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Abstract

Some authors present models in which they show that the Nash bargaining solution fails to be Maskin monotonic and hence cannot be implemented in Nash equilibrium. We find these results misleading and discuss how implementability of the Nash bargaining solution can be discussed in utility space. Arguing that the status quo should be treated as initial endowments we keep it fixed and show that with this assumption the Nash bargaining solution satisfies Maskin monotonicity. The key property used in the proof is independence of irrelevant alternatives which also turns out to be a necessary condition for implementability. We also show that the Nash bargaining solution satisfies a sufficient condition for implementability independent of the number of agents.

Keywords: Nash Bargaining, Implementation, IIA, Maskin Monotonicity
JEL Classification: C71, C78, D71

1 Introduction

The seminal references in the bargaining literature are the two papers by Nash (1950, 1953). Nash (1950) introduces the model of the bargaining problem consisting of a set of feasible payoffs and a status quo point and presents a solution which is uniquely determined by a list of properties and assigns to every bargaining problem a point in its payoff region. This axiomatic approach is further clarified in Nash (1953) where it is also complemented by the formulation of a noncooperative game, Nash's demand game, in which players strategies are individual utility levels and

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the payoff depends on the feasibility of the players' demands, the Nash equilibria of which are considered as candidates for a solution of the bargaining problem. With complete information, the set of Nash equilibria of this game turns out to coincide with the set of individually rational and strongly Pareto efficient points in the payoff region. Introducing uncertainty about the exact shape of the set of feasible payoffs, however, again the Nash bargaining solution emerges.

Following Nash, many different solution concepts have been proposed and axiomatically characterized (e. g. Raiffa (1953), Harsanyi (1955), Kalai and Smorodinsky (1975), Kalai (1977a), Kalai (1977b), and Perles and Maschler (1981), see Thomson (1994) for a good survey of this field). Also, his noncooperative approach has initiated a whole literature concerned with noncooperative foundations of cooperative solution concepts.[1] This line of research is commonly referred to as the Nash program. The classic contribution for bilateral bargaining is due to Binmore, Rubinstein, and Wolinsky (1986) who obtain the Nash bargaining solution as the limit of the unique subgame perfect equilibrium of an alternating offer game a la Rubinstein (1982) when the probability of breakdown of the negotiation goes to zero. Krishna and Serrano (1996) generalize their approach to the case of more than two players. Howard (1992) obtains the Nash bargaining solution as the unique utility payoff of the subgame perfect equilibria of a finite extensive form game with perfect information. Trockel (1995) has shown that for the two-player case the Nash bargaining solution can be obtained as the unique equilibrium in dominant strategies of a game which is very similar to Nash's demand game.

While the Nash program is intended to support cooperative solution concepts by constructing noncooperative games based on the data of the cooperative game, the implementation literature focuses on the issue of information concerning the underlying data of the cooperative game itself. Given that the planner cannot observe agents' preferences, she needs to design a mechanism such that for all possible preference profiles agents' behavior, as modelled by some equilibrium notion, in the noncooperative game induced by the mechanism leads to exactly the outcomes prescribed by the solution to be implemented. This literature has been initiated by Hurwicz (1972). His negative result on implementation in dominant strategies (cf. also Gibbard (1973) and Satterthwaite (1975)) were overcome by Maskin's (1977) contribution on Nash implementation which was made thorough by Repullo (1987) and Saijo (1988). The implementation literature has since provided results on implementation in various equilibrium concepts[2]. Howard (1992) shows how the Nash

[1] See also Harsanyi and Selten (1972) who view Nash's noncooperative game with complete information as the starting point of the bargaining problem and his axiomatic approach as a way to select a particular equilibrium of this game.

[2] E. g. subgame perfect equilibrium (cf. Moore and Repullo (1988) and Abreu and Sen (1990)), backward induction (cf. Herrero and Srivastava (1992)), perfect equilibrium, (cf. Sjöström (1993)), equilibrium in undominated strategies (cf. Palfrey and Srivastava (1991)), and virtual implemen-

bargaining solution can be implemented in subgame perfect equilibrium of a game with a finite number of stages and perfect information.

He as well as Serrano (1996), who offers a model which intends to reconcile the different approaches of the Nash program and the implementation approach, claim the Nash bargaining solution fails to be Maskin monotonic (cf. Maskin (1977)) and hence cannot be implemented in Nash equilibrium. Trockel (1997) suspects that this is due to the particular choice of the outcome space of the mechanisms these authors consider and suggests a different approach which yields an implementation of the Nash bargaining solution (even in weakly dominant strategies). In this note a third approach is taken to discuss whether the Nash bargaining solution is monotonic. The main difference with the literature lies in the fact that the discussion stays completely in utility space, which seems to be more faithful to the original formulation of the bargaining problem by Nash (1950) and his attempt towards a noncooperative foundation of his bargaining solution (Nash (1953)). Also, we clearly distinguish the set of feasible payoffs, which is represented in the agents' preferences, and the status quo, which is seen as initial endowments and hence, in a first step, assumed to be fixed as it is customary in the implementation literature.

The paper is organized as follows. We start by giving a definition of the class of problems under consideration and of the Nash bargaining solution. Then we briefly review the arguments given in the literature to show that this solution fails to be Maskin monotonic. In the next section we present our approach and show that the Nash bargaining solution is Maskin monotonic if we fix the status quo. In fact, it turns out that the axiom of independence of irrelevant alternatives is crucial for this result. We show that if we restrict attention to individually rational bargaining solutions, independence of irrelevant alternatives of the bargaining solution and Maskin monotonicity of the associated social choice function are equivalent, Following this, we consider sufficient conditions for implementability in Nash equilibrium. Since the Nash bargaining solution fails to satisfy no veto power (cf. Maskin (1977)) and strong monotonicity (cf. Danilov (1992) and Yamato (1992)), we turn to a condition presented by Moore and Repullo (1990) and made more operational by Sjöström (1991), that is both necessary and sufficient for implementation in Nash equilibrium with three or more agents. This condition has the additional advantage that a slightly stronger version that has been independently discovered by Dutta and Sen (1991) assures implementability with two players. We can show that both their conditions hold for a class of bargaining solutions including Nash's. We conclude with some remarks on lines of further research.

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tation (cf. Matsushima (1988) or Abreu and Sen (1991)).

2 The Bargaining Problem and Nash's Solution

Let U be a set, the universe of players, with $|U| \geq 2$. U can be finite or infinite. Let $\mathcal{I} := \{I \subseteq U \,:\, |I| \in \mathbb{N}, |I| \geq 2\}$. $\mathcal{I}$ is the set of possible sets of players that we will consider. For $I \in \mathcal{I}$ with $|I| = N$ we often identify I with the set $\{1, \dots N\}$. For $I \in \mathcal{I}$ the space of utility allocations is $\mathbb{R}^I$. Define $\mathcal{A} := \bigcup_{I \in \mathcal{I}} \mathbb{R}^I$.

A bargaining problem for player set $I \in \mathcal{I}$ is (S, d) where $S \subset \mathbb{R}^I$ and $d \in \mathbb{R}^I$. The interpretation is that S is the set of feasible utility allocations the players can agree upon and d is the status quo, i. e., the utility payoff the players obtain if they do not reach an agreement.

Definition 2.1
A bargaining problem (S, d) with player set I is regular if the following conditions are satisfied.

(i) $S \subset \mathbb{R}^I$ is nonempty, closed, convex, and comprehensive[3].

(ii) $d \in \operatorname{int} S$.

(iii) The set $S_{\geq d} := \{x \in S \,:\, x \geq d\}$ is compact.

For fixed player set $I \in \mathcal{I}$, and fixed status quo $d \in \mathbb{R}^I$ we will denote the set of all regular bargaining problems with this player set and status quo by $\mathcal{B}_d^I$. Also we write $\mathcal{B}^I := \bigcup_{d \in \mathbb{R}^I} \mathcal{B}_d^I$ and $\mathcal{B} := \bigcup_{I \in \mathcal{I}} \mathcal{B}^I$ for the set of all regular bargaining problems with fixed player set I and with any player set from $\mathcal{I}$, respectively.

A bargaining solution ψ is a mapping $\psi : \mathcal{B} \to \mathcal{A}$, such that $\psi\big((S, d)\big) \in S$ for all $(S, d) \in \mathcal{B}$. If ψ is a bargaining solution, we will also use ψ for its restriction to subsets of $\mathcal{B}$, in particular, for fixed player set $I \in \mathcal{I}$ we view ψ as a mapping $\psi : \mathcal{B}^I \to \mathbb{R}^I$.

