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Axiomatizations of Game Theoretical Solutions
for One–Output Cost Sharing Problems

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Abstract

Moulin and Shenker (1992, 1993) discussed two cost sharing rules for one-output cost sharing problems. Average cost pricing divides the total cost generated by the aggregate demand proportionally to the demands among the agents. Serial cost sharing depends on both, the demand profile and the cost function evaluated at certain distinguished "derived" demands. Other cost sharing rules, which are based on the costs for the aggregate demands of all subgroups of agents only, can be obtained by considering solution concepts on the associated cost games. In the cost sharing context the Shapley value, prenucleolus, and anti-prenucleolus are called Shapley rule, nucleolus rule, and anti-nucleolus rule respectively. These game theoretical rules possess "covariance under strategical equivalence" as a common property. Together with "equal treatment of equals" and two versions of additivity on the one hand and some consistency property on the other hand the Shapley rule can be characterized. The nucleolus rules can be axiomatized analogously by changing the definition of the "reduced cost sharing problem" adequately. In the special case of concave cost functions a strong version of consistency can be used to axiomatize the nucleolus rule. Moreover, the anti-nucleolus rule is shown to be a core selector in this case. Cost functions for which the proposed game theoretical solutions coincide for fixed aggregate demand with average cost pricing are characterized by a simple functional equation. Cost functions with globally coinciding cost sharing rules are shown to be linear or parabolic.

Keywords: Cost sharing problem and rule, game, solution concept, Shapley value, nucleolus, average cost pricing

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0. Introduction

The problem how to allocate the total production cost of a single homogeneous good is discussed in this paper. A cost function C, i.e. a nondecreasing function on the nonnegative reals which intersects the origin, is used to model the production costs depending on the demanded quantity. Here $C(y)$ should be interpreted as the cost for producing y units of the good. If the agents or players in a finite group N possess the production technology and agent i has demand q_i, then the question how to divide the total cost $C(\Sigma_{i \in N} q_i)$ among the players is answered in general by proposing any cost sharing rule. Several applications occur in the literature (see, e.g. Shenker (1989, 1990).

A cost sharing rule assigns an allocation to any cost sharing problem (N,C,q), where q is the demand profile of the agents in N and C is the cost function. This allocation has to be feasible in the sense that the proposed aggregate payment of the players covers the total cost. An example is average cost pricing. With respect to this rule every agent i has to pay her proportional part of the total cost. This means that average cost pricing only depends on the total cost and the demand profile, regardless of the special shape of the cost function. A further cost sharing rule, serial cost sharing, also takes into account the evaluation of the cost function at several distinguished arguments (see Moulin–Shenker (1992)). These authors (see Moulin–Shenker (1993)) showed that both rules can be characterized by certain intuitive properties. They also proposed to consider the cost games generated by the cost sharing problems and apply classical solution concepts for games with transferable utility to these cost games. Indeed, they mentioned the Shapley value and nucleolus as possible cost sharing rules for the cost sharing problems. The present paper, which is organized as follows, discusses the game theoretical solutions in detail.

In Section 1 the necessary notation is presented. It is shown (see Remark 1.3 and Figure 1) that individual rationality, i.e. every player has to pay not more than her individual cost, cannot be a general property of a cost sharing rule. Indeed, individual costs may not cover total cost. Consequently individual rationality is dropped as a property for cost sharing rules in general. In view of this fact we define the nucleolus rule to be the prenucleolus of the corresponding cost game. Well–known intuitive properties for cost sharing rules, e.g. Pareto optimality, "equal treatment of equals", the null property (players with zero demands do not have to pay), monotonicity, and additivity properties, are defined formally. Moreover, it is shown that similar well–accepted properties for solution concepts on games imply the properties on cost sharing

problems. To translate a cost sharing rule into a solution concept on the arising cost games this rule should yield the same result in case it is applied to different cost sharing problems which possess the same underlying cost game. A slightly stronger property is "covariance under strategical equivalence". Average cost pricing does not satisfy covariance. The solution concept which arises from a covariant cost sharing rule inherits many properties of the rule as shown in Corollary 1.8. Covariance is the basic property of the game theoretical cost sharing rules discussed in the following sections.

In Section 2 three different axiomatizations of the Shapley rule are presented. The common properties of all characterizations are covariance and equal treatment of equals (i.e. agents with coinciding demands are proposed to pay coinciding amounts). Together with the null property and additivity (i.e. if the cost function is the sum of two cost functions, then everybody has to pay the sum of the fees of both cost sharing problems) or minmax additivity (a modification of additivity) Theorems 2.1 and 2.2 are similar to the results of Shapley (1953) and Dubey (1975). Here covariance is strongly used and cannot be replaced by Pareto optimality as in the classical context. Especially Theorem 2.2 is no trivial analogon to Dubey's result, which is shown by the fact that in the cost sharing context these properties are not strong enough to uniquely determine the Shapley rule on the restricted class of cost sharing problems which generate monotonic simple cost games. Note that Dubey's result applies to this class of games. In a further axiomatization null property and additivity are replaced by some reduction property which is analogous to that introduced by Hart–Mas–Colell (1989). Clearly consistency can only be applied to classes of cost sharing problems with varying agent sets.

In Sections 3 and 4 further versions of consistency are introduced and it is shown that both the nucleolus rule and the anti–nucleolus rule (which coincides with the pre–nucleolus of the dual cost game) can be axiomatized by some version of consistency, covariance, and equal treatment of equals (see Theorems 3.4 and 4.2). As in the game theoretical context the infinity assumption on the set of potential agents is needed as a prerequisite (see Sobolev 1975). Moreover, two interesting results concerning cost sharing problems with concave cost functions are presented. A strong version of consistency together with equal treatment and covariance uniquely determines the nucleolus even in case of a finite universe of players (see Theorem 3.3). In general the anti–nucleolus is, other than the nucleolus rule, not a core selector, even in case that the core is nonempty. In Corollary 4.4 it is proved that the anti–nucleolus is a core selector in case of concave cost functions. This fact also shows that the class of underlying cost games does not cover, even up to strategical equivalence, the class of all concave games. This is in contrast to the general case: every game is, up to strategical equivalence, an under

lying game of some cost sharing problem (see Lemma 1.6).

Finally, in Section 5 the cost functions which yield coinciding average cost pricing, Shapley rule, and nucleolus rules when applied to a cost sharing problem with a fixed aggregate demand are characterized by a functional equation (see Theorem 5.1). Concerning this class of cost sharing problems any of the proposed cost sharing rules can be characterized by "separable costs" (i.e. coincidence with average cost pricing for linear cost functions), monotonicity, and Pareto optimality. For average cost pricing this characterization holds in general (see Moulin–Shenker (1993)). Moreover, Theorem 5.3 shows that linear and parabolic cost functions are the unique functions which globally yield coinciding cost sharing rules. Section 6 presents diagrams which summarize the main results.

1. Notation and preliminary results

Let U denote the nonvoid set of potential agents; it can be finite or infinite. A triple (N,C,q) is called **cost sharing problem** (CSP), if N is a finite nonvoid set (the set of **agents** of the CSP), C is a nondecreasing function on the nonnegative reals $\mathbb{R}_{\geq 0}$ such that $C(0) = 0$ (the **cost function** of the CSP), and $q = (q_i)_{i \in N} \in \mathbb{R}^N$ is such that $q_i \geq 0$ for $i \in N$ (the **demand profile** of the CSP). Let $\Gamma(U)$ denote the set of all cost sharing problems with the foregoing properties. A **cost sharing rule** (CSR) on a subset Γ of $\Gamma(U)$ associates a vector $\sigma(N,C,q) \in \mathbb{R}^N$ satisfying

$$\sum_{i \in N} \sigma_i(N,C,q) \geq C(q(N)) \qquad \text{(feasibility)}$$

with each CSP $(N,C,q) \in \Gamma$. As usual $x(S) = \sum_{i \in S} x_i$ denotes the aggregate weight of S for $x \in \mathbb{R}^N$ and $S \subseteq N$. Feasibility means that at least the total cost is covered by the agents. We do not require $\sigma_i(N,C,q) \geq 0$ but in all our examples this property automatically holds. Well–known and intuitively justified properties of a CSR σ on Γ are as follows:

