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Rationalizability of the Nash Bargaining Solution

by

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Abstract

This paper discusses the possibilities to rationalize the Nash Bargaining solution. For one of the two discussed concepts of rationalizability a solution is offered. A minimal set of requirements on a social planner's preference on the set of a group's utility allocations is presented which ensures that his uniquely determined choice is the Nash Bargaining solution. As this solution maximizes the Nash product it is a rationalizable choice in the sense of Richter (1971).

0 Introduction

Nash's (1950, 1953) seminal papers on two-person cooperative games presented a fundamental solution concept known today as the *Nash Bargaining Solution.*

Nash also tried to rationalize this axiomatically defined concept in two different ways.

On the one hand he characterized the Nash solution as the unique maximizer of the product of the two player's payoffs from feasible outcomes, the so called *Nash product.* This characterization my be seen as a result on rationalizability in the sense of Richter (1971).

On the other hand Nash initiated what is known in modern game theory as the *"Nash program"*, i.e. he tried to realize the Nash outcome of a two-person cooperative game as an equilibrium of some suitably definded non-cooperative game between the two players. While his simple demand game failed due to the large amount of equilibria, his smoothed demand game approach resulted in an approximate solution for this problem and thereby established (approximately) rationalizability of the Nash solution in the sense of Bernheim (1984) and Pearce (1984).

What remained to be done in terms of analyzing rationalizability of the Nash solution is the following:
First, the approximate realization of the Nash solution as a non-cooperative equilibrium had to be sharpened to an exact one. This has been done in different ways by Howard (1992) and by Trockel (1998). I shall briefly comment on this and on the underlying rationality concept in section 2.

Secondly, it had to be clarified what kind of assumptions on a social planner's preference relation over player's utility allocation space result in representability by the Nash product. A discussion of the underlying rationality concept is the content of section 3. Section 4 concludes with the result which is an application of section 3 to the Nash solution.

1 Rationalizable strategies

Even if we know what rational behavior of an isolated individuum is it does not imply what rational behavior in a context of several interacting rational individuals could mean. While we shall consider isolated rational choice in section 3 we consider rational choice in strategic contexts now.

A *non-cooperative strategic n-person game* is a tupel $\Gamma = (S_1, \ldots, S_n; u_1, \ldots, u_n)$, where for each player $i \in \{1, \ldots, n\}$ the non-empty set S_i is interpreted as his set of available strategies and the map $u_i : \prod_{j=1}^{n} S_j \longrightarrow I\!R$ represents his payoff dependent on the other players' menue of chosen strategies.

The players' choices of strategies are assumed to be made independently of each other and without knowledge of the other players' choices. Rationality of all players (in the sense of section 3) is assumed to be common knowledge among them.

Now any rationalizable strategy should be one whose selection is consistent with all players' rationality. Accordingly, Bernheim (1984) and Pearce (1984) called a strategy of some player *rationalizable* if it survives iterated removal of all those strategies from the strategy set of the players that are never a best response to the remaining players' combined strategy choices.

It is not necessary for our purpose to dwell on this notion or to go into more details because, as we shall see, Nash equilibria of non-cooperative strategic games are n-tupels of rationalizable strategies. Therefore, realizing the Nash solution of a cooperative two-person game as a Nash equilibrium of some two-person non-cooperative game establishes automatically rationalizability in this sense.

Now Howard (1992) constructed a multi-stage game, each subgame-perfect equilibrium of which attributes to each player exactly what he would get in the Nash solution of the underlying cooperative game. *Subgame perfect equilibria* were defined by Selten (1965) and are particularly interesting Nash equilibria in that they induce Nash equilibria in all subgames of the original game.

Trockel (1998) contains a modification of Nash's simple demand game whose unique Nash equilibrium, which is even in dominant strategies, exactly realizes the outcome of the Nash solution of the underlying cooperative game.

As a result of these articles the Nash solution can be realized by both players choosing rationalizable strategies of some suitable non-cooperative two-person game.

2 Rationalizable choice rules

Rationality of an individual who has to make a decision, which out of a set of alternatives it chooses, is usualy formalized by a complete, transitive binary relation on the set of alternatives, called *preference relations.* Completeness guaranties universal comparability of alternatives while transitivity reflects consistency of direct and indirect comparisons of alternatives.

An element x of a set X of alternatives is a *$\succsim$-best element* of X, if and only if $x \succsim y$ for all $y \in X$. Now for any family $\mathcal{F}$ of subsets of X a *choice rule* is a correspondence $\varphi : \mathcal{F} \Rightarrow X : A \Rightarrow \varphi(A) \subset A$. Here $\varphi(A)$ represents the chosen elements where all alternatives in A were available. If all the choice sets $\varphi(A), A \in \mathcal{F}$ are singletons the choice rule is a *choice function.*

According to Richter (1971) a choice rule is *rationalizable* if it always picks exactly the $\succsim$-best elements of each member set of $\mathcal{F}$. The preference $\succsim$ on X is called *representable by the utility function* $u : X \longrightarrow \mathbb{R}$ if for all x, $y \in X$ one has

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Obviously, any real valued function u on X defines a preference on X which is represented by u.