We will be especially interested in the Nash bargaining solution ν (cf. Nash (1950)) which is a mapping $\nu : \mathcal{B} \to \mathcal{A}$, defined by

$$\nu\big((S, d)\big) = \arg\max \left\{ \prod_{i \in I} x(i) - d(i) \,:\, x \in S_{\geq d} \right\}$$

for all $(S, d) \in \mathcal{B}$.

Next, we list some properties of bargaining solutions that we will use later in the paper. [4] The first is actually the key property (axiom) used in the characterization of the Nash bargaining solution (cf. Nash (1950, Assumption 7) or Nash (1953, Axiom V)) and will also play a prominent role in our discussion.

[3] A set $A \subseteq \mathbb{R}^I$ is comprehensive, if $A - \mathbb{R}^I_+ = A$.

[4] The domain we consider will always be $\mathcal{B}$, while in general of course, all properties can be formulated relative to different domains; whether a particular bargaining solution satisfies a certain property usually depends on the chosen domain.

Definition 2.2
A bargaining solution ψ satisfies **independence of irrelevant alternatives (IIA)** *if for all $I \in \mathcal{I}$ and all pairs $\big((S,d),(\tilde{S},d)\big) \in \mathcal{B}^I \times \mathcal{B}^I$ if $\tilde{S} \subseteq S$ and $\psi\big((S,d)\big) \in \tilde{S}$ then $\psi\big((\tilde{S},d)\big) = \psi\big((S,d)\big)$.*

Definition 2.3
A bargaining solution ψ is **individually rational** *if $\psi\big((S,d)\big) \in S_{\geq d}$ for all $(S,d) \in \mathcal{B}$. It is* **strongly individually rational** *if $\psi\big((S,d)\big) \in S_{\gg d} := \{x \in S \,:\, x \gg d\}$ for all $(S,d) \in \mathcal{B}$.*

Definition 2.4
A bargaining solution ψ is **efficient** *if for all $(S,d) \in \mathcal{B}$ the solution $\psi\big((S,d)\big)$ is weakly Pareto efficient in S, i.e, there does not exist $x \in S$ such that $x \gg \psi\big((S,d)\big)$. It is* **strongly efficient** *if for all $(S,d) \in \mathcal{B}$ the solution $\psi\big((S,d)\big)$ is strongly Pareto efficient in S, i.e, there does not exist $x \in S$ such that $x \gneqq \psi\big((S,d)\big)$.*

Remark 2.5
The Nash solution satisfies all of the above properties on the domain $\mathcal{B}$.

3 Non-monotonicity according to the Literature

Generally speaking, the arguments in the literature showing that the Nash bargaining solution is not implementable in Nash equilibrium rely on the following procedure. Starting with a class of bargaining problems in $\mathcal{B}$ an underlying economic environment is specified. It is assumed that the grand coalition can achieve certain allocations in this economic environment and that players have utility functions on these allocations. The set S of feasible utility allocations of a bargaining problem is interpreted as the image of the set of feasible allocations under agents utility functions. Thus, the same economic environment gives rise to different bargaining problems depending on agents' utility functions. Typically, the possible preferences considered lead to bargaining problems for which both the set of feasible payoffs and the status quo may vary. Now the planner is supposed to know all relevant data on the economic environment but players' utility functions. The Nash bargaining solution corresponds to a social choice correspondence (or function) for the class of economies consisting of the economic environment and a class of possible preference tuples. It is the monotonicity of this social choice correspondence which is discussed.

The demonstration of the non-monotonicity of the Nash bargaining solution is then done by means of examples (cf. Howard (1992, p. 145) and (Serrano (1996, Example 1, p. 4–5)).

We will take a different in some sense dual approach which focuses on the problem that the set of possible outcomes may not be known, while the utility consequences for all outcomes are.

4 Maskin Monotonicity and IIA

Nash formulated the bargaining problem in utility space. While his extremely abstract approach has been criticized (e. g. by Roemer (1986, 1988), see also Roemer (1996, Chpts. 2 and 3)), the discussion of the possibility of implementation of the Nash bargaining solution should take place in the same framework used in his original formulation. Therefore, we will not leave utility space but will choose exactly the space of utility allocations $\mathbb{R}^I$ as the outcome space for our implementation problem.

Following the model of Serrano (1996), we will choose agents' preferences to represent different possible sets of feasible payoffs. The status quo will be treated differently, though, namely assumed to be fixed. This is how initial endowments in economies are customarily treated in the implementation literature, and we feel that the status quo in a bargaining problem plays a similar role than initial endowments. In particular, what changes for a player if her component of the status quo changes are not her preferences between different possible agreements, but rather how she evaluates possible outcomes of the bargaining problem relative to her outside opportunities, or what she will accept as individually rational. Note, however, that in our model feasibility is unaffected by the status quo.

The question is, of course, how one should think of this setup. The crucial assumption is that the planner, who should be thought of as an arbitrator asked to solve their bargaining problem by a group of players, can assign utilities to the players. Alternatively, one can think of a fixed economy with given utility functions, where our outcomes are the images of some allocations in the economy under these utility functions; this is exactly how Serrano (1996) presents his model. The planner also knows the actual situation of everybody, i. e. the status quo for all possible bargaining problems she may be asked to act as arbitrator in. A particular bargaining problem should be thought of as some project a group of agents is planning to stage, where the characteristics of this project, i. e. the set of possible payoff vectors it can generate, is not known to the planner.

So we fix a set $I \in \mathcal{I}$ and a point $d \in \mathbb{R}^I$ and consider the set $\mathcal{B}_d^I$ of regular bargaining problems with player set I and status quo d.

The outcome space will be $\mathbb{R}^I$. The preferences of player $i \in I$ when the set of feasible payoffs is $S \subset \mathbb{R}^I$ are given by the utility function $u_S^i : \mathbb{R}^I \to \mathbb{R}$ defined by

$$u_S^i(x) = \begin{cases} x(i) & \text{if } x \in S \\ d(i) & \text{if } x \notin S\,. \end{cases} \qquad (1)$$

The interpretation of (1) is that a point outside the feasible payoff set is equivalent to the failure to reach agreement and hence results in the agent falling back upon her status quo utility, while feasible payoff vectors give to every player exactly her component. We prefer this straightforward interpretation and hence have chosen the definition of utility functions as given in (1) rather than Serrano's (1996, p. 2) formulation, which amounts to assigning a utility of $-\infty$ to all infeasible payoff vectors. The advantage of his formulation is that changes in the status quo do not affect preferences at all, which strengthens the analogy between status quo and initial endowments.

We can index the possible preference profiles in a very natural way by the set $\mathcal{B}_d^I$. Indeed, the set of possible preference or utility profiles given I and $d \in \mathbb{R}^I$ is $\mathcal{U}_d^I = \{u_S : (S,d) \in \mathcal{B}_d^I\}$, where $u_S = (u_S^i)_{i \in I}$. Also we will write $\mathcal{U}_d^i$ for all possible utility functions for agent $i \in I$. Note that this implies that we have a strong restriction on the domain of preferences since we only consider preferences that can be represented by utility functions of the form given in equation (1). Moreover, the preferences of the agents are not independent of each other since they stem from the same bargaining problem, that is, the set of possible preference profiles does not have a product structure.

In this model we can interpret any bargaining solution $\psi : \mathcal{B}_d^I \to \mathbb{R}^I$ as a social choice function mapping a utility profile (u_S) to the outcome $\psi((S,d)) \in \mathbb{R}^I$. Therefore, we will, in an abuse of notation, also use ψ for that social choice function. To make sure that this abuse of notation does not backfire later, we shall include the status quo in the argument of the social choice function, that is we write $\psi(u_S; d)$.

We will now demonstrate that the social choice function ν is Maskin monotonic (see Definition 4.1 below). In fact, it turns out that the key property of the Nash bargaining solution which yields Maskin monotonicity is independence of irrelevant alternatives (see Definition 2.2 above). Indeed, we will show that any bargaining solution which satisfies IIA together with individual rationality is Maskin monotonic. More importantly, we will also show that Maskin monotonicity of a social choice correspondence ψ implies that the corresponding bargaining solution satisfies IIA, provided it is individually rational.