(i) **Pareto optimality (PO):** $\sum_{i \in N} \sigma_i(N,C,q) = C(q(N))$ for all $(N,C,q) \in \Gamma$

(ii) **Ranking (RAN):** $q_i \leq q_j$ implies $\sigma_i(N,C,q) \leq \sigma_j(N,C,q)$ for all $(N,C,q) \in \Gamma$

(iii) **Separable costs (SC):** If $C(y) = \lambda \cdot y$ for $y \geq 0$ and $(N,C,q) \in \Gamma$, then $\sigma(N,C,q) = \lambda \cdot q$

(iv) **Equal treatment of Equals (ET):** $q_i = q_j$ implies $\sigma_i(N,C,q) \leq \sigma_j(N,C,q)$ for all $(N,C,q) \in \Gamma$

(v)   **Null property (NP):** $q_i = 0$ implies $\sigma_i(N,C,q) = 0$ for all $(N,C,q) \in \Gamma$

(vi)   **Anonymity (AN):** If $(N,C,q),(N,C,\pi q) \in \Gamma$ for some permutation π of N, then
$$\sigma_{\pi i}(N,C,\pi q) = \sigma_i(N,C,q) \text{ for all } i \in N$$

(vii)   **Monotonicity (MON):** If $(N,C,q),(N,\tilde{C},q)\in\Gamma$ with $C \leq \tilde{C}$, then $\sigma(N,C,q) \leq \sigma(N,\tilde{C},q)$

(viii)   **Additivity (ADD):** If $(N,C,q),(N,\tilde{C},q),(N,C+\tilde{C},q) \in \Gamma$, then
$$\sigma(N,C,q) + \sigma(N,\tilde{C},q) = \sigma(N,C+\tilde{C},q)$$

(ix)   **Minmax additivity (MMADD):** If $(N,C,q),(N,\tilde{C},q),(N,C\wedge\tilde{C},q),(N,C\vee\tilde{C},q)\in \Gamma$ (where $\{^\wedge_\vee$ denotes $\{^{\text{"minimum"}}_{\text{"maximum"}})$, then

$$\sigma(N,C,q) + \sigma(N,\tilde{C},q) = \sigma(N,C\wedge\tilde{C},q) + \sigma(N,C\vee\tilde{C},q)$$

Example 1.1:

For every CSP (N,C,q) define
$$a_i(N,C,q) = (q_i/q(N)) \cdot C(q(N)) \text{ (where } 0/0 = 1).$$

The mapping a is called **average cost pricing (rule)**.

Note that average cost pricing distributes the total cost $C(q(N))$ among the agents according to the Aumann–Shapley unit price (see Aumann–Shapley (1974)). Note furthermore that average cost pricing is completely determined by the demand profile and the total cost. There is an example of another CSR, namely **serial cost sharing** (see Moulin–Shenker (1992)), which depends on both, the demand profile and the cost function evaluated at n arguments. Moulin–Shenker (1993, Examples 3 and 4 of Section 3) suggested to look at cost sharing rules which mainly depend on the cost function (the demand profile is only used implicitly). They proposed the Shapley value and the nucleolus of a certain underlying game as feasible cost sharing rules. To formulate this more explicitly it is necessary to define the **underlying (TU–) cost game** $(N,v^{C,q})$ of an arbitrary CSP (N,C,q). The **coalitional function** $v^{C,q}$ is given by
$$v^{C,q}(S) = C(q(S)) \text{ for } S \subseteq N.$$
In general a game is a pair (N,v) such that $v: 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$ hold true. The **Shapley (cost sharing) rule** φ assigns to every CSP the Shapley value of its underlying game (for the definition of the Shapley value Shapley (1953) is referred to). There are two cost sharing rules based on the prenucleolus of the underlying game or its dual. Here is the precise

Definition 1.2:

1. Let (N,v) be a game (considered as cost game) and let $X(v) = \{x\in\mathbb{R}^N|x(N)=v(N)\}$

be the set of **preimputations of v**. The **prenucleolus** $\bar{\nu}(v)$ of (N,v) is the unique pre-imputation which successively minimizes the largest excesses. Here $x(S)-v(S)$ is the **excess of S at x (with respect to v)** for $x \in \mathbb{R}^N$. To put it more formally, let

$$Z = \{x \in X(v) \mid \Theta((x(S)-v(S))_{S \subseteq N}) \underset{\text{lex}}{\leq} \Theta((y(S)-v(S))_{S \subseteq N}) \text{ for } y \in X(v)\},$$

where Θ applied to a vector $(x(S)-v(S))_{S \subseteq N}$ of excesses at x orders the components of this vector nonincreasingly. The set Z is a singleton and its unique element $\bar{\nu}(v)$ is the prenucleolus of v (see Maschler–Peleg–Shapley (1979)). In the definition of the **nucleolus** (see Schmeidler (1969)) $X(v)$ is replaced by the subset of **individually rational** preimputations. (A vector $x \in \mathbb{R}^N$ is individually rational if $x_i \leq v(\{i\})$ for all $i \in N$.)

2. The **nucleolus rule on a subset** Γ of $\Pi(U)$ assigns to each CSP $(N,C,q) \in \Gamma$ the prenucleolus of its underlying game $(N,v^{C,q})$. We denote this cost sharing rule by ν.

3. The **anti–nucleolus rule** ν^* on a subset Γ of $\Pi(U)$ assigns to each CSP $(N,C,q) \in \Gamma$ the negative of the prenucleolus of the negative of its underlying game $(N,v^{C,q})$, i.e.

$$\nu^*(N,C,q) = -\bar{\nu}(-v^{C,q}).$$

Remark 1.3:

(i)   Note that the nucleolus in the sense of Schmeidler does not necessarily exist, i.e. the set of **imputations** (individually rational preimputations) of a game can be the empty set. This is also true for underlying games of cost sharing problems. In Figure 1 two 2–agent cost sharing problems with demand profiles q and $\tilde{q}$ and the same "S–shaped" cost function C are sketched. Note that S–shaped cost functions (i.e. decreasing marginal costs for "small" aggregate demand and, due to, e.g., weak capacity bounds, increasing marginal costs for "large" aggregate demand) are typical for many economical applications. For the demand profile q feasibility of a CSR σ requires at least one individual fee which is higher than the individual cost whereas in case of $\tilde{q}$ the total cost can be divided among the players in such a way that both pay less than their individual costs. Therefore the version of a nucleolus rule based on individually rational feasible payoffs does not establish a cost sharing rule.

[Insert Figure 1 about here!]

(ii) Using the notion of "interpersonal comparisons of utility" the prenucleolus can be justified for cost games (see Maschler (1992)). Though the definition of the

anti–nucleolus rule, i.e. successively maximizing minimal excesses, seems to be counter intuitive, there are examples of cost sharing situations for which the anti–nucleolus rule can be justified (see Potters–Sudhölter (1995)). Therefore we also analyze the anti–nucleolus rule ν^* in this paper.

In order to characterize the game theoretical cost sharing rules (Shapley rule, nucleolus rule, and anti–nucleolus rule), which are determined by the cost function evaluated at the aggregate demands of the coalitions, one additional property referring to the underlying games is needed. Indeed, average cost pricing satisfies all properties (i)–(ix) (see Moulin–Shenker (1993)) and certain reduction properties as shown in the following sections. A game theoretical solution rule is determined by the underlying game. A stronger version of this property is presented in .

Definition 1.4:

A cost sharing rule σ on a set $\Gamma \subseteq \Gamma(U)$ satisfies **covariance (COV)**, if the following condition holds:

If $(N,C,q) \in \Gamma$ and $\alpha > 0$, $\beta \in \mathbb{R}^N$ are such that there is $(N,\tilde{C},\tilde{q}) \in \Gamma$ with

$$\alpha \cdot v^{C,q} + \beta = v^{\tilde{C},\tilde{q}},$$

then $\quad \sigma(N,\tilde{C},\tilde{q}) = \alpha \cdot \sigma(N,C,q) + \beta.$

Covariance on a set of games is an intuitive property. Two games (N,v) and (N,w) which coincide up to **strategical equivalence** (i.e., $w = \alpha \cdot v + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}^N$) should be treated accordingly by a solution concept for games. In the context of cost sharing problems COV seems less intuitive. Nevertheless, as long as the underlying game serves as an adequate description of the cost sharing problem, covariance can be interpreted as in the original game theoretical context.

