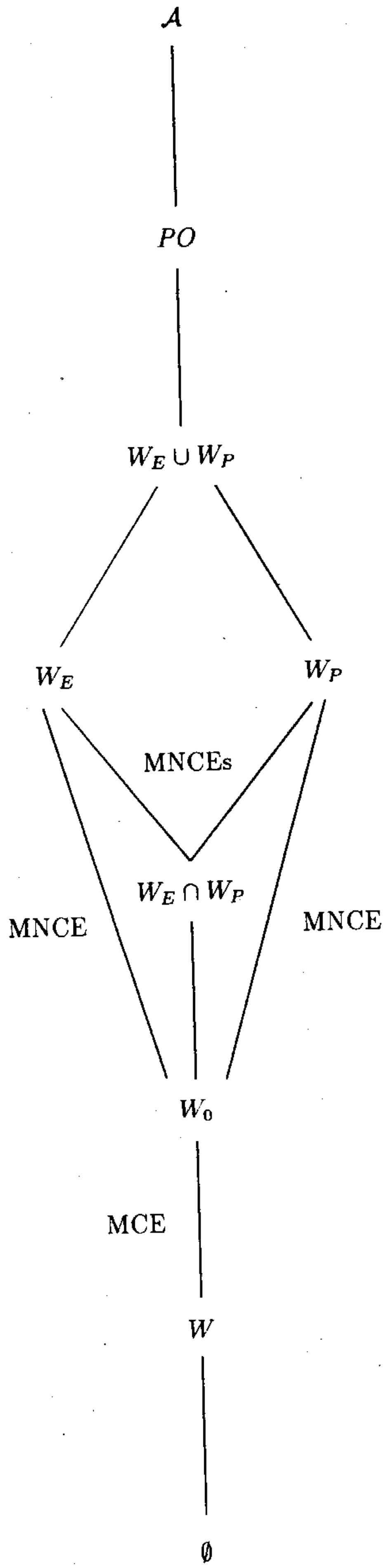
$$\forall S \text{ with } |S| = 2 \ : \ x^S \in \Phi_1(E^{S,x}) \quad \text{or}$$
$$\forall S \text{ with } |S| = 2 \ : \ x^S \in \Phi_2(E^{S,x})$$

Proof: Let $E \in \mathcal{F}$, $x \in \mathcal{A}(E)$ and $R_i := R_{\Phi_i,E,x}$ for $i = 1, 2$. Then, starting from $R_1 \cup R_2 = R_{\neq} := \{(i,j) \in N(E)^2 | i \neq j\}$ all we have to show is $R_1 = R_{\neq}$ or $R_2 = R_{\neq}$. This will be proved by induction over $n := |N(E)|$. For $n = 2$ the proof is trivial since both relations are symmetric. Let the statement be true for $n = k - 1 \geq 2$. Let R'_1, R'_2 and $R'_{\neq}$ be the restrictions of R_1, R_2 and $R_{\neq}$ to $N(E) \setminus \{n\}$. Then $R'_1 = R'_{\neq}$ or $R'_2 = R'_{\neq}$. Without loss of generality we can assume $R'_1 = R'_{\neq}$. Then there are two cases.

1. There is an $i \neq n$ such that iR_1n.
   Then, by transitivity for all $j \in N(E)$ we get jR_1n and thus $R_1 = R_{\neq}$.

2. There is no $i \neq n$ such that iR_1n.
   That means, that for all $i \neq n$ we get iR_2n. But this implies jR_2k for all $j, k \in N(E) \setminus \{n\}$ with $j \neq k$ since jR_2n and nR_2k hold. Hence, $R_2 = R_{\neq}$. □

Theorem 4.7 *Let Φ_1 and Φ_2 be COCONS and transitive. Then $\Phi := \Phi_1 \cup \Phi_2$ is COCONS.*



	CONS	COCONS	TRANS	N-EMPTY
$\mathcal{A}$	×	×	×	×
PO	×	on $\mathcal{E}^2$	on $\mathcal{E}^2$	×
$W_E \cup W_P$	×	on $\mathcal{E}^2_E$	no	on $\mathcal{E}^2$
W_E, W_P	×	on $\mathcal{E}^2$	on $\mathcal{E}^2_E/\mathcal{E}^2$	on $\mathcal{E}^2_E/\mathcal{E}^2$
$W_E \cap W_P$	×	on $\mathcal{E}^2$	on $\mathcal{E}^2_E$	no
W_0	×	on $\mathcal{E}^2$	on $\mathcal{E}^2$	no
W	no	on $\mathcal{E}^2$	on $\mathcal{E}^2$	no
$\emptyset$	×	×	×	no

By using the foregoing lemma the result is obvious.

Proposition 4.8 *W_P is transitive on $\mathcal{E}^2$ and W_E is transitive on $\mathcal{E}^2_E$.*

The proofs for the chosen domains are straightforward. To see that W_E is not transitive on $\mathcal{E}^2$ consider the following example: Take a Pareto optimum x for a three-agent economy such that the derived values are $w_1 = 5$, $w_2 = 1$, $w_3 = 10$ and $t = -6$ resp. $v_1 = 2$, $v_2 = 0$ and $v_3 = 8$. Then, the reduced economies for $S_{1,2} = \{1,2\}$ and $S_{2,3} = \{2,3\}$ have $x^{S_{1,2}}$ resp. $x^{S_{2,3}}$ as equal sharing equilibrium outcome. But x is not an equal sharing equilibrium outcome since both non-bancruptcy declaring agents 1 and 3 do not pay the same amount (-3 resp. -2).

Corollary 4.9 *$W_E \cup W_P$ is COCONS on $\mathcal{E}^2$.*

Like in the case of consistency we can define minimal converse consistent extensions (MCCE) and maximal conversely consistent subsolutions (MCCS). Whereas MCCEs are unique, MCCSs need not be unique, since the union of conversely consistent and transitive solution concepts is conversely consistent, but might not be transitive.

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