We start by introducing the following piece of notation. For $i \in I$ and $x \in \mathbb{R}^I$ let

$$L(x, u_S^i) := \{x' \in \mathbb{R}^I : u_S^i(x) \geq u_S^i(x')\} \tag{2}$$

be the lower contour set for utility function u_S^i at the outcome x. It is easy to see that for our context for $x \in S$ with $x(i) \geq d(i)$, we have $L(x, u_S^i) = \mathbb{R}^I \backslash S_x^i$, where for $(S,d) \in \mathcal{B}_d^I$ and $z \in S$ the set S_z^i is defined as

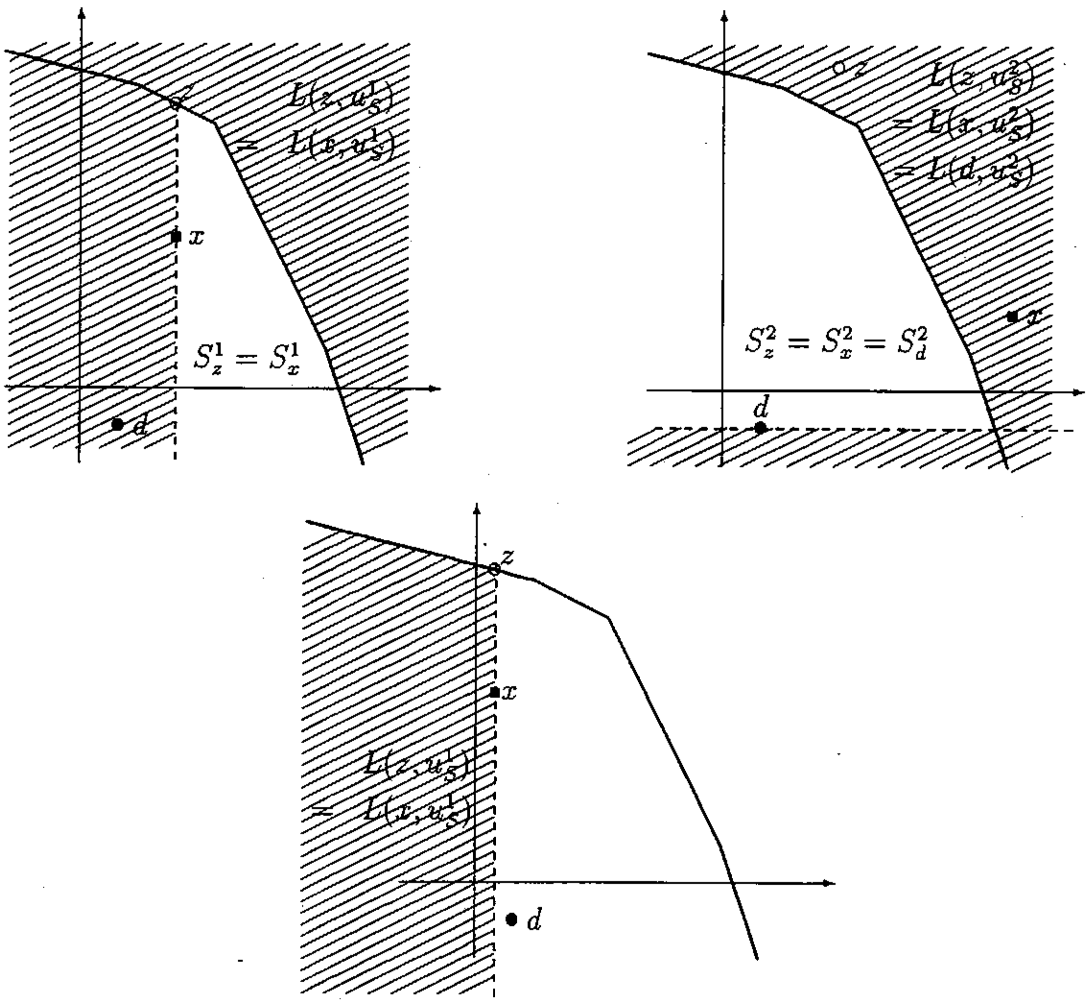
$$S_z^i := \{x \in S : x(i) > z(i)\}. \tag{3}$$

For $x \notin S$ the lower contour set is $L(x, u_S^i) = \mathbb{R}^I \backslash S_d^i$, and for $x \in S$ with $x(i) < d(i)$, it is

$$L(x, u_S^i) = \{z \in S : z(i) \leq x(i)\}.$$

The three different cases are depicted in Figure 1.

Figure 1: Lower contour sets for u_S



Definition 4.1
A social choice function ψ is **(Maskin) monotonic** *if for all pairs of utility profiles $(u_S, u_{\tilde S}) \in \mathcal{U}_d^I \times \mathcal{U}_d^I$ we have*

$$z \in \psi(u_S; d) \text{ and } L(z, u_S^i) \subseteq L(z, u_{\tilde S}^i) \text{ for all } i \in I \text{ implies } z \in \psi(u_{\tilde S}; d) .$$

Proposition 4.2
Let ψ be a bargaining solution satisfying IIA and individual rationality. Then the corresponding social choice function ψ on $\mathcal{U}_d^I$ is Maskin monotonic.

Proof:
Consider two situations $(u_S; d)$ and $(u_{\tilde S}; d)$ satisfying the condition in the definition of Maskin monotonicity, i. e., such that $z = \psi(u_S; d)$ and $L(z, u_S^i) \subseteq L(z, u_{\tilde S}^i)$ for all $i \in I$. Individual rationality of ψ yields $z \geq d$.

We first show that $z \in \tilde S$. This is clear for $z = d$ by assumption (ii) in the definition of regular bargaining problems (cf. Definition 2.1). For $z \neq d$ there exists a player $j \in I$ for whom $z(j) > d(j)$. Now assume $z \notin \tilde S$. Then we have $L(z, u_{\tilde S}^j) =$

$\mathbb{R}^I \backslash S^j_d$. Since $d \in \operatorname{int} \tilde{S}$, there exists $p \in \tilde{S}$ such that $d(j) < p(j) < z(j)$. But this leads to a contradiction because then $p \in L(z, u^j_S)$ but $p \notin L(z, u^j_{\tilde{S}})$.

Given $d \leq z$, $z \in S$ and $z \in \tilde{S}$, we can, for all $i \in I$, rewrite $L(z, u^i_S) \subseteq L(z, u^i_{\tilde{S}})$ as $\mathbb{R}^I \backslash S^i_z \subseteq \mathbb{R}^I \backslash \tilde{S}^i_z$ which yields $\tilde{S}^i_z \subseteq S^i_z$. From the latter holding for all $i \in I$ we can deduce that $\tilde{S} \subseteq S$.

Hence we have the two bargaining problems (S, d) and $(\tilde{S}, d)$ with $\tilde{S} \subseteq S$, $z = \psi\big((S,d)\big)$ and $z \in \tilde{S}$. These are exactly the conditions in the IIA axiom. Hence $z = \psi\big((\tilde{S}, d)\big)$, i. e., $z = \psi\,(u_{\tilde{S}}; d)$ as required by Maskin monotonicity.

◇

Since the Nash bargaining solution is individually rational and satisfies IIA, we immediately have the following corollary.

Corollary 4.3
The Nash bargaining solution is Maskin monotonic.

Other solutions that our result shows to be Maskin monotonic are nonsymmetric Nash solutions as introduced by Kalai (1977a), the egalitarian solution of Kalai (1977b)[5] or utilitarian solutions (cf. Harsanyi (1955)) whenever the domain is restricted such that they are individually rational.

The following example of a bargaining solution satisfying IIA but not individual rationality the corresponding social choice function of which fails to satisfy Maskin monotonicity demonstrates, that we cannot dispense with individual rationality in the prerequisites of Proposition 4.2.