If a CSR σ on a set Γ of cost sharing problems satisfies COV, then there is a unique continuation $\bar{\sigma}$ on the set $\mathscr{G}(\Gamma)$ of games which are strategically equivalent to the underlying game of some CSP in Γ. To put it more formally, let $(N,C,q) \in \Gamma$, $\beta \in \mathbb{R}^N$, and $\alpha > 0$. Then $\quad \bar{\sigma}(\alpha \cdot v + \beta) = \alpha \cdot \sigma(N,C,q) + \beta.$

The solution concept $\bar{\sigma}$ on $\mathscr{G}(\Gamma)$ is called **continuation of** σ. A **solution concept** $\bar{\sigma}$ **on a set** $\mathscr{G}$ of cost games associates a vector $\bar{\sigma}(v) \in \mathbb{R}^N$ satisfying feasibility ($\sum_{i \in N} \bar{\sigma}_i(v) \geq v(N)$)

with each game $(N,v) \in \mathscr{G}$. Conversely, if $\mathscr{G}(\Gamma) = \mathscr{G}$, then $\bar{\sigma}$ induces a CSR σ on Γ by $\sigma(N,C,q) = \bar{\sigma}(v^{C,q})$. Clearly $\bar{\sigma}$ is feasible, iff σ is feasible. Well–known properties for a solution concept $\bar{\sigma}$ on $\mathscr{G}$ are as follows (with $(N,v),(N,w) \in \mathscr{G}$):

(a) **Pareto optimality (PO):** $\sum\limits_{i \in N} \bar{\sigma}_i(v) = v(N)$

(b) **Equal treatment property (ET):** $v(S \cup \{i\}) = v(S \cup \{j\})$ for $S \subseteq N \setminus \{i,j\}$ (i and j are interchangeable) implies $\bar{\sigma}_i(v) \leq \bar{\sigma}_j(v)$

(c) **Null property (NP):** $v(S \cup \{i\}) = v(S)$ for $S \subseteq N$ (i is a **nullplayer**) implies $\bar{\sigma}_i(v) = 0$

(d) **Anonymity (AN):** If (N,w) arises from (N,v) by a permutation of the players then $\bar{\sigma}$ "respects this permutation"

(e) **Additivity (ADD):** If $(N, v+w) \in \mathcal{G}$, then $\bar{\sigma}(v) + \bar{\sigma}(w) = \bar{\sigma}(v+w)$

(f) **MM additivity (MMADD):** $v \wedge w, v \vee w \in \mathcal{G}$ implies $\bar{\sigma}(v) + \bar{\sigma}(w) = \bar{\sigma}(v \wedge w) + \bar{\sigma}(v \vee w)$.

(g) **Covariance (COV):** If $w = \alpha \cdot v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$, then $\sigma(\alpha v + \beta) = \alpha \sigma(v) + \beta$

Clearly the following statements for a covariant CSR σ on Γ together with its continuation $\bar{\sigma}$ hold true:

1. $\bar{\sigma}$ satisfies $\overline{\text{COV}}$;
2. $\bar{\sigma}$ satisfies $\overline{\text{PO}}$, iff σ satisfies PO;
3. If $\bar{\sigma}$ satisfies $\overline{\text{NP}}$, then σ satisfies NP;
4. If $\bar{\sigma}$ satisfies $\overline{\text{AN}}$, then σ satisfies AN;
5. If $\bar{\sigma}$ satisfies $\overline{\text{ADD}}$, then σ satisfies ADD;
6. If $\bar{\sigma}$ satisfies $\overline{\text{MMADD}}$, then σ satisfies MMADD;

Ranking has an analogon for solution concepts, too. Indeed, many well–known solution concepts, e.g. the Shapley value and the proposed versions of nucleoli, **preserve the desirability relation** in the sense of Maschler–Peleg (1966). Preservation of desirability on $\mathcal{G}(\Gamma)$ implies RAN on Γ.

Note that the Shapley rule and both nucleolus rules satisfy COV, PO, RAN, SC, ET, NP, and AN (see, e.g., Peleg (1989)). Moreover, the Shapley rule satisfies ADD and MMADD (see Shapley (1953) and Dubey (1975)). For completeness reasons we present an example which shows that neither φ nor ν or ν^* satisfy MON.

Example 1.5:

Let $C(y) = 5 \cdot \sqrt{y}$, $\tilde{C}(y) = \min\{C(y), y\}$, $N = \{1,2\}$, and $q = (9,16)$. An easy computation shows that

$$\varphi(N,C,q) = \nu^*(N,C,q) = \nu(N,C,q) = (10,15),$$

whereas
$$\alpha(N,C,q) = \alpha(N,\tilde{C},q) = \nu(N,\tilde{C},q) = \nu^*(N,\tilde{C},q) = (9,16).$$

Hence the game theoretical cost sharing rules do not satisfy MON even in the case of concave cost functions.

For every finite nonvoid subset N of U let Γ_N denote the set of cost sharing problems with agent set N. The following assertion shows the "power" of COV.

Lemma 1.6: Every game (N,v) is strategically equivalent to some underlying game $(N,v^{C,q})$ of a CSP (N,C,q), i.e. $\mathscr{Y}(\Gamma_N) = \{(N,v) \mid v \text{ is a game}\}$.

The proof of this lemma is constructive and several versions of it will be used frequently.

Proof:

Let $d(v) = \max \{v(S)-v(T) \mid S,T \subseteq N\}$ be the maximal difference of worths of coalitions. Assume for simplicity reasons $N = \{1,...,n\}$, take $d \geq d(v)$, and define $q,\beta \in \mathbb{R}^N$ by
$$q_i = 2^{i-1}, \quad \beta_i = 2^{i-1} \cdot d \text{ for } i \in N.$$

Moreover, let C be defined by
$$C(y) = \max \{v(S) + \beta(S) \mid S \subseteq N \text{ and } q(S) \leq y\} \text{ for } y \geq 0.$$

Clearly C is a cost function ($C(0) = 0$ and C is nondecreasing). In order to verify that $v^{C,q} = v + \beta$ holds true, it remains to prove
$$v(S)+\beta(S) \leq v(T)+\beta(T) \text{ for } S,T \subseteq N \text{ with } S \neq T \text{ and } q(S) \leq q(T). \quad (1)$$

By definition of q there is a unique $i \in N$ such that $i \in T\backslash S$ and $T\backslash\{1,...,i\} = S\backslash\{1,...,i\}$. The definition of β directly shows
$$v(S)+\beta(S) \leq \max_{R \subseteq N} v(R) + \beta(S \cup \{1,...,i-1\}) \quad (2)$$

and
$$v(T)+\beta(T) \geq \min_{R \subseteq N} v(R) + \beta(T \cap \{i,...,n\}). \quad (3)$$

The observation
$$\min_{R \subseteq N} v(R) - \max_{R \subseteq N} v(R) + \beta(T \cap \{i,...,n\}) - \beta(S \cup \{1,...,i-1\}) =$$

$$-d(v) + \beta_i - \beta(\{1,...,i-1\}) = -d(v) + d(2^{i-1} - \sum_{j=0}^{i-2} 2^{j-1}) =$$

$$-d(v) + d \geq 0$$

completes the proof. q.e.d.

A stronger version of this result shows that $\overline{ET}$ and ET are equivalent for a covariant CSR and its continuation.

Lemma 1.7:

For every game (N,v) there is a vector $\beta \in \mathbb{R}^N$, a demand profile $q \in \mathbb{R}^N_{\geq 0}$, and a cost function C such that $v + \beta$ is the underlying game of (N,C,q), interchangeable players of v possess coinciding demands, and nullplayers possess zero demands.

Proof:

Take d as in the last proof, assume that $N = \{1,...,n\}$ and assume that the equivalence classes of interchangeable players are integer intervals, i.e. if i is interchangeable with j and $j > i$, then i and j are interchangeable with k for $k \in \{i,...,j\}$. This can be achieved by just renaming the players. For $j \in N$ define $q_j = 2^{i-1}$ and $\beta_j = 2^{i-1} \cdot d$ in case j is no nullplayer, where i is the player of minimal index interchangeable with j. Moreover, put $q_j = 0$ for every nullplayer $j \in N$. The proof can be completed analogously to that of Lemma 1.6.   q.e.d.