To see the relation between the two notions of rationalizability consider a one-person strategic game (S_1, u_1). Then S_1 is the set of alternatives X and u_1 a utility function on X. The best elements with respect to the preference relation $\succsim$ induced on X by u are exactly the Nash equilibria of the game and thus rationalizable in the sense of Bernheim and Pearce. The restriction of u resp. $\succsim$ on any subset of $S_1 = X$ induces a choice rule on the power set of X which is rationalizable in the sense of Richter.

3 Rationalizability of the Nash Solution

We start this section with the notion of a *two-person bargaining game.* This is generally defined as a compact, strictly convex, non-empty subset B of $\mathbb{R}^2_+$ which is comprehensive with respect to $\mathbb{R}^2_+$, i.e. $x \in B \Rightarrow \{x' \in \mathbb{R}^2 \mid 0 \leq x' \leq x\} \subset B$, together with a point $d \in B$ such that $d + \mathbb{R}^2_{++} \cap B \neq \{\}$.

The points in B represent feasible utility allocations for the two players. Often those are thought to be the evaluations of outcomes of an underlying economic or social scenario by cardinal utility functions, for instance von Neumann-Morgenstern utilities. This interpretation suggests to allow for any affine transformations of both players' utilities to get a different "equivalent" bargaining game B'. Therefore we set without loss of generality $d = 0$. Still we have the choice of a unit for each player as a degree of freedom in determining B.

Let $\mathcal{B}$ denote the set of all bargaining games (with $d = 0$), i.e. the set of all subsets B of $\mathbb{R}^2_+$ as described above. Then a *bargaining solution* is a map $\varphi : \mathcal{B} \longrightarrow \mathbb{R}^2_+ : B \longrightarrow \varphi(B) \in B$. Thus a bargaining solution is a choice function on the family $\mathcal{B}$ of subsets of $\mathbb{R}^2_+$.

The *Nash bargaining solution* φ_N is defined by

$$B \longrightarrow \underset{x \in B}{\operatorname{argmax}}\ x_1 * x_2.$$

Before we formulate this paper's main result, we have to define some properties of binary relations.

A binary relation $\succsim$ on $\mathbb{R}^2_{++}$ is called *unit-invariant* if and only if, for any x, y, $z \in \mathbb{R}^2_{++}$ one has:

$$x \succsim y \Leftrightarrow z * x \succsim z * y.$$

Here $*$ denotes coordinatewise multiplication.

A binary relation R von $\mathbb{R}^2$ is called *translation-invariant* if and only if for any u, v, $w \in \mathbb{R}^2$ one has:

$$u \, R \, v \Leftrightarrow w + u \, R \, w + v$$

In the following we make use of derived binary relations, which we define now.

For $\succsim$ resp. R we define the derived *strict* and *indifference relations* $\succ$ resp. P and $\sim$ resp. I as follows:

$$x \succ (P)y :\Leftrightarrow x \succsim (R)y \text{ and not } y \succsim (R)x$$
$$x \sim (I)y :\Leftrightarrow x \succsim (R)y \text{ and } y \succsim (R)x.$$

A binary relation $\succsim$ on a subset X of $\mathbb{R}^2$ is called *indifference-invariant* if and only if for any x, x', y, $y' \in X$ one has:

$$x \succ y,\ x' \sim x,\ y' \sim y \Rightarrow x' \sim y'.$$

A binary relation $\succsim$ on $X \subset \mathbb{R}^2$ is called *neutral* if and only if

$$(x_1, x_2) \succsim (y_1, y_2) \Rightarrow (x_2, x_1) \succsim (y_2, y_1).$$

A binary relation $\succsim$ on $X \subset \mathbb{R}^2$ is called *monotone* if and only if for all x, $y \in X$:

$$y \in x + \mathbb{R}^2_{++} \Rightarrow y \succ x.$$

A binary relation $\succsim$ on an open subset X of $\mathbb{R}^2$ is called *lower continuous* if and only if for all $x \in X$ the sets $\{x' \in X \,|\, x \succ x'\}$ are open and the sets $\{x' \in X \,|\, x \succsim x'\}$ are closed.

After these preparations we are able to state our result.

Proposition: Let $\succsim$ be a lower continuous, unit-invariant, indifference-invariant, neutral, monotone binary relation on $\mathbb{R}^2_{++}$. Then $\succsim$ rationalizes the Nash solution, i.e. $\varphi_N(B)$ is the unique $\succsim$-best element in B for all $B \in \mathcal{B}$.