Example 4.1
Let $I = \{1, 2\}$ and $d = (0, 0)$ and consider the solution $\bar{\psi}$ on $\mathcal{B}^I_d$ defined by

$$\bar{\psi}\big((S,d)\big) = \begin{cases} \nu\big((S,d)\big)\,, & \text{if } \nu\big((S,d)\big)(1) \geq \frac{1}{2} \\ (\max\{x \in \mathbb{R} \,:\, (x,-1) \in S\}, -1)\,, & \text{otherwise}\,. \end{cases}$$

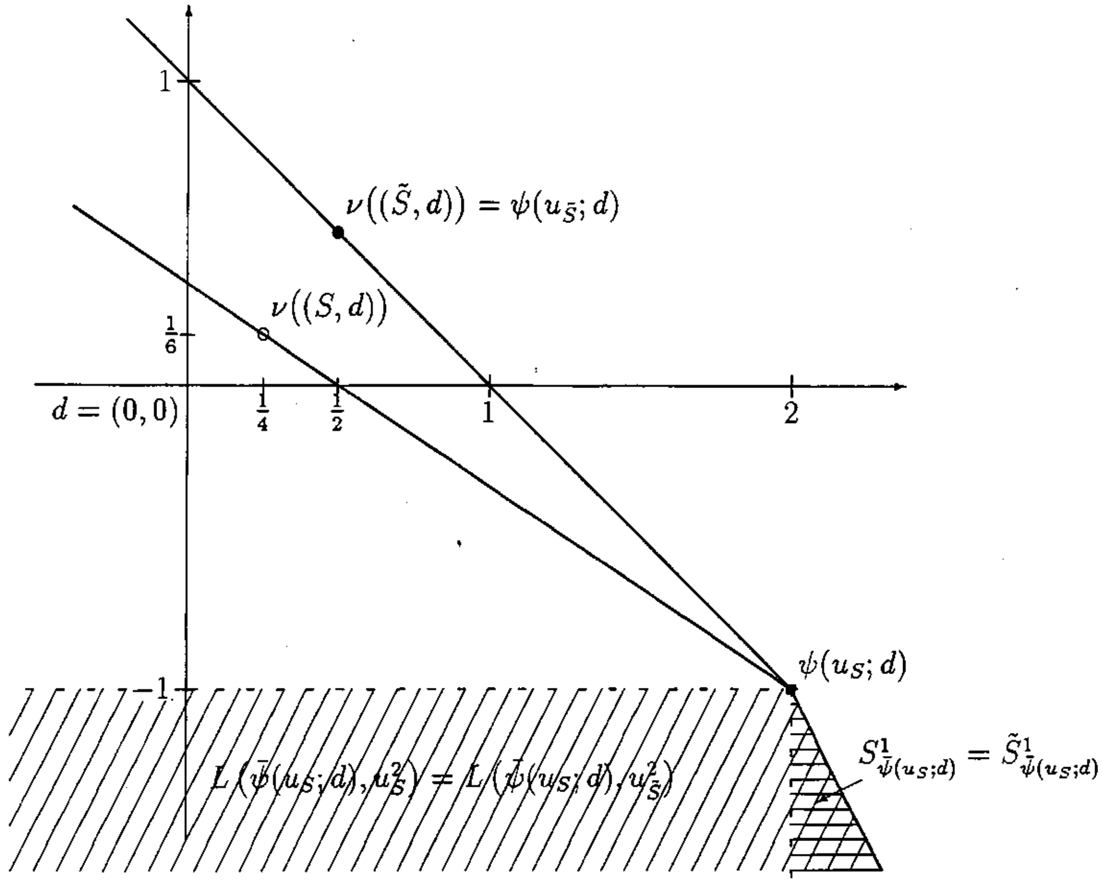
It is easy to check that $\bar{\psi}$ satisfies IIA. It does not satisfy individual rationality for player 1, for all bargaining problems for which the Nash bargaining solution yields less than $\frac{1}{2}$.

To see that the associated social choice function is not Maskin monotonic consider the two situations depicted in Figure 2. The lower contour sets of $\bar{\psi}(u_S; d)$ coincide for the two situations u_S and $u_{\tilde{S}}$, but contrary to what Maskin monotonicity would require we have $\bar{\psi}(u_S; d) \neq \bar{\psi}(u_{\tilde{S}}; d)$.

◇

[5] This solution satisfies IIA and individual monotonicity, but not scale covariance. For bargaining solutions satisfying the latter, IIA and individual monotonicity are incompatible (see Kalai and Smorodinsky (1975)).

Figure 2: A solution being neither individually rational nor Maskin monotonic



In the proof of Proposition 4.2 the key step was to use IIA. The next result shows that IIA is in fact a necessary condition for a bargaining solution to be Maskin monotonic, given that we restrict attention to individually rational solutions.

Proposition 4.4
Let $I \in \mathcal{I}$, $d \in \mathbb{R}^I$ and ψ be a bargaining solution on $\mathcal{B}_d^I$ satisfying individual rationality. If the corresponding social choice function is Maskin monotonic, ψ satisfies IIA.

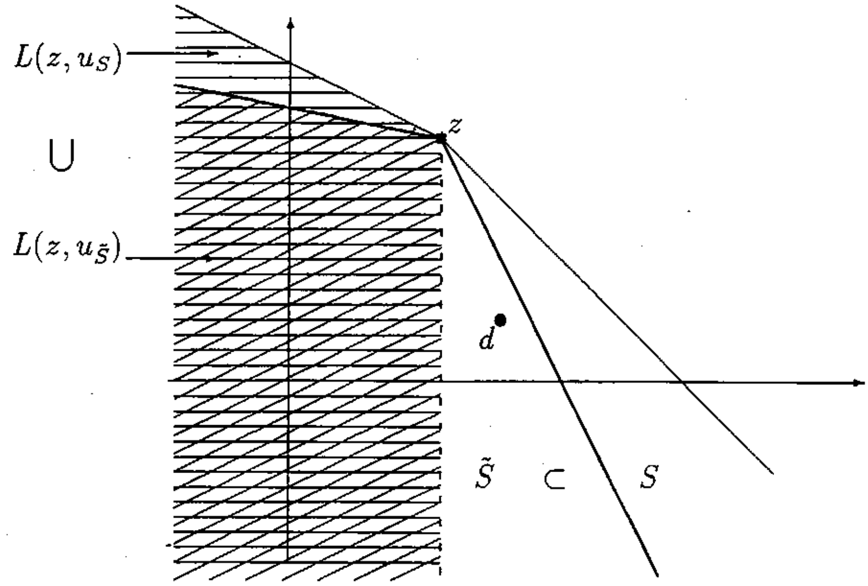
Proof:
Let I, d and ψ be as in the formulation of the proposition and assume ψ is Maskin monotonic. Let (S,d) and $(\tilde{S},d)$ be two bargaining problems in $\mathcal{B}_d^I$ fulfilling the conditions of the IIA axiom, i. e., such that $\tilde{S} \subseteq S$ and $\psi((S,d)) = z \in \tilde{S}$. We have to show that $z = \psi((\tilde{S},d))$.

We know $\psi(u_S;d) = z$ and $z \in \tilde{S}$. Furthermore, since $\tilde{S} \subseteq S$ we have, for all $i \in I$ that $\tilde{S}_z^i \subseteq S_z^i$ which implies $\mathbb{R}^I \backslash S_z^i \subseteq \mathbb{R}^I \backslash \tilde{S}_z^i$. The latter is equivalent to $L(u^i;z) \subseteq L(u_{\tilde{S}};z)$, because ψ is individually rational and hence $z > d$. Thus, Maskin monotonicity of the social choice function ψ yields $\psi(u_{\tilde{S}}; d) = z$ and therefore $z = \psi((\tilde{S}, d))$.

◇

We need individual rationality of the bargaining solution because if there exists an agent $j \in I$ with $z(j) < d(j)$, the fact that $\tilde{S} \subseteq S$ implies $L(z, u_S) \supseteq L(z, u_{\tilde{S}})$ and we cannot use Maskin monotonicity to get the desired result. This situation is depicted in Figure 3.

Figure 3: Problems without individual rationality



Proposition 4.4 tells us that other bargaining solutions in the literature like the Kalai–Smorodinsky solution proposed by Raiffa (1953) and axiomatically characterized[6] by Kalai and Smorodinsky (1975), or the Maschler–Perles solution axiomatically introduced by Perles and Maschler (1981) and Maschler and Perles (1981), are not implementable in Nash equilibrium in our framework since they fail to satisfy IIA and hence fail to be Maskin monotonic.

Serrano (1996) shows that in his model, the core is the only solution concept which is Maskin monotonic. Clearly, if we also consider set valued solution concepts we can extend the definition of IIA, and the result of Proposition 4.4 will hold for set valued solution concepts as well. So the core has to satisfy IIA on the domain $\mathcal{B}$, which is indeed easy to see, given that for a bargaining problem $(S, d) \in \mathcal{B}$ it is the Pareto-frontier of the set $S_{\geq d}$.