A direct consequence of this result is

Corollary 1.8: Let σ be a CSR on Γ_N satisfying COV and $\bar{\sigma}$ its continuation on $\mathcal{A}(\Gamma_N)$.

(i)   Then σ satisfies ET, if and only if $\bar{\sigma}$ satisfies $\overline{ET}$.
(ii)   Then σ satisfies NP, if and only if $\bar{\sigma}$ satisfies $\overline{NP}$.

2. The Shapley rule

In this section we provide axiomatizations for the Shapley rule based on the classical characterizations of the Shapley value due to Shapley (1953), Dubey (1975), and Hart–Mas–Colell (1989). The analogon to Shapley's result is the following

Theorem 2.1:   The Shapley rule is the unique cost sharing rule on Γ_N satisfying ET, NP, ADD, and COV.

Proof:

By Lemma 1.6 $\mathcal{G} = \mathcal{A}(\Gamma_N) = \{(N,v)|$ v is a game$\}$. Shapley (1953) proved that the Shapley value is the unique solution concept on $\mathcal{G}$ satisfying $\overline{ET}$, $\overline{NP}$, $\overline{PO}$, and $\overline{ADD}$. Therefore it suffices to prove uniqueness. Let σ be a CSR with the desired properties and $\bar{\sigma}$ its continuation. By Corollary 1.8 it suffices to show that $\bar{\sigma}$ satisfies $\overline{ADD}$ and $\overline{PO}$. To verify the first property take (N,v_1), $(N,v_2) \in \mathcal{G}$ and repeat the construction of the proof of Lemma 2.6. Namely, take $d \geq d(v_1)vd(v_2)$, define β and q as in the mentioned proof and let C_i be the constructed cost function for $v_i+\beta$, i.e.

$$C_i(y) = \max \{v_i(S) + \beta(S)| \ S \subseteq N \text{ and } q(S) \leq y\} \text{ for } i = 1,2.$$

Additivity of σ and $\overline{COV}$ of $\bar{\sigma}$ implies

$$\bar{\sigma}(N,v+w) \qquad\qquad = \bar{\sigma}(N,v+w+2\beta)-2\beta \qquad =$$

$$= \sigma(N,C_1+C_2,q)-2\beta \qquad = \sigma(N,C_1,q)+\sigma(N,C_2,q)-2\beta \quad =$$

$$= \bar{\sigma}(N,v_1+\beta)+\bar{\sigma}(N,v_2+\beta)-2\beta \quad = \bar{\sigma}(N,v_1) + \bar{\sigma}(N,v_2)$$

and hence $\overline{ADD}$ of $\bar{\sigma}$.

It remains to show $\overline{PO}$. By $\overline{COV}$ and $\overline{ADD}$ it is sufficient to show that this property holds for all unanimity games. Recall that (N,u_S), defined by $u_S(T) = \{ \begin{smallmatrix} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise} \end{smallmatrix}$ is a **unanimity game** for every nonvoid subset S of N. We proceed by induction on $|S| =$ s. For s = 1 $\overline{COV}$ is used. Indeed, $2 \cdot u_S = u_S + \beta$, where $\beta_i = 1$ for $\{i\} = S$ and $\beta_i = 0$ otherwise. Assume $\overline{PO}$ is already shown for s < t for some t > 1. For s = t choose different agents i,j $\in$ S and define

$$T = S\backslash\{i\} \text{ and } R = S\backslash\{j\}.$$

Then $u_T + u_R = u_S + v$, where $v(Q) = \{ \begin{smallmatrix} 1, & \text{if } T \subseteq Q \text{ or } R \subseteq Q \\ 0, & \text{otherwise} \end{smallmatrix}$. Additivity of $\bar{\sigma}$ and the inductive hypothesis applied to both u_T and u_R complete the proof. (Recall that $\bar{\sigma}$ is feasible.)

<div align="right">q.e.d.</div>

An analogon to Dubey's result is

Theorem 2.2: The Shapley rule is the unique cost sharing rule on Γ_N satisfying ET, NP, MMADD, and COV.

Proof:

The Shapley value satisfies $\overline{MMADD}$ (Dubey (1975)), thus the Shapley rule satisfies MMADD. Similarly to the last proof it remains to show that $\bar{\sigma}$ satisfies $\overline{MMADD}$ and $\overline{PO}$. The last part of the just mentioned proof can be copied literally as soon $\overline{MMADD}$ is shown. Indeed $u_S = u_T \wedge u_R$ and $v = u_T \vee u_R$, thus

$$u_S+\beta = (u_T+\beta)\wedge(u_R+\beta) \text{ and } v+\beta = (u_T+\beta)\vee(u_R+\beta).$$

It remains to show $\overline{MMADD}$. Copy a part of the last proof to define q, β, C_1 and C_2 and observe that the equalities

$$\bar{\sigma}(N,v_1\vee v_2) + \bar{\sigma}(N,v_1\wedge v_2) = \bar{\sigma}(N,v_1\vee v_2+\beta)+\bar{\sigma}(N,v_1\wedge v_2+\beta)-2\cdot\beta \quad =$$

$$= \bar{\sigma}(N,(v_1+\beta)\vee(v_2+\beta))+\bar{\sigma}(N,(v_1+\beta)\wedge(v_2+\beta))-2\cdot\beta \qquad =$$

$$= \sigma(N,C_1\vee C_2,q) + \sigma(N,C_1\wedge C_2) - 2\cdot\beta = \sigma(N,C_1,q)+\sigma(N,C_2,q)-2\cdot\beta \quad =$$

$$= \bar{\sigma}(N,v_1+\beta) + \bar{\sigma}(N,v_2+\beta) - 2\cdot\beta = \bar{\sigma}(N,v_1) + \bar{\sigma}(N,v_2),$$

which are valid by MMADD and $\overline{\text{COV}}$, complete the inductive step. The induction can be initialized as in the last proof.   q.e.d.

To prove that both characterizations are axiomatizations, we present examples which show the logical independence of both sets of properties. A weighted Shapley values (see Kalai–Samet (1988)) satisfies $\overline{\text{NP}}$, $\overline{\text{COV}}$, $\overline{\text{ADD}}$, and $\overline{\text{MMADD}}$ but does not satisfy ET in general for $n \geq 2$. Therefore the corresponding CSR possesses all desired properties except ET. The CSR σ defined by

$$\sigma_i(N,C,q) = v(\{i\}) + (v(N) - \sum_{j \in N} v(\{j\}))/n,$$

where $v = v^{C,q}$, satisfies all axioms except NP for $n \geq 2$. For $n \geq 3$ the nucleolus rule does not satisfy ADD or MMADD but it possesses all other properties. Finally, average cost pricing can be used as an example which shows the independence of COV.

Remark 2.3:

Dubey (1975) used $\overline{\text{MMADD}}$ to characterize the Shapley value on monotone simple games. A **monotone simple game** (N,v) is a game such that $v(S) \in \{0,1\}$, $v(N) = 1$ (the game is simple) and $v(S) \leq v(T)$ for $S \subseteq T \subseteq N$ (the game is monotone). The analogon does not hold in the cost sharing context. Indeed, on the set

$$\Gamma = \{(N,C,q) \mid C \text{ is a cost function with } C(y) \in \{0,1\}\}$$

of cost sharing problems possessing cost functions with domain 0,1 the Shapley rule is not uniquely determined by ET, NP, MMADD, and COV, even together with PO. To see this, define

$$\sigma_i(N,C,q) = \begin{cases} 0, & \text{if } i \text{ is a nullplayer of the underlying game} \\ 1/k, & \text{otherwise} \end{cases},$$

where k denotes the number of non nullplayers of the underlying game. Clearly σ satisfies NP, ET, and PO. MMADD and COV are not strong enough to rule out the CSR σ on this small set of cost sharing problems. One reason can be seen in the fact that the class of underlying games is too small. Indeed, every underlying game is not only a monotone simple game but also a weighted majority game.

In the end of this section it is shown that NP and ADD (or MMADD) can be replaced by a reduction property in the sense of Hart–Mas–Colell. Certainly Γ_N has to be re-placed by some richer class of cost sharing problems, namely by $\Gamma(U)$. The next defini-tion recalls the notions of a "reduced game" and "consistency". Moreover, the analoga for cost sharing problems are presented.

