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A Note on Existence of Equilibria in Generalized Economies

by

Bezalel Peleg

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University of Bielefeld

33501 Bielefeld, Germany

Abstract

If the budget constraints of a generalized economy are continuous and balanced, then there exists a generalized Walras equilibrium.

A Note on Existence of Equilibria in Generalized Economies

Let $N = \{1, ..., n\}$ be a set of traders and let R^l_+ be the commodity space. A *generalized economy* is a $(2n + 1)$-tuple $E =< w^1, ..., w^n; u^1, ..., u^n; t >$ where $w^i \in R^l_+$ is the initial endowment of trader $i \in N$; $u^i : R^l_+ \to R$ is the utility function of $i \in N$; and $t \in R^l$ is the net trade vector of E (with the outside world). We shall assume in the sequal that

$$\sum_{i=1}^{n} w^i + t \in R^l_{++} (R^l_{++} = \{x \in R^l \mid x_j > 0 \text{ for } j = 1, ..., l\}) \quad (1.1)$$

$$u^i \text{ is continuous, quasi-concave, and strictly monotonic for all } i \in N \quad (1.2)$$

Generalized economies were introduced in Thomson (1992). Let $E =< w^1, ..., w^n; u^1, ..., u^n; t >$ be a generalized economy.

Definition 1.1 *A system of budget constraints for E is an n-tuple of function $v^i : \Delta^l \to R_+, i \in N$, that satisfy*

$$v^i \textit{ is continuous for all } i \in N; \quad (1.3)$$

$$\sum_{i=1}^{n} v^i(p) = \sum_{i=1}^{n} p \cdot w^i + p \cdot t \textit{ for all } p \in \Delta^l . \quad (1.4)$$

(Here Δ^l is the ($l - 1$)-dimensional standard simplex.)

Example 1.2 *Let* $w = \sum_{i=1}^{n} w^i \in R^l_{++}$ *and define*

$$v^i(p) = p \cdot w^i + \frac{p \cdot w^i}{p \cdot w} p \cdot t \textit{ for all } i \in N \textit{ and } p \in \Delta^l. \quad (1.5)$$

Thus, we obtain Thomson's proportional rule. It satisfies (1.3) and (1.4).

Definition 1.3 *A competitive equilibrium of E relative to the budget constraints $v^1, ..., v^n$ is an (n+1)-tuple $< x^1, ..., x^n; p >$ that satisfies*

$$x^i \in R^l_+ \textit{ for } i \in N \textit{ and } \sum_{i=1}^{n} x^i = w + t \textit{ (here } w = \sum_{i=1}^{n} w^i). \quad (1.6)$$

$$p \in \Delta^l . \quad (1.7)$$

$$p \cdot x^i \leq v^i(p) \text{ for } i \in N. \quad (1.8)$$

$$[y \in R^l_+, i \in N, \text{ and } u^i(y) > u^i(x^i)] \Rightarrow p \cdot y > v^i(p). \quad (1.9)$$

Theorem 1.4 *Let $E =< w^1, ..., w^n; u^1, ..., u^n; t >$ be a generalized economy that satisfies (1.1) and (1.2). If $v^1, ..., v^n$ is a system of budget constraints for E that satisfies (1.3) and (1.4), then there exists a competitive equilibrium of E relative to $v^1, ..., v^n$.*

Korthues [1995] proved the existence of proportional equilibrium, that is, equilibrium relative to (1.5).

Proof: STEP 1: Let $q = 2[\sum_{i=1}^{n} w^i + t]$ and let $Q = \{x \in R^l_+ \mid x \leq q\}$. We claim that if the (n+1)-tuple $< x^1, ..., x^n; p >$ satisfies (1.6), (1.7), (1.8) and

$$[y \in Q, i \in N, \text{ and } u^i(y) > u^i(x^i)] \Rightarrow p \cdot y > v^i(p), \tag{1.10}$$

then it also satisfies (1.9), that is, it is an equilibrium. Assume, on the contrary, that there exist $y \in R^l_+$ and $i \in N$ such that $u^i(y) > u^i(x^i)$ and $p \cdot y \leq v^i(p)$. Clearly, $y \neq 0$. Hence, there is $z \in R^l_+, z \leq y, z \neq y$, such that $u^i(z) > u^i(x^i)$. For every $0 < \lambda < 1$ $u^i(\lambda z + (1-\lambda)x^i) \geq u^i(x^i)$. Thus, by strict monotonicity, $u^i(\lambda y + (1-\lambda)x^i) > u^i(x^i)$. However, $x^i \leq w + t << q$. Hence, for $\lambda > 0$ sufficiently small, $\lambda y + (1-\lambda)x^i \leq q$ contradicting (1.10).