[6]The Kalai–Smorodinsky solution is characterized by replacing IIA in the characterization of the Nash bargaining solution by an axiom of individual monotonicity. It is important to notice, that contrary to what may be suggested by the terminology it is IIA and not individual monotonicity of a bargaining solution which is related to Maskin monotonicity.

If we try to also treat variations of the status quo with a similar approach, we fail. In the original model of Serrano (1996) where the utility of infeasible payoff vectors is $-\infty$ the reason can be seen directly, since we see immediately, that the status quo does not enter at all in the definition of agents' preferences, while it obviously has an influence on the Nash bargaining solution.

In our model, though preferences do change with changing status quo, still an example very similar to Example 1 in Serrano (1996) shows, that Maskin monotonicity is violated.

Example 4.2
Consider the two bargaining problems (S, d) and $(S, \tilde{d})$ both in $\mathcal{B}^{\{1,2\}}$ with payoff set $S = \left\{x \in \mathbb{R}^{\{1,2\}} : \sum_{i\in\{1,2\}} x(i) \leq 1\right\}$, $d = 0$ and $\tilde{d} = (0, 0.5)$. Then we have $z = \nu\big((S,d)\big) = (0.5, 0.5)$ and $\tilde{z} = \nu\big((S,\tilde{d})\big) = (0.75, 0.25) \neq z$. The hypothesis of the monotonicity condition is fulfilled, however.

◇

One might wonder if this example could be rectified by reformulating the utility functions to capture changes in the status quo in a different manner. A first idea might be to rewrite the utility functions u^i_S as follows $u^i_{(S,d)} : \mathbb{R}^I \to \mathbb{R}$ defined by

$$u^i_{(S,d)}(x) = \begin{cases} x(i) - d(i) & \text{if } x \in S \\ 0 & \text{if } x \notin S\,. \end{cases} \tag{4}$$

It is immediately clear, however, that this does not change the **preferences** at all.

As we stressed above, the status quo is not related to preferences over different payoffs but to the evaluation of payoffs in the bargaining problem relative to the failure to reach agreement, i. e. to some outside option. So a second idea would be to explicitly include this in the outcome space. To do so, choose the outcome space to be $\mathbb{R}^I \cup cup\{\omega\}$, where $\omega \notin \mathbb{R}^I$ represents the status quo or conflict outcome.[7] The utility functions are $u^i_S : \mathbb{R}^I \cup cup\{\omega\} \to \mathbb{R}$ defined by

$$u^i_S(x) = \begin{cases} d(i) & \text{if } x = \omega \\ x(i) & \text{if } x \in S \\ d(i) & \text{if } x \notin S \cup \{\omega\}\,. \end{cases} \tag{5}$$

As before, Example 4.2 shows, that Maskin monotonicity does not hold for this variation of the model, either.

5 Sufficient Conditions

Given that we have seen that with fixed status quo the Nash bargaining solution satisfies monotonicity as a necessary condition for implementability in Nash equi-

[7]We distinguish between the conflict outcome ω and the utility payoff $d \in S$.

librium, we next inquire whether it also satisfies sufficient conditions.

The classical sufficient condition for implementability with at least three agents is no veto power (cf. Maskin (1977), Repullo (1987) and Saijo (1988)). In our context, no veto power reads as follows.

Definition 5.1
A social choice function ψ satisfies **no veto power** *if for all utility profiles $u_S \in \mathcal{U}_d^I$, if there exists a point $z \in \mathbb{R}^I$ and an agent $i \in I$, such that $L(z, u_S^j) = \mathbb{R}^I$ for all $j \in I\backslash\{i\}$ then $z = \psi(u_S; d)$.*

Unfortunately, this is not fulfilled by the Nash bargaining solution as can be seen with the following example.

Example 5.1
Take $I = \{1,2,3\}$, $d = 0$, and S the comprehensive hull[8] of the convex hull[9] of the points $(0,1,1)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. The point $(0,1,1)$ is weakly preferred to any other point in S by players 2 and 3. Therefore it should be in the solution according to the condition of no veto power. But clearly it is not the Nash bargaining solution, which is the point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

◇

So we have to consider other sufficient conditions given in the literature. In a model with finite sets of alternatives and linear orderings Danilov (1992) provides such a condition, called strong monotonicity, for implementability with three or more agents. His results have been generalized by Yamato (1992) by allowing for infinite alternative sets and arbitrary preference relations.

To define the condition of strong monotonicity we first have to introduce the concept of an essential alternative. So let I and $d \in \mathbb{R}^I$ be given and let ψ be a bargaining solution on $\mathcal{B}_d^I$. Let $i \in I$ and $A \subseteq \mathbb{R}^I$. An alternative $a \in A$ is essential for agent i in the set A with respect to the social choice function ψ if there exists a utility profile u_S such that $a = \psi(u_S; d)$ and $L(a, u_S^i) \subseteq A$. The set of all alternatives which are essential for agent i in the set A with respect to ψ is denoted $\mathrm{Ess}^i(A, \psi)$.

A social choice function ψ satisfies strong monotonicity if for all pairs of utility profiles u_S and $u_{\tilde{S}}$ if $a = \psi(u_S; d)$ and if $\mathrm{Ess}^i\big(L(a, u_S^i), \psi\big) \subseteq L(a, u_{\tilde{S}}^i)$, for all $i \in I$, then $a = \psi(u_{\tilde{S}}; d)$.

Proposition 5.2
The Nash bargaining solution fails to be strongly monotonic.

Proof:
We first show that for all utility profiles u_S and all $i \in I$ we have

$$\mathrm{Ess}^i\Big(L\big(\nu(u_S; d), u_S^i\big), \nu\Big) \subseteq \mathbb{R}^I \backslash S\,. \tag{6}$$

[8]For $A \subseteq \mathbb{R}^I$, the comprehensive hull is defined as $\mathrm{compH}(A) = A - \mathbb{R}_+^I$.
[9]For $A \in \mathbb{R}^I$ the convex hull is defined as $\mathrm{cH}(A) = \bigcap \{C \subseteq \mathbb{R}^I \,:\, A \subseteq C, \text{ and } C \text{ convex}\}$.

To see this fix $i \in I$ and u_S and let $a = \nu((u_S; d)$. Since the Nash bargaining solution is individually rational we know that $L(a, u_S^i) = \mathbb{R}^I \backslash S_a^i$. Therefore, $\mathrm{Ess}^i\Big(L\big(a, u_S^i\big), \nu\Big) \subseteq \mathbb{R}^I \backslash S_a^i$. For $b \in S$ with $b(i) < d(i)$ we know that $b \notin \mathrm{Ess}^i\Big(L\big(a, u_S^i\big), \nu\Big)$ since by individual rationality there can be no preference profile $u_{\tilde{S}}$ for which $b = \nu(u_{\tilde{S}}; d)$. On the other hand, if $b \in S$ such that $d(i) \leq b(i) \leq a(i)$, the condition $L(b, u_{\tilde{S}}) \subseteq L(a, u_S)$ for essential alternatives in $L(a, u_S)$ implies $S_a^i \in \tilde{S}$, that is in particular, $a \in \tilde{S}$. But then $b \neq \nu(u_{\tilde{S}}; d)$ because for the set S the Nash product was maximized in a so, since $b \in S$, it cannot be maximized in b over the set $\tilde{S} \ni a$.

Now it is easy to construct an example which shows that ν is not strongly monotonic. Let $a \in \mathbb{R}^I_{\gg d}$ and $b \in \mathbb{R}^I$ with $d \ll b \ll a$. Let $S = \mathrm{compH}\,(a)$ and $\tilde{S} = \mathrm{compH}\,(b)$. Then we have $a = \nu(u_S; d)$ and $\mathrm{Ess}^i\Big(L\big(a, u_S^i\big), \nu\Big) \subseteq \mathbb{R}^I \backslash \mathrm{compH}\,(a) \subseteq \mathbb{R}^I \backslash \mathrm{compH}\,(b) \subseteq L(a, u_{\tilde{S}}^i)$ for all $i \in I$ but $a \neq \nu(u_{\tilde{S}}; d) = b$.

◇

Proposition 5.2 is most discomforting given the fact that Yamato presents strong monotonicity as a necessary and sufficient result and stresses the fact that he allows for restricted domains of preferences. It turns out, however, that our preference domain is more restrictive than those he considers in his necessity result (cf. Yamato (1992, Theorem 1, p. 487)), in particular, his condition D (cf. Yamato (1992, p. 487)) is not satisfied. This condition requires that for every $a \in \mathbb{R}^I$, for every preference profile u_S, every agent $i \in I$ and every element $b \in L(a, u_S^i)$ there be an alternative preference profile $u_{\tilde{S}}$ such that $L(a, u_S^i) = L(b, u_{\tilde{S}})$ and $L(b, u_{\tilde{S}}) = \mathbb{R}^I$, for all $j \in I \backslash \{i\}$. This is not the case in our setup.