Definition 2.4: Let σ be a CSR on $\Gamma(U)$ and $\bar{\sigma}$ be a solution concept on $\mathcal{G}(U) = \mathcal{G}(\Gamma(U))$.

(i)   Let $(N,v) \in \mathcal{G}(U)$ and $\emptyset \neq S \subseteq N$. The $\bar{\sigma}$–reduced game $(S,v_{\bar{\sigma},S})$ (on S with respect to $\bar{\sigma}$) is defined by

$$v_{\bar{\sigma},S}(T) = v(T \cup N\backslash S) - \Sigma_{i \in N\backslash S}\, \sigma_i(T \cup N\backslash S, v) \text{ for } \emptyset \neq T \subseteq N$$

and $\quad v_{\bar{\sigma},S}(\emptyset) = 0$.

(ii) $\bar{\sigma}$ satisfies **consistency (CON) in the sense of Hart–Mas–Colell**, if the restriction to S of the solution is a solution of the $\bar{\sigma}$–reduced game, i.e.

$$\bar{\sigma}(S,v_{\bar{\sigma},S}) = \sigma(N,v)_S \text{ for every game } (N,v) \text{ of } \mathcal{G}(U).$$

(iii)   For a CSP (N,C,q) and $\emptyset \neq S \subseteq N$ define the σ–reduced function $C_{\sigma,S} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$C_{\sigma,S}(y) = C(y+q(S^c)) - \min\{\,\Sigma_{i \in S^c}\, \sigma_i(T \cup S^c, C, q_{T \cup S^c}) \,|\, {\textstyle{T \subseteq N \text{ and } q(T) = \atop \max\{q(R)\,|\,R \subseteq N,\ q(R) \leq y\}}}\,\}$$

for $y > 0$ and $C_{\sigma,S}(0) = 0$.

Here $S^c = N\backslash S$ is the complement of S. Note that the σ–reduced function is not necessarily a cost function (monotonicity of C is not guaranteed.).

(iv)   The CSR σ satisfies **consistency (CON)**, if for $(N,C,q) \in \Gamma(U)$ and $\emptyset \neq S \subseteq N$ the following condition holds:

If $C_{\sigma,S}$ is a cost function and for every $\emptyset \neq T \subseteq S$

$$C_{\sigma,S}(q(T)) = C(q(T)+q(S^c)) - \Sigma_{i \in S^c}\, \sigma_i(T \cup S^c, C, q_{T \cup S^c})$$

is valid, then $\sigma(S,C_{\sigma,S},q_S) = \sigma(N,C,q)_S$.

Definition 2.4 (i),(ii) is due to Hart–Mas–Colell (1989). They showed that the Shapley value is uniquely determined on $\mathcal{G}(U)$ by $\overline{\text{CON}}$ and some weak versions of $\overline{\text{ET}}$, $\overline{\text{COV}}$, and $\overline{\text{PO}}$. It is obvious that $\overline{\text{CON}}$ together with $\overline{\text{COV}}$ (on $\mathcal{G}(U)$) implies CON for the associated CSR on $\Gamma(U)$. If the reduced function $C_{\sigma,S}$ is a cost function, then the arising aggregate costs for the coalitions can be interpreted as follows. Coalition $T \subseteq S$ determines its new cost (supposing all members of S^c agree on the cost sharing rule σ) to be the total cost for the aggregate demand of itself and S^c diminished by the agreegate fee which will be paid by S^c in the new situation. It should be noted that coalition T imagines a situation in which only the own agents and the agents of S^c are present. A different reduced situation will be discussed in the following two sections.

Theorem 2.5:   The Shapley rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and CON.

Proof:

It suffices to show the uniqueness part. Let σ be a CSR with the desired properties. Then the continuation $\bar{\sigma}$ on $\bar{\mathcal{G}} = \mathcal{G}(U)$ of σ satisfies $\overline{\text{COV}}$ and ET by Corollary 1.8. Moreover, $\bar{\sigma}$ is $\overline{\text{PO}}$ on 1–agent cost sharing problems by $\overline{\text{COV}}$. The proof is finished as soon it is verified that $\bar{\sigma}$ satisfies $\overline{\text{CON}}$ and $\overline{\text{PO}}$. In order to show $\overline{\text{CON}}$ take $(N,v) \in \bar{\mathcal{G}}$ and $\emptyset \neq S \subseteq N$. First of all it is proved that there is an underlying game (N,w) of some CSP $(N,\tilde{C},\tilde{q})$ which is strategically equivalent to (N,v) such that the $\bar{\sigma}$–reduced game $w_{\bar{\sigma},S}$ is the underlying game of the σ–reduced CSP $(N,\tilde{C}_{\sigma,S},\tilde{q}_S)$. Indeed, assume without loss of generality that $N = \{1,...,n\}$ and $S = \{n+1-s,..,n\}$. Moreover, take $q = (2^{i-1})_{i \in N}$, β, and C as defined in the proof of Lemma 1.6, i.e. $(N,C,q) \in \Gamma(U)$ such that $u = v+\beta = v^{C,q}$. Define $\tilde{\beta}_i = 0$ for $i \in S^c$ and $\tilde{\beta}_i = 2^i \cdot d$ for $i \in S$, where $d = d(u) + \max_{R \subseteq S^c}$
$\sum_{i \in S^c} \sigma_i(R \cup S^c, C, q_{R \cup S^c})$. For $y \geq 0$ define

$$\tilde{C}(y) = \max\{w(T) \mid T \subseteq N \text{ and } \tilde{q}(T) = \max\{\tilde{q}(R) \mid R \subseteq N, \tilde{q}(R) \leq y\}\},$$

where $w = u + \tilde{\beta}$. Clearly $\tilde{C}(y) = C(y)$ for $y \leq q(S^c)$ and $\tilde{C}$ is nondecreasing, thus a cost function. The straightforward observation that

$$\tilde{C}(\tilde{q}(T)) = C(q(T)) + \tilde{\beta}(T) \text{ for } T \subseteq N$$

shows that $v^{\tilde{C},\tilde{q}} = w$ holds true. By construction and COV of σ the σ–reduced function $\tilde{C}_{\sigma,S}$ is a cost function and the underlying game of $(S,\tilde{C}_{\sigma,S},\tilde{q}_S)$ coincides with $w_{\bar{\sigma},S}$, thus $\bar{\sigma}$ satisfies $\overline{\text{CON}}$ by $\overline{\text{COV}}$. Property $\overline{\text{PO}}$ is a direct consequence of $\overline{\text{CON}}$ applied to singletons S. Indeed, note that $\overline{\text{COV}}$ implies $\overline{\text{PO}}$ for 1–person games   q.e.d.

For completeness reasons examples are presented which show the logical independence of ET, COV, and CON. There is a positively weighted Shapley value on $\mathcal{G}(U)$ which does not satisfy $\overline{\text{CON}}$ in case there are at least two potential agents. (Note that the notion of positively weighted Shapley values introduced for the set of games with fixed player set (see Kalai–Samet (1988)) can easily be generalized to $\mathcal{G}(U)$ (see Potters–Sudhölter (1995)).) By definition weighted Shapley values satisfy $\overline{\text{COV}}$. The proof that positively weighted Shapley values satisfy $\overline{\text{CON}}$ is straightforward and skipped. In view of these considerations the associated CSR on $\Gamma(U)$ shows the independence of ET. In view of the definition it can easily be seen that average cost pricing satisfies CON, thus independence of COV is guaranteed for $|U| \geq 2$. Finally the nucleolus rule shows that CON cannot be dropped as a prerequisite of Theorem 2.5 in case $|U| \geq 3$.

3. The nucleolus rule

There is a characterization of the prenucleolus on the family of all games with player set contained in an infinite universe due to Sobolev (1975). He needs a consistency property based on a certain reduced game introduced by Davis–Maschler (1965). The corresponding notion and its analogon in the cost sharing situation is content of

Definition 3.1: Let σ be a CSR on $\Gamma \subseteq \Gamma(U)$ and $\bar{\sigma}$ be a solution concept on $\mathscr{G} \subseteq \mathscr{G}(U)$.