STEP 2: For $i \in N$ and $p \in \Delta^l$ we denote

$$\hat{B}^i(p) = \{x \in Q \mid p \cdot x \leq v^i(p)\} \tag{1.11}$$

Then $\hat{B}^i(\cdot)$ is upper hemicontinuous. Furthermore, if $v^i(p) > 0$ then $\hat{B}^i(\cdot)$ is lower hemicontinuous at p. Indeed, upper hemicontinuity of $\hat{B}^i(\cdot)$ follows from the continuity of v^i. Assume now that $v^i(p) > 0, x \in \hat{B}^i(p), p(k) \in \Delta^l, k = 1, 2, ...,$ and $p(k) \to p$. Let $y(k)$ be defined by

$$[0, y(k)] = \hat{B}^i(p(k)) \cap [0, x]$$

Clearly, if $p \cdot x < v^i(p)$ then $y(k) = x$ for k sufficiently large. Thus, assume $p \cdot x = v^i(p)$. We claim that $y(k) \to x$. Assume, on the contrary, that there exists a subsequence $y(k_j) \to y$ and $y < x$. Then $p \cdot y < p \cdot x$ (because $p \cdot x = v^i(p) > 0$). Hence $p(k_j) \cdot y(k_j) \to p \cdot y < p \cdot x$. Thus, for j sufficiently large, $p(k_j) \cdot y(k_j) = v^i(p(k_j))$, and, therefore, $v^i(p(k_j)) \to p \cdot y$ while $p \cdot y < v^i(p)$, which is impossible because $v^i(\cdot)$ is continuous.

STEP 3: For $i \in N$ and $p \in \Delta^l$ define

$$\hat{D}^i(p) = \{x \in \hat{B}^i(p) \mid u^i(x) \geq u^i(y) \text{ for all } y \in \hat{B}^i(p)\}. \tag{1.12}$$

$\hat{D}^i(p)$ is i's demand correspondence for the bounded economy. Further, let

$$F^i(p) = \begin{cases} \hat{D}^i(p), \text{ if } v^i(p) > 0 \\ \hat{B}^i(p), \text{ if } v^i(p) = 0 \end{cases} \tag{1.13}$$

Then $F^i(p)$ is upper hemicontinuous. The proof is straightforward. We also notice that if $x \in F^i(p)$ then $p \cdot x = v^i(p)$.

STEP 4: Let $\hat{e}(p) = \sum_{i=1}^{n} F^i(p) - w - t$ and $y \in \hat{e}(p)$. Then $y = \sum_{i=1}^{n} x^i - w - t$ where $x^i \in F^i(p)$, $i \in N$. Thus

$$p \cdot y = \sum_{i=1}^{n} p \cdot x^i - p \cdot w - p \cdot t = \sum_{i=1}^{n} v^i(p) - p \cdot w - p \cdot t = 0 \qquad (1.14)$$

Therefore, by Lemma AIV.1 of Hildenbrand and Kirman (1988) there exists $\overline{p} \in \Delta^l$ such that $\hat{e}(\overline{p}) \cap R^l_- \neq \emptyset$. Thus, there exists a feasible allocation $\overline{x} = < \overline{x}^1, ..., \overline{x}^n >$ such that $\overline{x}^i \in F^i(\overline{p})$, $i \in N$.

STEP 5: $< \overline{x}^1, ..., \overline{x}^n; \overline{p} >$ is a competitive equilibrium of the bounded economy. By strict monotonicity of $u^1, ..., u^n$, $\overline{p}_j > 0$, $j = 1, ..., l$. Hence, by (1.13), $\overline{x}^i \in \hat{D}^i(\overline{p})$ for $i \in N$. Q.E.D.

References

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