Consider the status quo $d \in \mathbb{R}^I$, a preference profile u_S and a point $a \in \mathbb{R}^I$ with $a(i) > d(i)$ and $S_a^i \neq \emptyset$. Then we have $L(a, u_S^i) = \mathbb{R}^I \backslash S_a^i \neq \mathbb{R}^I$. Take $b \in \mathrm{compH}\,(a) \subset L(a, u_S^i)$. Then there exists no preference profile $u_{\tilde{S}}$ such that $L(b, u_{\tilde{S}}^i) = L(a, u_S^i)$. There are three possible cases:

(i) If $b(i) < d(i)$ we have $L(b, u_{\tilde{S}}^i) \subset \big\{x \in \mathbb{R}^I : x(i) \leq b(i)\big\}$. But then $a \notin L(b, u_{\tilde{S}}^i)$ though $a \in L(a, u_S^i)$.

(ii) If $b \notin \tilde{S}$, $b(i) \geq d(i)$, we have $L(b, u_{\tilde{S}}^i) \supset \mathbb{R}^I \backslash \tilde{S}$. By comprehensiveness of $\tilde{S}$ it has to be the case that also $a \notin \tilde{S}$. Therefore, since $\tilde{S}$ is closed, there exists an open neighborhood of a not contained in $\tilde{S}$. But every open neighborhood of a has nonempty intersection with S_a^i. Thus there exists $c \in S_a^i$ with $c \notin L(a, u_S^i)$ but $c \in L(b, u_{\tilde{S}}^i)$.

(iii) If $b \in \tilde{S}$, $b(i) \geq d(i)$, we have $L(b, u_{\tilde{S}}^i) = \mathbb{R}^I \backslash \tilde{S}_b^i$. Then either $\tilde{S} = \emptyset$, in which case $L(b, u_{\tilde{S}}^i) = \mathbb{R}^I \neq L(a, u_S^i)$, or $\tilde{S}_b^i \neq \emptyset$. In the latter case there exists $c \in \tilde{S}_b^i$ with $b(i) < c(i) < a(i)$. For this c we have $c \notin L(b, u_{\tilde{S}}^i)$ but $c \in L(a, u_S^i)$.

Note that the above arguments go through even if we would allow to vary the status quo. In this case just replace $d(i)$ in the three cases by $\tilde{d}(i)$.

Given the negative result concerning strong monotonicity, we have to turn to the necessary and sufficient conditions presented by Moore and Repullo (1990). While their condition is more complicated, it has the advantage that with an additional requirement it also covers the case of two agents[10] which for the case of bargaining is especially desirable. We will follow Sjöström (1991) who presents a reformulation of their conditions[11] and a tractable way to check whether they are satisfied by a given social choice correspondence.

The condition introduced by Moore and Repullo (1990), which we need to understand Sjöström's (1991) modified condition, adapted to our model reads as follows.

Definition 5.3
A social choice correspondence ψ satisfies **condition μ** *if there is a set $D \subseteq \mathbb{R}^I$ and for all triplets $(i, u_S, z) \in I \times \mathcal{U}_d^I \times \mathbb{R}^I$ such that $z = \psi(u_S; d)$, there is a set $C^i(z, u_S)$ such that $x \in C^i(z, u_S) \subseteq L(z, u_S^i) \cap D$ and the following three conditions are satisfied*

(i) *For all $u_{\bar{S}} \in \mathcal{U}_d^I$, $C^i(z, u_S) \subseteq L(z, u_{\bar{S}}^i)$ for all $i \in I$ implies $z = \psi(u_{\bar{S}}; d)$.*

(ii) *For all $u_{\bar{S}} \in \mathcal{U}_d^I$, $c \in C^i(z, u_S) \subseteq L(c, u_{\bar{S}}^j)$, and $D \subseteq L(c, u_{\bar{S}}^j)$ for all $j \in I \backslash \{i\}$ implies $c \in \psi(u_{\bar{S}}; d)$.*

(iii) *For all $u_{\bar{S}} \in \mathcal{U}_d^I$, $c \in D \subseteq L(c, u_{\bar{S}}^i)$ for all $i \in I$ implies $c \in \psi(u_{\bar{S}}; d)$.*

Theorem 1 of Moore and Repullo (1990) states that if $|I| \geq 3$, ψ on $\mathcal{U}^I$ can be implemented in Nash equilibrium if and only if it satisfies condition μ.

Sjöström (1991) presents an equivalent condition which is easier to check. To present his condition, we need to define two additional sets.

Definition 5.4
$\bar{D}$ is the union of all sets $D \subseteq \mathbb{R}^I$ satisfying condition (iii) in Definition 5.3.

Definition 5.5
For all $(i, u_S^i, z) \in I \times \mathcal{U}_d^i \times \mathbb{R}^I$ such that $z = \psi(u_S; d)$, $\bar{C}^i(z, u_S^i)$ is the union of all sets $C^i \subseteq L(z, u_S^i) \cap \bar{D}$ satisfying condition (ii) in Definition 5.3 with $D = \bar{D}$.

[10] The condition they present for this particular case has also been discovered independently by Dutta and Sen (1991).

[11] Even though Sjöström (1991) assumes that the space of preferences has a product structure, this is not used anywhere in his paper. Therefore, his results apply to our setup as well as those of Moore and Repullo (1990) who explicitly allow for arbitrary domains of preferences (cf. their footnote 4, p. 1086).

Definition 5.6

A social choice function ψ satisfies **condition** M *if for all triplets* $(i, u_S, z) \in I \times \mathcal{U}_d^I \times \mathbb{R}^I$ *such that* $z = \psi(u_S; d)$*, the following holds:* $z \in \bar{C}^i(z, u_S^i)$ *and condition (i) of Definition 5.3 is satisfied for* $C^i(z, u_S) = \bar{C}^i(z, u_S^i)$.

Lemma 3 of Sjöström (1991) states that condition M is equivalent to condition μ. Hence by Theorem 1 of Moore and Repullo (1990) it is necessary and sufficient for implementability when there are at least three agents. We will use this result to show that the Nash bargaining solution is implementable in Nash equilibrium when $|I| \geq 3$. Actually, we will again demonstrate that this is the case for a class of bargaining solutions including the Nash solution. We start with two lemmata concerning the sets $\bar{D}$ and $\bar{C}^i(x, u_S^i)$, respectively.

Lemma 5.7

Let ψ be a bargaining solution satisfying strong efficiency. Then for the corresponding social choice function ψ on ${\mathcal{U}^{\mathcal{I}}}_d$ we have $\bar{D} = \mathbb{R}^I$.

Proof:

Since the bargaining solution ψ is strongly efficient the corresponding social choice function ψ satisfies unanimity.[12] To see this, let $c \in \mathbb{R}^I_{\geq d} := \{x \in \mathbb{R}^I : x \geq d\}$ and $u_S \in \mathcal{U}_d^I$ such that $\mathbb{R}^I \subseteq L(c, u_S^i)$, for all $i \in I$, i. e., such that c is everybody's best element in $\mathbb{R}^I$ under the utility profile u_S. From $\mathbb{R}^I \subseteq L(c, u_S^i)$ for $i \in I$ it follows that $S_c^i = \emptyset$. This for all $i \in I$ together with comprehensiveness yields $S = \text{compH}(c)$ and hence $c = \psi\big((S, d)\big) = \psi(u_S; d)$, because c is the only strongly Pareto efficient point in S.

By unanimity of ψ, condition (iii) of Definition 5.3 is satisfied for all $D = \mathbb{R}^I$ and thus $\bar{D} = \mathbb{R}^I$.[13]

◇

Lemma 5.8

Let ψ be a bargaining solution satisfying IIA, strong efficiency, and strong individual rationality. Then for the corresponding social choice function ψ on ${\mathcal{U}^{\mathcal{I}}}_d$ we have $\bar{C}^i(z, u_S^i) = L(z, u_S^i) = \mathbb{R}^I \backslash S_z^i$ *for all $i \in I$, all $u_S^i \in \mathcal{U}_d^i$, and all $z \in \psi(u_S; d)$.*[14]

Proof:

By Lemma 5.7 $\bar{D} = \mathbb{R}^I$, and by its proof, the social choice function ψ satisfies unanimity.