(i) For every game (N,v), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **reduced game** $(S, v_{S,x})$ of v (at x with respect to S) is defined by

$$v_{S,x}(T) = \begin{cases} 0 & , \text{ if } T = \emptyset \\ v(N) - x(S^c) & , \text{ if } T = N \\ \min \{v(T \cup R) - x(R) | R \subseteq S^c\} , & \text{ otherwise} \end{cases} \quad \text{for } T \subseteq N.$$

(ii) $\bar{\sigma}$ is **consistent** (satisfies $\overline{\text{CONS}}$), if $v_{S,\sigma(v)} \in \mathscr{G}$ and $\sigma(S, v_{S,\sigma(v)}) = \sigma(N,v)_S$ for $v \in \mathscr{G}$ and $\emptyset \neq S \subseteq N$.

(iii) For every CSP (N,C,q), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **reduced function** $C_{S,x} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined by

$$C_{S,x}(y) = \begin{cases} \min \{C(y+q(R)) - x(R) | R \subseteq S^c\}, & \text{if } 0 \leq y < q(S) \\ C(y+q(S^c)) - x(S^c) & , \text{if } y \geq q(S) \end{cases}. \quad (1)$$

(Note that the reduced function, though it is nondecreasing, need not be a cost function, since $C_{S,x}(0)$ might be negative.)

(iv) σ is **strongly consistent** (satisfies SCONS), if $(S, C_{S,x}, q_S) \in \Gamma$ and $\sigma(S, C_{S,x}, q_S) = x_S$, where $x = \sigma(N,C,q)$, for $(N,C,q) \in \Gamma$.

(v) σ is **consistent** (satisfies CONS), if for $(N,C,q) \in \Gamma$, $\emptyset \neq S \subseteq N$, and $x = \sigma(N,C,q)$ the following condition is satisfied:

If $C_{S,x}(q_i) \geq 0$ for $i \in S$, then $(S, (C_{S,x})_+, q_S) \in \Gamma$ and $x_S = \sigma(S, (C_{S,x})_+, q_S)$.

(Here y_+ denotes the positive part of the real number y.)

An interpretation of CONS is similar to that of CON (see Section 2). The main difference is that a coalition T takes an "optimistic" view of the world: every coalition R of members of S^c (who agreed upon the proposal given by σ) can be taken as a coalition of partners. Therefore the total cost generated by the aggregate demand of the union T $\cup$ R can be "reduced" by the fee which will be paid by R. Moreover, it should be noted that T imagines a situation with respect to the grand coalition in which all members of S^c have already paid their fees. The "pessimistic" view of the world will be discussed in Section 4.

Remark 3.2:

(i) A solution concept $\bar{\sigma}$ on $\mathcal{G}(U)$ which satisfies $\overline{\text{CONS}}$ induces a CSR σ on $\Gamma(U)$ which satisfies CONS but not necessarily SCONS, since $C_{S,x}(0) = 0$ is not necessarily true for $x = \sigma(N,C,q)$. Indeed, the prenucleolus and thus the nucleolus rule are consistent. Nevertheless, the nucleolus rule does not satisfy SCONS as examples show. The objection that the definition of the reduced function (see (1)) should be modified by, e.g., taking the positive part of this function, can be countered as follows. In the interpretation presented above it is hard to motivate that only certain coalitions take the optimistic view of the world and others throw away money. Moreover, the idea that (at least) the nucleolus rule should satisfy CONS excludes this modification as the following example shows:

Let $N = \{1,2,3\}$, $q=(1,2,4)$, $C(y) = y^2$. The nucleolus rule applied to this CSP yields $x = \nu(N,C,q) = (7,14,28)$.

With $S = \{1,2\}$ the reduced function satisfies $C_{S,x}(1) = -3$, $C_{S,x}(2) = 4$, and $C_{S,x}(3) = 21$, thus the positive part of C determines the game (S,v) defined by

$$v(\{1\}) = v(\emptyset) = 0, \; v(\{2\}) = 4, \text{ and } v(S) = 21.$$

Clearly $\bar{\nu}(v) = (16,25)/2 \neq (7,14) = x_S$.

(ii) It should be noted that it is not known whether there is any Pareto optimal CSR on $\Gamma(U)$ satisfying SCONS, unless $|U| \leq 2$. SCONS is only used in the context of cost sharing problems with concave cost functions (see Theorem 3.3).

(iii) Note that average cost pricing satisfies CONS on $\Gamma(U)$. A proof of this property is straightforward and skipped.

For any set Γ of cost sharing problems let Γ^* denote the subset of cost sharing problems with **concave** cost functions within Γ.

Theorem 3.3: The nucleolus rule is the unique cost sharing rule on $\Gamma^*(U)$ satisfying ET, COV, and SCONS.

Proof:

Clearly each underlying game (N,v) of some CSP (N,C,q) with concave C is **concave**:

$$v(S) + v(T) \geq v(S \cup T) + v(S \cap T) \text{ for } S,T \subseteq N.$$

Therefore the nucleolus $\bar{\nu}(v)$ belongs to the **core**

$$\mathcal{C}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \leq v(S) \text{ for } S \subseteq N\}.$$

This consideration shows that any reduced function is itself a cost function in case ν is

the CSR. Moreover, it is not necessary to distinguish whether y exceeds or does not exceed q(S) in equality (1). The first row of (1) can be taken as definition in this special case. Therefore reducing does not destroy concavity. Up to now we have shown that CONS and SCONS cannot be distinguished for the nucleolus rule on cost sharing problems with concave cost functions. By the choice of the set of cost sharing problems every reduced CSP is a member of $\Gamma^*(U)$, thus Sobolev's (1975) result shows that the nucleolus rule possesses the desired properties.

To show uniqueness let σ be any CSR with the desired properties and $\bar{\sigma}$ its continuation. By COV σ is Pareto optimal on 1–agent cost sharing problems. CONS guarantees PO in general. The proof is complete as soon it is verified that $\bar{\sigma}(v)$ belongs to the prekernel of v for every $v \in \mathscr{G}(\Gamma^*(U))$. (Recall that the **prekernel** of a game (N,v) is the set

$$\{x \in X(v) \mid s_{ij}(x) = s_{ji}(x) \text{ for } i,j \in N, i \neq j\},$$

where $s_{ij}(x) = \max \{x(S) - v(S) \mid j \notin S \ni i\}$ is the **maximal surplus from i over j at x**.) Indeed, Maschler–Peleg–Shapley (1972) showed that the prekenel of a concave game is a singleton consisting of the prenucleolus only. If (N,C,q) is any CSP with a concave cost function and $n \geq 2$, take different agents i and j of N, take $S = \{i,j\}$, and consider the reduced CSP $(S, C_{S,x}, q_S)$, where $x = \sigma(N,C,q)$. The vector x is a member of the prekernel of $v^{C,q}$, if and only if x_S is the **standard solution** for $v = v^{S,C_{S,x}}$ for $i,j \in N$ with $i \neq j$. (Recall that $\bar{\sigma}$ is called **standard** on $\mathscr{G}$, if for every game $(N,v) \in \mathscr{G}$ with $n = 2$ $x_i = (v(N)+v(\{i\})-v(\{j\}))/2$ is true, where $x = \sigma(v)$ and $N = \{i,j\}$.) To show that our continuation $\bar{\sigma}$ is standard, let (N,C,q) be a 2–agent CSP in $\Gamma^*(U)$ (i.e. $|U| \geq 2$ is assumed implicitly; otherwise the proof is already finished). Without loss of generality let $q_i \leq q_j$. Define $\tilde{q}_k = q_i$ for $k \in N$ and

$$\tilde{C}(y) = \begin{cases} C(y) &, y \leq C(q_i) \\ C(q_i) + (y-q_i) \cdot (C(q(N))-C(q_j))/q_i &, y \geq C(q_i) \end{cases},$$

where $0/0 = 1$ (if $q_i = 0$) in this context. Since C is concave, $C(q_i)+C(q_j) \geq C(q(N))$, thus $\tilde{C}$ is concave. Therefore $(N,\tilde{C},\tilde{q}) \in \Gamma^*(U)$. Moreover, $v^{C,q}+\beta = v^{\tilde{C},\tilde{q}}$, where $\beta_i = 0$ and $\beta_j = C(q_i) - C(q_j)$. By ET $\sigma_i(N,\tilde{C},\tilde{q})$ coincides with $\sigma_j(N,\tilde{C},\tilde{q})$. COV and PO complete the proof of standardness.
<div style="text-align:right">q.e.d.</div>

Note that this characterization is in fact a characterization of the prekernel which can be generalized to the set of concave cost games with player sets contained in U.