Proof: Following Trockel (1989) we exploit the fact that the map $L : (\mathbb{R}^2_{++}, *) \longrightarrow (\mathbb{R}^2, +) : (x_1, x_2) \longrightarrow (\ln x_1, \ln x_2)$ is an isomorphism between topological groups. It induces in a natural way a bijection between binary relations on $\mathbb{R}^2_{++}$ and on $\mathbb{R}^2$ which moreover respects monotony. So for any binary relation $\succsim$ on $\mathbb{R}^2_{++}$ we get a binary relation $R := L^*(\succsim)$ via

$$x \succsim y \Leftrightarrow L(x) \, R \, L(y).$$

Observe, that $R = L^*(\succsim)$ is translation-invariant if and only if $\succsim$ is unit-invariant. Also lower continuity, neutrality and indifference-invariance are preserved under the map L^*. These properties of R allow us to apply the Proposition in Neuefeind and Trockel (1995) to conclude that R is representable by a continuous linear utility function. That means there exist some $p \in I\!R^2$ such that for any $v, w \in I\!R^2$ we have

$$vRw \Leftrightarrow p \bullet v \geq p \bullet w.$$

Here $\bullet$ denotes the inner product in $I\!R^2$. Monotony of R yields $p > 0$ (i.e. $p \geq 0$ and $p \neq 0$). Neutrality moreover yields $p_1 = p_2$. Without loss of generality we normalize p such that $p_1 + p_2 = 1$, thus $p_1 = p_2 = 1/2$. Accordingly, we get for $x := L^{-1}(v)$, $y := L^{-1}(w) \in I\!R^2$:

$$x \succsim y \Leftrightarrow vRw \Leftrightarrow p \bullet v \geq p \bullet w \Leftrightarrow p \bullet L(x) \geq p \bullet L(y) \Leftrightarrow p_1 \ln x_1 + p_2 \ln x_2 \geq p_1 \ln y_1 + p_2 \ln y_2 \Leftrightarrow \ln(x_1^{p_1} * x_2^{p_2}) \geq \ln(y_1^{p_1} * y_2^{p_2}) \Leftrightarrow x_1^{p_1} * x_2^{p_2} \geq y_1^{p_1} * y_2^{p_2} \Leftrightarrow x_1^{1/2} * x_2^{1/2} \geq y_1^{1/2} * y_2^{1/2} \Leftrightarrow x_1 * x_2 \geq y_1 * y_2.$$

Hence our binary relation is represented by the Nash product and, therefore, the Nash solution is rationalized by the preference relation $\succsim$. $\diamond$

We conclude with some remarks about the meaning of the assumptions we imposed on the social planner's preference relation.

Note that we did not assume rationality as represented by completeness and transitivity. Indifference-invariance is a very weak form of consistency, which does not imply transitivity and is even compatible with cycles. Continuity is a meaningful technical assumption which reflects adequacy of modelling an observable reality. Next recall that we are in a purely welfaristic context. Unit-invariance means that the planners preference is not affected by change of the players' cardinal utility representations. Neutrality just means what is says and monotony reflects the social planner's preference for allocations which are better for the players.

So a planner who is able to correctly observe the players' utility levels, who is without own interest but neutral and benevolent is automatically rational and picks the Nash solution.

References

Bernheim, B.D. (1984), "Rationalizable strategic behavior", **Econometrica, 52**, 1007–1028.

Howard, J.V. (1992), "A social choice rule and its implementation in perfect equilibrium", **Journal of Economic Theory, 56**, 142–158.

Nash, J.F. (1950), "The Bargaining Problem", **Econometrica, 18**, 155–162.

Nash, J.F. (1953), "Two-person cooperative games", **Econometrica, 21**, 128–140.

Neuefeind, W. and W. Trockel (1995) "Coninuoues linear representability of binary relations", **Economic Theory, 6**, 351–356.

Pearce, D.G. (1984), "Rationalizable strategic behavior and the problem of perfection", **Econometrica, 52**, 1022–1050.

Richter, M. (1971), "Rational choice", Chapter 2 in: Chipman, J., L. Hurwicz and H. Sonnenschein (eds.): **Preference, Utility and Demand**, Harcourt Brace Jovanovich, New York.

Selten, R. (1965), "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit", **Zeitschrift für die gesamte Staatswissenschaft, 121**, 301–324, 667–689.

Trockel, W. (1989), "Classification of budget-invariant preferences", **Economics Letters, 30**, 7-10.

Trockel, W. (1998), "An exact inplementation of the Nash Bargaining Solution in dominant strategies", in: Abromovich, Y. and N. Yannelis (eds.): **Functional Analysis and Economic Theory, Special Volume in Honour of Aliprantis**, Springer, New York, forthcoming.