[12] A social choice correspondence ψ satisfies unanimity, if any allocation c which is most preferred by all agents is chosen by ψ.

[13] Cf. Sjöström (1991, p. 335) the example after the definition of B^* (which is $\bar{D}$ in our notation.

[14] Note, that by the example given directly after the definition of the set $C_i^*(a, R_i)$ in Sjöström (1991, p. 335), which corresponds to $\bar{C}^i(z, u_S^i)$ in our notation, this is exactly how $\bar{C}^i(z, u_S^i)$ would be if ψ would satisfy no veto power.

We are going to employ the algorithmic method to construct $\bar{C}^i(z, u^i_S)$ presented by Sjöström (1991, p. 336, bottom). Fix $(i, u^i_S, z) \in I \times \mathcal{U}^i_d \times \mathbb{R}^I$ such that $z = \psi(u_S; d)$. $C^i_1(z, u_S) = L(z, u^i_S)$. That is $C^i_1(z, u_S) = \mathbb{R}^I \backslash S^i_z$. Continuing with the algorithm next we compute

$$C^i_2(z, u_S) = \left\{ x \in C^i_1(z, u_S) \; : \; x \in \psi(u_{\tilde{S}}; d) \begin{array}{l} \text{for all } u_{\tilde{S}} \in \mathcal{U}^I_d \text{ satisfying} \\ (1)\; C^i_1(z, u_S) \subseteq L(x, u^i_{\tilde{S}}) \\ (2)\; \mathbb{R}^I \subseteq L(x, u^j_{\tilde{S}}) \text{ for all } j \neq i \end{array} \right\}.$$

Take $x \in C^i_1(z, u_S) = \mathbb{R}^I \backslash S^i_z$. Let $u_{\tilde{S}} \in \mathcal{U}^i_d$ such that the conditions in the definition of $C^i_2(z, u_S)$ are satisfied. Then we have $C^i_1(z, u_S) = \mathbb{R}^I \backslash S^i_z \subseteq L(x, u^i_{\tilde{S}})$. If $x \notin \tilde{S}$ we would have $L(x, u^i_{\tilde{S}}) = \mathbb{R}^I \backslash \tilde{S}^i_d$. But since $\tilde{S}^i_d \not\subseteq S^i_z$ because of the strong individual rationality of ψ, this cannot be the case. Hence it must be that $x \in \tilde{S}$. Then $L(x, u^i_{\tilde{S}}) = \mathbb{R}^I \backslash \tilde{S}^i_x$ and thus $\tilde{S}^i_x \subseteq S^i_z$.

We have to distinguish two cases. First, if $x \neq z$, we have that $\tilde{S}^i_x \subseteq S^i_z$ implies $\tilde{S}^i_x = \emptyset$. Together with the second condition in the definition of $C^i_2(z, u_S)$ this implies $\mathbb{R}^I \subseteq L(x, u^i_{\tilde{S}})$ for all $i \in I$ and hence by unanimity $x \in \psi(u_{\tilde{S}}; d)$.

Second, consider $x = z$. Then $\tilde{S}^i_x \subseteq S^i_z$ and $\mathbb{R}^I \subseteq L(x, u^j_{\tilde{S}})$ for all $j \neq i$ implies $\tilde{S} \subseteq S$. Since we also have $x = z \in \tilde{S}$, IIA yields $x \in \psi(u_{\tilde{S}}; d)$.

Together, we have that $C^i_2(z, u_S) = C^i_1(z, u_S)$ and therefore by Proposition 4 of Sjöström (1991) also $\bar{C}^i(z, u^i_S) = C^i_1(z, u_S)$. Thus, $\bar{C}^i(z, u^i_S) = L(z, u^i_S) = \mathbb{R}^I \backslash S^i_z$. ◇

Proposition 5.9

Let ψ be a bargaining solution satisfying IIA, strong efficiency, and strong individual rationality. Then the corresponding social choice function ψ on $\mathcal{U}^I{}_d$ is implementable in Nash equilibrium if $|I| \geq 3$.

Proof:

We will check that Sjöström's (1991) condition M holds which is a sufficient condition for implementability in Nash equilibrium with $|I| \geq 3$.

Let $(i, u_S, z) \in I \times \mathcal{U}^I_d \times \mathbb{R}^I$ be such that $z = \psi(u_S; d)$. Since $\bar{C}^i(z, u^i_S) = \mathbb{R}^I \backslash S^i_z$, obviously we have $z \in \bar{C}^i(z, u^i_S)$. It remains to demonstrate that condition (i) of Definition 5.3 is satisfied for $C^i(z, u_S) = \bar{C}^i(z, u^i_S)$, that is, that for all utility profiles $u_{\tilde{S}} \in \mathcal{U}^I_d$ such that $\bar{C}^i(z, u^i_S) \subseteq L(z, u^i_{\tilde{S}})$ for all $i \in I$ we have $z = \psi(u_{\tilde{S}}; d)$.

As before, $\bar{C}^i(z, u^i_S) \subseteq L(z, u^i_{\tilde{S}})$ excludes the possibility that $z \notin \tilde{S}$. Therefore, we have $z \in \tilde{S}$, and hence $\bar{C}^i(z, u^i_S) \subseteq L(z, u^i_{\tilde{S}})$ is equivalent to $\mathbb{R}^I \backslash S^i_z \subseteq \mathbb{R}^I \backslash \tilde{S}^i_z$ or $\tilde{S}^i_z \subseteq S^i_z$. This condition holding for all $i \in I$ implies $\tilde{S} \subseteq S$. Again, we can apply IIA: $\tilde{S} \subseteq S$ and $\psi(S, d) = z \in \tilde{S}$ implies $z = \psi(\tilde{S}, d) = \psi(u_{\tilde{S}}; d)$.

◇

Implementability of the Nash bargaining solution ν for $|I| \geq 3$ comes as an immediate corollary, given that ν is strongly efficient, strongly individually rational and satisfies IIA.

Corollary 5.10
The Nash bargaining solution is implementable in Nash equilibrium when the number of agents is at least three.

In the context of bargaining, the case of $|I| = 2$ deserves special attention. For this case, Sjöström (1991) presents condition $M2$, which is a strengthened version of condition M. The condition reads as follows.

Definition 5.11
The social choice function ψ satisfies **condition $M2$** *if it satisfies condition M and in addition the following is true.*[15]
(iv) For all $(\hat{z}, u_{\hat{S}}, z, u_S) \in \mathbb{R}^I \times \mathcal{U}_d^I \times \mathbb{R}^I \times \mathcal{U}_d^I$ such that $\hat{z} \in \psi(u_{\hat{S}}; d)$ and $z = \psi(u_S; d)$, there is an alternative $\phi(\hat{z}, u_{\hat{S}}, z, u_S) \in \bar{C}^1(\hat{z}, u_{\hat{S}}^1) \cap \bar{C}^2(z, u_S^2)$ such that, for all $u_{\bar{S}} \in \mathcal{U}_d^I$, if $\bar{C}^1(\hat{z}, u_{\hat{S}}^1) \subseteq L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^1\right)$ and $\bar{C}^2(z, u_S^2) \subseteq L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^2\right)$, then $\phi(\hat{z}, u_{\hat{S}}, z, u_S) \in \psi(u_{\bar{S}}; d)$.

By Lemma 4 of Sjöström (1991) condition $M2$ is equivalent to condition $\mu 2$ which by Theorem 2 of Moore and Repullo (1990) is necessary and sufficient for implementability when $|I| = 2$.

Proposition 5.12
Let ψ be a bargaining solution satisfying IIA, strong efficiency, and strong individual rationality. Then the corresponding social choice function ψ on $\mathcal{U}^{\mathcal{I}}{}_d$ is implementable in Nash equilibrium for all $I \in \mathcal{I}$.

Proof:
Implementability for $|I| \geq 3$ follows from Proposition 5.10. Furthermore, in the proof of that proposition we have already shown that ψ satisfies condition M. Hence, we only need to demonstrate that condition (iv) in Definition 5.11 holds true.