Though the analogon of Lemma 1.6 does not hold (i.e. it is not true that every concave cost game coincides, up to strategical equivalence, with the underlying game of some CSP with a concave cost function (see Section 4)), COV cannot be dropped, because average cost pricing satisfies ET and SCONS but does not coincide with the nucleolus rule if U contains different agents. Sudhölter (1993) presents examples which show that ET is independent unless U is a singleton. The Shapley rule satisfies ET and COV but SCONS fails for $|U| \geq 3$.

It should be remarked that it is not known whether CONS is strong enough to replace SCONS in Theorem 3.3. Nevertheless, the present approach shows that within this restricted family of cost sharing properties strong consistency (which seems to be more intuitive than CONS) can easily be satisfied. The general case requires CONS. SCONS is not satisfied even for average cost pricing.

Theorem 3.4:   If U is infinite, then the nucleolus rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and CONS.

In the proof of this theorem Sobolev's (1975) result is applied.

Proof:

It is sufficient to show uniqueness. Let σ be a CSR with the desired properties and let $\bar{\sigma}$ be its continuation. Sobolev showed that $\bar{\sigma}$ has to coincide with $\bar{\nu}$, if $\bar{\sigma}$ satisfies $\overline{COV}$, $\overline{AN}$, and $\overline{CONS}$. Orshan (1993) showed that $\overline{AN}$ can be replaced by $\overline{ET}$. In view of Corollary 1.8 it suffices to show that $\bar{\sigma}$ satisfies $\overline{CONS}$. A further modification of the proof of Lemma 1.6 (which is similar to that of Theorem 2.5) shows that $\bar{\sigma}$ is consistent. Indeed, if $(N,v) \in \mathscr{Y}(U)$ and S is a nonvoid coalition in N, then there is a demand profile q and a cost function C such that $v^{C,q} = v+\beta$ and $C_{S,x}$ $(x = \sigma(N,C,q))$ is a cost function. COV completes the proof.   q.e.d.

There are examples which show the logical independence of the properties as well as the necessity of the infinity assumption (see Sudhölter (1993)).

4. The anti–nucleolus rule

In this section we present an axiomatization of the anti–nucleolus by ET, COV, and dual consistency. Moreover, it is shown that the anti–nucleolus rule is in the core, if the cost function is concave.

Definition 4.1: Let σ be a CSR on $\Gamma \subseteq \Gamma(U)$ and $\bar\sigma$ be a solution concept on $\mathcal{G} \subseteq \mathcal{G}(U)$.

(i)   For every game (N,v), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **dual reduced game** $(S,{}^*v_{S,x})$ of v (at x with respect to S) is defined by

$${}^*v_{S,x}(T) = \begin{cases} 0 & \text{, if } T = \emptyset \\ v(N) - x(S^c) & \text{, if } T = N \\ \max \{v(T\cup R) - x(R) \mid R \subseteq S^c\}, & \text{otherwise} \end{cases} \quad \text{for } T \subseteq N.$$

(ii) $\bar\sigma$ is **dual consistent** (satisfies $\overline{\text{DCONS}}$), if ${}^*v_{S,\sigma(v)} \in \mathcal{G}$ and $\sigma(S,{}^*v_{S,\sigma(v)}) = \sigma(N,v)_S$ for $v \in \mathcal{G}$ and $\emptyset \neq S \subseteq N$.

(iii)   For every CSP (N,C,q), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **dual reduced function** $C_{S,x} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined by

$$C^*_{S,x}(y) = \begin{cases} 0 & \text{, if } y = 0 \\ C(q(N)) - x(S^c) & \text{, if } y > \bar y \\ \max \{C(y+q(R)) - x(R) \mid R \subseteq S^c\}, & \text{if } 0 < y \leq \bar y \end{cases},$$

where $\bar y = \max \{q(T) \mid T \subsetneq S\}$. (Note that the dual reduced function need not be a cost function, since $C^*_{S,x}(\bar y)$ might be larger than $C^*_{S,x}(\epsilon+q(S))$.)

(iv) σ is **dual consistent** (satisfies DCONS), if for $(N,C,q) \in \Gamma$, $\emptyset \neq S \subseteq N$, and $x = \sigma(N,C,q)$ the following condition is satisfied:

If $C^*_{S,x}$ is nondecreasing, then $(S,C^*_{S,x},q_S) \in \Gamma$ and $x_S = \sigma(S,C^*_{S,x},q_S)$.

Note that nondecreasing in (iv) only means $C^*_{S,x}(\bar y) \leq C(q(N)) - x(S^c)$. The main difference between CONS and DCONS is that the "optimistic" view is replaced by the "pessimistic" view of coalition $T \subseteq S$. Every coalition R of members of S^c (who agreed upon the proposal given by σ) has to be considered as possible coalition of partners of T. Therefore the total cost generated by the aggregate demand of the union $T \cup R$ can be "reduced" by the fee which will be paid by R in the "worst" case.

Note that $\overline{\text{DCONS}}$ for underlying games implies DCONS for the cost sharing problems. The proof of the next theorem is similar to that of Theorem 3.4. Mainly games have to be replaced by their "duals". Recall that if (N,v) is a game, then (N,v^*), defined by $v^*(S) = V(N) - v(S^c)$ for $S \subseteq N$ is its dual.

Theorem 4.2:   If U is infinite, then the anti–nucleolus rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and DCONS.

Proof:

By duality (the anti–prenucleolus coincides with the prenucleolus of the dual game) it is sufficient to show uniqueness. Therefore we only need to show that $\bar\sigma$ satisfies $\overline{\text{DCONS}}$. The proof of $\overline{\text{DCONS}}$ is very similar to the proof of $\overline{\text{CONS}}$ in Theorem 3.4. Therefore we skip the details and apply Sobolev's and Orshan's results.   q.e.d.

In order to show the independence of the properties again Sudhölter (1993) is referred to.

To provide the second result of this section the definition of the least core of a cost game (N,c) is recalled. The **least core of** (N,v) is the set

$$\mathcal{LC}(v)=\{x\in X(v)\,|\,x(S)-v(S)\leq\max\,\{y(T)-v(T)\,|\,\emptyset\neq T\subsetneq N\},\;\emptyset\neq S\subsetneq N\text{ for }y\in X(v)\}$$

of preimputations which minimize the maximal nontrivial excess. As a matter of fact the nucleolus $\bar{\nu}(v)$ is a member of the least core.

Lemma 4.3:   If C is a concave cost function and (N,C,q) is a CSP, then the least core of the dual of its underlying game is contained in its core, i.e.

$$\mathcal{LC}(v^*)\subseteq\mathcal{C}(v),\text{ where }v = v^{C,q}.$$

Proof:

Let x be any preimputation of v which does not belong to the core of v. Then there is a coalition S in N with positive excess, i.e. $v(S) - x(S) < 0$. Define $\mathcal{D} = \mathcal{D}(x)$ to be the set of maximal excess coalitions (i.e. $\mathcal{D} = \{S \subseteq N\,|\,v(S)-x(S)\leq v(T)-x(T)\text{ for }T \subseteq N\}$). The set $\mathcal{D}$ possesses the "near–ring property" in the sense of Maschler–Peleg–Shapley (1972), thus $\bar{S} = \underset{S\in\mathcal{D}}{\cup}\,S \in \mathcal{D}$. Conversely, let $\mathcal{M}=\{S \subseteq N\,|\,v(S)-x(S)\geq v(T)-x(T)\text{ for }T\subseteq N\}$ denote the set of minimal excess coalitions. For $i \in \bar{S}^c$ we easily obtain

$$x_i < v(\bar{S}\cup\{i\}) - v(\bar{S}). \tag{1}$$

Claim: For $S \in \mathcal{M}$ either $\bar{S}^c \subseteq S$ or $S \subseteq \bar{S}^c$ is true.

If $q(S) \leq q(\bar{S})$, then assume $S \cup \bar{S} \neq N$ and take $i \in (S\cup\bar{S})^c$. The fact

$$x_i < v(\bar{S}\cup\{i\}) - v(\bar{S}) \leq v(S\cup\{i\}) - v(S),$$

which is true by concavity of C and (1), implies

$$(v(S\cup\{i\}) - x(S\cup\{i\})) - (v(S)-x(S)) > 0,$$

which establishes a contradiction to the fact $S \in \mathcal{M}$. Therefore $\bar{S}^c \subseteq S$ in this case.

If $q(S) > q(\bar{S})$, then take such S that $|S|$ is minimal. If $\bar{S} \cap S \neq \emptyset$, take $i \in \bar{S} \cap S$ and observe that

$$v(S) - v(S\backslash\{i\}) \leq v(\bar{S}) - v(\bar{S}\backslash\{i\})\qquad\qquad\text{(by concavity of C)}$$

is valid, hence

$$\begin{array}{ll}(v(S)-x(S)) - (v(S\backslash\{i\})-x(S\backslash\{i\})) & \leq \\ (v(\bar{S})-x(\bar{S})) - (v(\bar{S}\backslash\{i\})-x(\bar{S}\backslash\{i\})) & \leq 0,\end{array}$$

thus equality holds (remember that $S \in \mathcal{M}$, $\bar{S} \in \mathcal{D}$). Minimality of S and the first part of this proof show that $q(S\backslash\{i\}) \leq q(\bar{S})$ and $\bar{S}^c \subseteq S$. If $S\cap\bar{S} = \emptyset$, then $S \subseteq \bar{S}^c$.

Now the proof can be finished using Kohlberg's (1971) characterization of the nucleolus

by balanced collections of coalitions which can easily be weakened for the least core (see, e.g., Sudhölter (1996)). Indeed, if y is a preimputation of v, then y belongs to the least core of v^*, if and only if $\mathscr{A}(y)$ is **weakly balanced**. Weakly balanced means that the barycenter of the grand coalition is a in the convex hull of the barycenters of the maximal excess coalitions, i.e. there are coefficients $\gamma_S \geq 0$ such that

$$\sum_{S \in \mathscr{D}} \gamma_S \cdot 1_S = 1_N.$$

By our claim x cannot be a member of the least core of v^*. q.e.d.

This last lemma shows that the set $\mathscr{G}(\Gamma^*(U))$ of games strategically equivalent to the underlying game of some CSP with concave cost function is a "small" subset of the set of all concave games with player set contained in U. (Note that concavity is closed under strategical equivalence, i.e. the game is concave, iff every game, which is strategically equivalent to the initial game, is concave.) This can be seen with the help of examples. There is a concave game such that the least core of its dual does not even intersect the core of the game (see Sudhölter (1996), Example 3.2.3).

Corollary 4.4: The anti–nucleolus of a CSP (N,C,q) with concave C is in the core of its underlying cost game $v^{C,q}$.

5. Coincidence of cost sharing rules

This section serves to classify the set of cost functions such that the mentioned cost sharing rules, namely the Shapley rule and both nucleolus rules, coincide for every demand profile. We shall say that a cost function C satisfies the **coincidence property** (CP) at α for some $\alpha \geq 0$, if

$$\mathscr{a}(N,C,q) = \mathscr{G}(N,C,q) \tag{1}$$

holds true for every agent set N with two agents and every demand profile $q \in \mathbb{R}^N_{\geq 0}$ such that the aggregate demand coincides with α, i.e. $q(N) = \alpha$.

Theorem 5.1:
(i) A cost function satisfies CP at α, if and only if
$$\alpha \cdot (C(y) - C(\alpha-y)) = (2y - \alpha) \cdot C(\alpha) \text{ for } 0 \leq y \leq \alpha. \tag{2}$$
(ii) If C satisfies CP at α and (N,C,q) is a CSP with aggregate demand α, then
$$\nu(N,C,q) = \nu^*(N,C,q) = \mathscr{G}(N,C.q) = \mathscr{a}(N,C,q).$$

The typical shape of a cost function satisfying CP is sketched in Figure 2. Note that in assertion (ii) of this theorem n = 2 is **not** assumed. The nucleolus rule, the anti–nucleolus rule, average cost pricing, and the Shapley rule coincide for any cost sharing problem with a cost function satisfying CP at the aggregate demand of the agents.

Proof of Theorem 5.1:

Let C be any cost function. For $\alpha = 0$ both assertions are trivially satisfied, thus $\alpha > 0$ is assumed from now on. Take any y with $0 < y < \alpha$ and define N = {1,2}, q = (y,α–y). Standardness of the Shapley value shows that

$$\varphi_1(N,C,q) = (C(q_1)+C(q(N))-C(q_2))/2 = (C(y)+C(\alpha)-C(\alpha-y))/2.$$

Average cost pricing yields

$$a_1(N,C,q) = (q_1/q(N))\cdot C(\alpha) = y/\alpha \cdot C(\alpha).$$

This shows that φ and a coincide on (N,C,q), if and only if

$$(C(y)+C(\alpha)-C(\alpha-y))/2 = y/\alpha \cdot C(\alpha),$$

which is equivalent to (2). For y $\in$ {α,0} equality (2) is trivially satisfied.

In order to prove (ii) let (N,C,q) be any CSP such that q(N) = α and C satisfies CP at α. Let x $\in \mathbb{R}^N$ abbreviate a(N,C,q), i.e. $x_i = (q_i/\alpha)\cdot C(\alpha)$ for i $\in$ N. The explicit formula for the Shapley value (see Shapley 1953) yields

$$\varphi_i(N,C,q) = \sum_{i\in S\subseteq N} \frac{(s-1)!\cdot(n-s)!}{n}(C(q(S))-C(q(S\backslash\{i\})))$$

$$= \sum_{i\in S\subseteq N} \frac{(s-1)!\cdot(n-s)!}{n}\cdot\frac{(C(q(S))-C(q(S\backslash\{i\}))+C(q(S^c\cup\{i\}))-C(q(S^c)))}{2}$$

(by taking "complements")

$$\underset{(2)}{=} \sum_{i\in S\subseteq N} \frac{(s-1)!\cdot(n-s)!}{n}\cdot\frac{(2q(S)-\alpha)+(2q(S^c\cup\{i\})-\alpha)}{2\cdot\alpha}$$

$$= \sum_{i\in S\subseteq N} \frac{(s-1)!\cdot(n-s)!}{n}\cdot\frac{q_i}{\alpha}\cdot C(\alpha)$$

$$= (q_i/\alpha)\cdot C(\alpha)\cdot \sum_{s=1}^{n} \frac{(s-1)!\cdot(n-s)!}{n}\cdot\binom{n-1}{s-1} \qquad \text{(by "counting subsets")}$$

$$= x_i.$$

For the nucleolus rules Kohlberg's approach is used. At x the excess x(S)–v(S) coincides with that of S^c, where $v = v^{C,q}$ (by (2)). Therefore the balancedness critera show that both nucleolus rules coincide with average cost pricing. q.e.d.

[Insert Figure 2 about here]

Corollary 5.2: A cost function satisfying CP at α is continuous, if it is resticted to {y | 0$\leq$y$\leq\alpha$}.

Proof:

This assertion can be shown with the help of (2) and monotonicity of a cost function C. If C satisfies CP at α then the slope of C is bounded from below by 0 by definition. Equality (2) shows that the slope is bounded from above by $2 \cdot C(\alpha)$ whence C is restricted to the interval $[0,\alpha]$, i.e $(C(y)-C(z))/(y-z) \leq 2 \cdot C(\alpha)$ for $0 \leq z < y \leq \alpha$. Indeed,

$$\frac{(C(y)-C(z))}{y-z} + \frac{(C(\alpha-z)-C(\alpha-y))}{(\alpha-z)-(\alpha-y)} = \frac{(C(y)-C(\alpha-y))-(C(z)-C(\alpha-z))}{y-z}$$

$$= 2 \cdot C(\alpha) \qquad \text{(by (2))}$$

and both summands are nonnegative. q.e.d.

For $\alpha \geq 0$ the proposed cost sharing rules (Shapley rule, both nucleolus rules, and average cost pricing) can be characterized on the set Γ of cost sharing problems with cost functions satisfying CP at α, where α is the aggregate demand. