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Conservation of Energy in Nonatomic Games

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Abstract

The Shapley–Value for games with a continuum of players of finitely many types can be uniquely characterized by the potential approach. The proof gives a clear insight in the problem and might be extended to a more general setup. Moreover like in physics there is a theorem of conservation (of energy). In this sense the Shapley–Value is the only efficient solution concept which conserves the ability of obtaining utility.

1 The Shapley–Value

Hart and Mas-Colell[1] offered in 1989 to extend the potential approach to games with a continuum of players of finitely many types. Here now is a proof of this idea. It is based on partial differential equations and might be extended to a more general setup.

1.1 Axiomatization

For $\bar{z} \in \mathbf{R}^n_+$ let $B^+_{\bar{z}} := \{z \in \mathbf{R}^n_+ : z_i \leq \bar{z}_i,\ i = 1, \cdots, n\}$. Then there is the following definition.

Definition 1.1 *A game with finitely many types is a tupel* $\Sigma = (\bar{z}, f)$, *where*

$\bar{z} \in \mathbf{R}^n_+,\ n \in \mathbb{N}$	grand coalition
$f:\ B^+_{\bar{z}} \subseteq D \to \mathbf{R},\ f(0) = 0$	characteristic funktion

Remark. The number of different types is n. A coalition is represented by a profile $z \in B^+_{\bar{z}}$, where z_i is the mass of players of type i. The worthes of the coalitions are given by the function f. Sometimes one designates a characteristic function already as a game, if per definitionem the grand coalition and the domain of definition can be recognized.

Now for the types there is the question of allocation .

Definition 1.2 *A solution concept is an operator which assigns to every game* $\Sigma = (\bar{z}, f)$ *exactly one element of* $\mathbf{R}^n$.

Remark. For arbitrary, but fixed $\bar{z} \in \mathbf{R}^n_+$ and arbitrary $f : B^+_{\bar{z}} \to \mathbb{R}$ with $f(0) = 0$ a solution concept may be viewed as a functional $\Phi_f : B^+_{\bar{z}} \to \mathbf{R}^n$.

[1] cf. [2], [3]

Definition 1.3 *A solution concept $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ is efficient, if for all $z \in B_{\bar{z}}^+$ it is true that*

$$< \Phi_f(z), z >= f(z) \tag{1}$$

Theorem 1.4 *Let $\bar{z} \in \mathbf{R}_+^n$ and $f \in C^1(B_{\bar{z}}^+)$ with $f(0) = 0$ be arbitrary. Then there is exactly one solution concept $\Psi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ which fulfills the following properties.*

(i) *Ψ_f is efficient.*

(ii) *Ψ_f is a gradient field, i.e., there exists a continously differentiable potential $V_f : B_{\bar{z}}^+ \to \mathbf{R}$ such that*

$$\Psi_f = \text{grad } V_f \tag{2}$$

Proof. Let $\Psi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ be defined as followed

$$\begin{aligned} \Psi_f(z) &:= \int_0^1 \text{grad } f(tz)\, dt \\ &= \text{grad} \int_0^1 \frac{1}{t} f(tz)\, dt \end{aligned}$$

Therefore Ψ_f is a gradient field. Hence (ii) is true. It remains to show effiency. For for arbitrary $z \in B_{\bar{z}}^+$ it is true that

$$\begin{aligned} < \Psi_f(z), z > &= < \int_0^1 \text{grad } f(tz)\, dt,\ z > \\ &= \int_0^1 < \text{grad } f(tz),\ z dt > \\ &= \int_0^z < \text{grad } f(x),\ dx > \\ &= \int_0^z df(x) \\ &= f(z) \end{aligned}$$

Now let $\Phi_f : B_z^+ \to \mathbf{R}^n$ be an additional solution concept with the demanded properties (i) and (ii). Then consider the difference $w_f := \Phi_f - \Psi_f$. Because of linearity of the gradient there is a continously differentiable potential $v_f : B_z^+ \to \mathbf{R}$, such that $w_f = \text{grad } v_f$. And for $z \in B_z^+$

$$\begin{aligned} < w_f(z), z > &= < \Phi_f(z) - \Psi_f(z), z > \\ &= < \Phi_f(z), z > - < \Psi_f(z), z > \\ &= f(z) - f(z) \\ &= 0 \end{aligned}$$

This implies a homogeneous linear partial differential equation for v_f.

$$\sum_{i=1}^{n} z_i \frac{\partial v_f(z)}{\partial z_i} = 0 \tag{3}$$

Every constant function fulfills this equation. Suppose there is a function $v_f : B_z^+ \to \mathbf{R}$ which is twice continously differentiable and not constant. Then there are two different points $z_1, z_2 \in B_z^+ \setminus \{0\}$ where v_f has the different values v_1 respectively v_2. Now one considers the two curves $\xi, \eta : \mathbf{R} \to \mathbf{R}^n$, defined by $\xi(t) = z_1 \exp(t)$, $\eta(t) = z_2 \exp(t)$. It is true while $\xi(t) \in B_z^+$

$$\begin{aligned} \frac{dv_f(\xi(t))}{dt} &= \sum_{i=1}^{n} \frac{\partial v_f(\xi(t))}{\partial \xi_i(t)} \frac{d\xi_i(t)}{dt} \\ &= \sum_{i=1}^{n} \xi_i(t) \frac{\partial v_f(\xi(t))}{\partial \xi_i(t)} \\ &= 0 \end{aligned}$$

Thus v_f is along the curve ξ and mutatis mutandis also along η constant. This means, the sets $\{(\xi(t), v_1) : t \in \mathbf{R},\ \xi(t) \in B_z^+\}$ and $\{(\eta(t), v_2) : t \in \mathbf{R},\ \eta(t) \in B_z^+\}$ are two contour lines of the function v_f. Hence it follows

$$\lim_{t \to \infty} \xi(t) = 0 \qquad \text{and} \quad \lim_{t \to \infty} \eta(t) = 0$$

but

$$\lim_{t \to \infty} v_f(\xi(t)) = v_1 \neq v_2 = \lim_{t \to \infty} v_f(\eta(t))$$

Thus is v_f not continous in zero in contradiction to the assumption. Therefore v_f can only be constant and hence w_f is equal to zero. Thus $\Phi_f = \Psi_f$. □

Remark. The demand for continuity in the origion is very strong. It gives the uniqueness. If one deals only with grand coalitions $z >> 0$ then this demand might be dropped with the result that there is a whole family of solution concepts wich fulfill the properties (i) and (ii) everywhere except in the origion.

Definition 1.5 *For arbitrary, but fixed $z \in \mathbf{R}^n_+$ and arbitrary $f \in C^1(B_z^+)$ with $f(0) = 0$ the solution concept $\Psi_f : B_z^+ \to \mathbf{R}^n$ defined by*

$$\Psi_f(z) = \int_0^1 \text{grad } f(tz)\, dt \tag{4}$$

is called the Shapley–Value.

Remark. In the class of infinite games with continously differentiable characteristic function f the Shapley–Value Ψ_f can be uniquely characterized by the properties (i) and (ii) of theorem 1.4.

1.2 Conservity

With analysis one can verify the following corollary.

Corollary 1.6 *Let $z \in \mathbf{R}^n_+$ and $f \in C^1(B_z^+)$ with $f(0) = 0$ be arbitrary. Then it is equivalent for a solution concept $\Psi_f : B_z^+ \to \mathbf{R}^n$*

(i) *Ψ_f is the Shapley–Value, i.e., Ψ_f is a gradient field, i.e., thereis a continously differentiable potential $V_f : B_z^+ \to \mathbf{R}$ such that $\Psi_f = \text{grad } V_f$.*

(iii) *Ψ_f is conservative, i.e., for every piecewise continously differentiable closed way $\gamma : [0, 1] \to B_z^+$ with $\gamma(0) = \gamma(1)$ it is true that*

$$\oint_\gamma < \Psi_f(z), dz >= 0 \tag{5}$$

If $f \in C^2(B_z^+)$ then (i), (ii) *are equivalent to*

(iii) Ψ_f *fulfills the integrability condition, i.e., for all* $i = 1, \cdots, n$ *and all* $z \in B_{\bar{z}}^+$ *holds*

$$\frac{\partial(\Psi_f)_i(z)}{\partial z_j} = \frac{\partial(\Psi_f)_j(z)}{\partial z_i} \tag{6}$$

With the following definition one can make the property of conservity more clear.

Definition 1.7 *For arbitrary, but fixed* $\bar{z} \in \mathbf{R}_+^n$ *and arbitrary* $f \in \mathcal{C}^1(B_{\bar{z}}^+)$ *with* $f(0) = 0$ *let* $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ *be a solution concept,* $z_a, z_e \in B_{\bar{z}}^+$ *two arbitrary coalitions and* $\gamma : [0,1] \to B_{\bar{z}}^+$ *a piecewise continously differentiable way with* $\gamma(0) = z_a$ *and* $\gamma(1) = z_e$*. Then*

$$W(\Phi_f, z_a, z_e, \gamma) := \int_{z_a, \gamma}^{z_e} < \Phi_f(z), dz > \tag{7}$$

is the expenditure of Φ_f *for the pair* (z_a, z_e) *with respect to the way* γ*.*

Then there is the following well known proposition.

Proposition 1.8 *With the assumptions of the preceeding definition the following is equivalent*

(i) Φ_f *is conservative.*

(ii) *The expenditure* $W(\Phi_f, z_a, z_e, \gamma)$ *is independent of the way* γ*.*

If Φ_f *is conservative, then there exists a potential* $V_f : B_{\bar{z}}^+ \to \mathbf{R}$*, such that* $\Psi_f = \text{grad } V_f$*. Hence particularly for the expenditure W*

$$W(\Phi_f, z_a, z_e, \gamma) = V_f(z_e) - V_f(z_a) \tag{8}$$

In the physical sense the expenditure corresponds to the work which is independent of the way just for conservative forces. The sign is an agreement with the author.

The expenditure has a clear interpretation[2]. Imagine the grand coalition of a given game $\Sigma = (\bar{z}, f)$ has come in terms with a certain solution concept Φ. Now let there be a dynamic situation in which $z(t)$ is the grand coalition at time t. Players might leave the game by receiving their payoffs according to Φ from a certain master of the game. And they can enter the game by transfering exactly the amount of utility to the master which they will afterwards get by Φ according to the new situation. It is only required that $z(t)$ is piecewise continously differentiable. That means changes in the grand coalition shall be smooth enough.

The master himself may be viewed as a "*deus ex machina*". Every arbitrary coalition $z \in B_{\bar{z}}^{+}$ can be the master if the players for example want to play the game on their own or want to gain utility on certain closed ways.

The expenditure $W(\Phi_f, z_a, z_e, \gamma)$ then is exactly the amount of utility which has to be transfered to the master if the coalition z_e comes into the game according to γ while starting with the coalition z_a.

If one deals with a solution concept which is not conservative then there is at least one way for which the expenditure is positive. This utility is deprived from the grand coalition $\bar{z}$. A repetition might be done such that more and more utility is deprived from the players. For conservative solution concepts on the other side the whole transferable utility of the players is the same at every time. This will be proofed formaly in the following.

2 Conservation of Energy

In classical physics conservative forces play an important role. Examples are the graviational force, the Coulomb force, the force of an linaear harmonic oszillator and so on. Conservaty implies conservation of energy which means that the whole mechanic energy is the same at every time. This theorem is the most important one in mechanics. To have a corresponding one in game theory would be highly desirable.

[2]cf. [7]

2.1 Characterization of Movement

First it is useful to watch at a time dependent grand coalition. The payoffs according to the arranged solution concept are the cause of movement of the grand coaliton. According to the interpretation above it is possible for the players to enter the game by paying the master or to get out by receiving a certain amount. Therefore the solution concept is the moving force. Quantitatively speaking players which will receive a hugh amount of utility are more interesting for the master.

The next aim is to give a complete foretell of this movement analog to classical mechanics. To do so one needs an axiomatization of the movement corresponding to Newton's Law.

Postulat *Every grand coalition has a scalar property given by a positive real number which is called psychical inertia.*

In general κ is time dependent as well as dependent of the grand coalition. But in the following the psychical inertia shall be independent of inner processes of the grand coalition and time. Hence κ might be viewed as a constant. Thus one has the following definition.

Definition 2.1 *The product of psychical inertia and velocity of the grand coalition is called impulse p.*

$$p = \kappa \dot{z}(t) \tag{9}$$

Thus all notions have been defined which are necessary for the game theoretical generalization of Newton's Laws.

Lex prima *The time dependent change of the impulse is equal to the moving force.*

$$F = \frac{d}{dt}(\kappa \dot{z}(t)) \tag{10}$$

This law of movement will be the fundamental dynamic equation of cooperative game theory. It has to be solved to give an exact foretell of the movement. Just for completeness there are two other laws.

Lex secunda *A grand coalition without influence stays still or in the state of straight uniform movement.*

Remark. In the physical sense this law is the theorem of conservation of impulse. With constant psychical inertia it is a special case of law one.

Lex tertia *If there are the $F_1, \cdots, F_n$ acting upon the same grand coalition, then they may be added.*

$$F = \sum_{i=1}^{n} F_i \tag{11}$$

2.2 Energy

The notion of energy is very important in mechanics. In the physical sense energy is the ability of doing work. There are different kinds of energy.

The game theoretical analogon to energy is the ability of obtaining utility (from the master). They are also two kinds. Let Φ_f be a conservative solution concept and V_f a potential of Φ_f. Set $V_f(0) = 0$, then the potential $V_f(z)$ describes exactly the amount of utility which has to be transfered to the master to bring the members of the coalition into the game. In this sense $-V_f(z)$ is the potential ability of the coalition z to obtain utility (from the master). It is the game theoretical analogon to the potential energy in physics. The sign says that this amount has to be paid from the master.

The next proposition makes it possible to extend the notion of potential energy to arbitrary solution concepts.

Proposition 2.2 *Let* $\Sigma = (\bar{z}, f)$ *be an arbitrary infinite game and let* $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ *be a continously differentiable solution concept. Then there exists a unique function* $\varphi_{\text{cons}} : B_{\bar{z}}^+ \to \mathbf{R}^n$ *such that the following two properties ar fulfilled*

(i) φ_{cons} *is conservative*

(ii) $< \varphi_{\text{cons}}(z), z > = < \Phi_f(z), z > \qquad \forall z \in B_{\bar{z}}^+$

This φ_{cons} *is called the corresponding conservative concept.*

Proof. Consider the game $\Sigma' = (\bar{z}, < \Phi_f(z), z >)$. Then by assumption $< \Phi_f(z), z >$ is continously differentiable. Hence by theorem 1.4 and corollary 1.6 there is a unique solution concept φ_{cons} which fulfills (i) and (ii). □

On the other side there is an analogon to kinetic energy as well. Once the grand coalitions has begun to move, it can only change direction or be stopped by transfering utility to or from the master. This type of energy is implied by the fact that $z(t)$ shall be piecewise continously differentiable. Changes of the grand coalition have to be smooth enough which implies that "queues" of players will arise. As in physics the kinetic energy T at time t is defined by $T = \frac{1}{2}\kappa \dot{z}^2(t)$.

Now there is a formal definition of energy resulting by a solution concept.

Definition 2.3 *Let* $\Sigma = (\bar{z}, f)$ *be an arbitrary infinite game and let* $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ *be a continously differentiable solution concept. Moreover let* φ_{cons} *be the included conservative concept and let* $V_{\text{cons}} : B_{\bar{z}}^+ \to \mathbf{R}$ *be such that* $\varphi_{\text{cons}} = \operatorname{grad} V_{\text{cons}}$. *Then one has the following kinds of energy.*

- *The kinetic energy is given by* $T = \frac{1}{2}\dot{z}^2(t)$.
- *The potential energy is given* $-V_{\text{cons}}$.
- *The whole energy is given by* $E = T + (-V_{\text{cons}})$.

2.3 Conservation of Energy

By time going on kinetic energy permanently changes to potential energy and vice versa. In conservative fields the whole energy is the same at every time. In the game theoretical context this means that the ability of obtaining utility (from a master) is conserved. That is the theorem of conservation in game theory.

Theorem 2.4 *Let $\Sigma = (\bar{z}, f)$ be an arbitrary infinite game and let $\Phi_f : B_{\bar{z}}^+ \rightarrow R^n$ be a conservative solution concept. Then the ability of obtaining utility is the same at every time.*

Proof. The law of movement is here

$$\Phi_f(z(t)) = F = \kappa \ddot{z}(t) \tag{12}$$

For two arbitrary points of time t_a, t_e it is true for the work

$$\begin{aligned} W(\Phi_f, z(t_a), z(t_e), z) &= \int_{z(t_a)}^{z(t_e)} < F(z(t)),\ dz(t) > \\ &= \int_{t_a}^{t_e} < F(z(t)),\ \dot{z}(t) > \ dt \\ &= \int_{t_a}^{t_e} < \kappa \ddot{z}(t),\ \dot{z}(t) > \ dt \\ &= \int_{t_a}^{t_e} \frac{d}{dt} \frac{\kappa \dot{z}^2(t)}{2}\ dt \\ &= \left[\frac{\kappa \dot{z}^2(t)}{2} \right]_{t_a}^{t_e} \\ &= T(t_e) - T(t_a) \end{aligned}$$

where $T(t) = \frac{\kappa \dot{z}^2(t)}{2}$ is the kinetic energy of the grand coalition at time t. On the other side by corollary 1.6 there is a potential $V_f : B_{\bar{z}}^+ \rightarrow \mathbb{R}$ such that $\Phi_f = \text{grad } V_f$. Therefrom by proposition 1.8

$$W(\Phi_f, z(t_a), z(t_e), z) = V_f(t_e) - V_f(t_a)$$

Hence

$$T(t_a) + (-V_f(t_a)) = T(t_e) + (-V_f(t_e)) \tag{13}$$

The sum of kinetic and potential energy is the same at every time. □

This theorem describes formaly the impossibility of being tricked with conservative solution concepts. Here it is not possible to gain utility from the players by acting as a master in contrary to non-conservative solution concepts.

With the additional constraint of effiency the Shapley–Value can be uniquely characterized.

Theorem 2.5 *Let $\Sigma = (\bar{z}, f)$ be an arbitrary infinite game and let $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ be a continously differentiable and efficient solution concept. Then Φ_f conserves the ability of obtaining utility, if and only if Φ_f is equal to the Shapley-Value Ψ_f.*

Proof. "$\Leftarrow$" Ψ_f is continously differentiable per definitionem and efficient by theorem 1.4. By corollary 1.6 and by theorem 2.4 the conservation of energy is true.

"$\Rightarrow$" Now let $\Phi_f : B_{\bar{z}}^+ \to \mathbf{R}^n$ be an additional continously differentiable and efficient solution concept. For the time dependent derivative of the kinetic energy T it is true that

$$\begin{aligned}\frac{d}{dt}T &= \frac{d}{dt}\left(\frac{1}{2}\kappa\dot{z}^2(t)\right)\\ &= <\kappa\ddot{z}(t), \dot{z}(t)>\\ &= <\Phi_f(z(t)), \dot{z}(t)>\end{aligned}$$

and the time dependent derivative of the potential energy $(-V_{\text{cons}})$ is

$$\frac{d}{dt}(-V_{\text{cons}}) = -<\text{grad } V_{\text{cons}}, \dot{z}(t)>$$

By theorem 1.4 and corollary 1.6 Φ_f is not conservative. Then there is a closed way γ, such that for the difference $w_f := \Phi_f - \text{grad } V_{\text{cons}}$ holds

$$\begin{aligned} \oint_\gamma < w_f(z), dz > &= \oint_\gamma < \Phi_f(z) - \text{grad } V_{\text{cons}}, dz > \\ &= \oint_\gamma < \Phi_f(z), dz > \\ &\neq 0 \end{aligned}$$

Therefrom there is a $t \in \mathbf{R}$, such that $< w_f(z(t)), \dot{z}(t) > \neq 0$. Hence

$$\begin{aligned} \frac{d}{dt}(T + (-V_f)) &= < \Phi_f(z(t)), \dot{z}(t) > - < \Psi_f(z(t)), \dot{z}(t) > \\ &= < w_f(z(t)), \dot{z}(t) > \\ &\neq 0 \end{aligned}$$

for at least one $t \in \mathbf{R}$. Therefore Φ_f does not conserve energy. □

Premises for these two theorems are an axiomatization of the movement and the definition of kinetic energy. If one is not concerned with the given definitions then this doesn't matter. It is possible to propose an arbitrary law of movement resulting in another definition of kinetic energy. But mutatis mutandis the theorems remain true.

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