To facilitate notation, we will assume $I = \{1, 2\}$. Let $(\hat{z}, u_{\hat{S}}, z, u_S) \in \mathbb{R}^I \times \mathcal{U}_d^I \times \mathbb{R}^I \times \mathcal{U}_d^I$ such that $\hat{z} \in \psi(u_{\hat{S}}; d)$ and $z = \psi(u_S; d)$. We know that $\bar{C}^1(\hat{z}, u_{\hat{S}}^1) = \mathbb{R}^I \backslash \hat{S}_{\hat{z}}^1$ and $\bar{C}^2(z, u_S^2) = \mathbb{R}^I \backslash S_z^2$. Thus $\bar{C}^1(\hat{z}, u_{\hat{S}}^1) \cap \bar{C}^2(z, u_S^2) = \mathbb{R}^I \backslash \left(\hat{S}_{\hat{z}}^1 \cup S_z^2\right)$. Let $\phi(\hat{z}, u_{\hat{S}}, z, u_S) = \max\{\hat{z}, z\} + (1,1)$, where, for $x, y \in \mathbb{R}^{\{1,2\}}$, $\max\{x, y\} := (\max\{x(1), y(1)\}, \max\{x(2), y(2)\}) \in \mathbb{R}^{\{1,2\}}$. Clearly, we have $\phi(\hat{z}, u_{\hat{S}}, z, u_S) > \hat{z}$ and $\phi(\hat{z}, u_{\hat{S}}, z, u_S) > z$ so that $\phi(\hat{z}, u_{\hat{S}}, z, u_S) \in \bar{C}^1(\hat{z}, u_{\hat{S}}^1) \cap \bar{C}^2(z, u_S^2)$.

Now let $u_{\bar{S}} \in \mathcal{U}_d^I$ such that $\bar{C}^1(\hat{z}, u_{\hat{S}}^1) \subseteq L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^1\right)$ and $\bar{C}^2(z, u_S^2) \subseteq L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^2\right)$. With arguments that are by now familiar we can conclude that $L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^1\right) = \mathbb{R}^I$ and $L\left(\phi(\hat{z}, u_{\hat{S}}, z, u_S), u_{\bar{S}}^2\right) = \mathbb{R}^I$. Therefore, una-

[15] The additional condition will be labeled (iv) to stress the connection to the conditions of Definition 5.3.

nimity yields $\phi(\hat{z}, u_{\hat{S}}, z, u_S) \in \psi(u_{\hat{S}}; d)$. Hence $M2$ is satisfied by ψ and thus ψ can be implemented also if $|I| = 2$.

◇

As before, the result for the Nash bargaining solution is a corollary.

Corollary 5.13
The Nash bargaining solution is implementable in Nash equilibrium.

Since the proof of Theorems 1 and 2 of Moore and Repullo (1990) is constructive, it provides a mechanism that implements the Nash bargaining solution. This mechanism is of the same type as the mechanism of Repullo (1987). Therefore the well-known criticism against this mechanism applies here as well, especially the arguments against the use of "integer games" (cf. Jackson (1992)). Another point that has been raised against this type of mechanism, namely the complexity introduced by the size of agents' message spaces, in particular the fact that they are required to report complete preference profiles, seems to have less bite in our model, however. This is due to the fact, that because of the restriction in the space of possible preference profiles, by reporting one's own preference, the preferences of all other agents are automatically given.

6 Concluding Remarks

The setup we chose to investigate whether the Nash bargaining solution is implementable is somewhat orthogonal to the standard approach in the literature. Rather than analyzing bargaining problems by viewing them as images of some underlying economic situation where the planner's basic problem is that she does not know players' utility functions, we work directly in utility space. In our model, however, this does not imply that there is no problem for the planner since we assume that she does not know the set of feasible utility allocations. Rather than interpreting implementation theory to be concerned with "real" economic problems, we take a more abstract view that we feel to be much closer to the Nash program. While there the question is, whether it is possible for every bargaining problem from a certain class to construct a noncooperative game the equilibrium of which yields the Nash bargaining solution, we basically reverse the order of the quantifiers and ask, if there exists a game form yielding the Nash bargaining solution as its equilibrium for every bargaining game from a certain class. Even though we have cast this in the terminology of implementation theory and worked with results from this area, maybe it would be more appropriate to see our note as a contribution to the literature on the Nash program. Tackling the problem in utility space, then, seems faithful to Nash's demand game which initiated this line of research. And stretching the parallels, one might even claim that the idea to consider the shape of the set S as unknown to the planner is inspired by the smoothed version of Nash's demand game, where the idea is that players may not be completely sure as to how this set looks like.

A comprehensive discussion of the scope and philosophy of implementation theory, though undoubtedly interesting and as far as we can see not available yet, clearly is beyond the scope of this note. So we restrict ourselves to just one other point, namely the interpretation of the outcome function in the implementing mechanism. In our model, the outcome functions maps into utility space. How is this to be interpreted? Having in mind, that agents value infeasible outcomes as if they were the status quo, it cannot be the case, that the planner hands out utils to the players. Rather, the outcome of the mechanism should be interpreted as a proposal to the agents to chose this utility allocation, for example by realizing some joint plan of actions. The main strength of such a proposal would be that it is a coordination device. If agents after having participated in the mechanism were not to follow the proposal, they would almost surely end up with a failure to cooperate, i.e. in the status quo. This interpretation stresses the fact, that the planner does not really need to have full control over players' utilities. Indeed, even without being able to observe if what agents did in the end resulted in the proposed utilities, she may have sufficient reason to believe that it did.

Accepting our framework, the most interesting point we make with our model seems the strong relation between the IIA axiom of Nash, which has often been criticized in the literature, and Maskin monotonicity. In fact, Proposition 4.4 could be seen as a defense of the IIA axiom on the grounds of asking for implementability of the bargaining solution.

In this context it may also be interesting to compare our results to those in the literature on rationality of bargaining solutions (cf. e. g. Peters and Wakker (1991), Bossert (1994), and Sánchez (1996)), where IIA also figures as a key condition.

As we have stressed above, the assumption of a fixed status quo, corresponding to constant initial endowments in the standard models, is a very common one in the implementation literature. Nevertheless, it seems unsatisfactory and the next logical step would be to investigate, how we can treat variations of the status quo. For implementation in the standard model of pure exchange economies as well as a model with production the problem of varying initial endowments has been discussed by Postlewaite (1979) and by Hurwicz, Maskin, and Postlewaite (1995). Since their framework is different from ours, we cannot directly use their results, of course. Given the negative results obtained towards the end of Section 4, we think, however, that their general insight, that with variable endowments the mechanism's message spaces have to depend on the endowment will also apply to our model.

Finally, we take up the discussion Howard (1992) starts in his introduction where he deals with the question of implementability of the Nash bargaining solution in different equilibrium concepts. Our paper refutes his assertion that implementation in Nash equilibrium is impossible, at least for the case of fixed status quo. For implementation in subgame perfect equilibrium his contribution provides a positive answer by constructing a particular mechanism. At the same time, it is true that the results on sufficient conditions for implementation in subgame perfect equilibrium of Moore and Repullo (1988, Theorem 2) and Abreu and Sen (1990, Theorem 2) require no veto power and at least three agents, and hence are not applicable to our model by Example 4.1. As for Nash implementation, however, no veto power is a sufficient but not a necessary condition. Therefore, Howard's result falls in the gap between these two and stresses the fact that it would be interesting to also have a full characterization for implementability in subgame perfect equilibrium. The situation with respect to implementation in equilibrium in undominated strategies discussed by Palfrey and Srivastava (1991) is exactly the same, i. e., one could either search for necessary and sufficient conditions or try to explicitly construct a mechanism implementing the Nash bargaining solution in undominated strategies. The results of Herrero and Srivastava (1992) regarding implementation in backward induction and those on implementation in perfect equilibrium by Sjöström (1993) are applicable neither in Howard's nor in our model because they deal with a finite set of alternatives. As to virtual implementation introduced by Matsushima (1988), the generally positive results of Abreu and Sen (1991, Theorem 1) while being exactly in line with Howard's setup do not directly apply to our model, since they work with lotteries over a finite set of alternatives. More significantly, however, there does not exist a result for the case of two agents. In summary, to apply general results on implementability in any equilibrium concept to our model one would like to have a full characterization of implementable social choice functions, i. e. conditions that are necessary and sufficient and include the two-player case.

Because such results are not yet available for most equilibrium concepts, it may be more fruitful to try to find explicit, preferably simple, or natural mechanisms, which implement the Nash bargaining solution. It may be possible to find such mechanisms taking advantage of the particular structure of the social choice function ν, in particular the restricted preference domain.

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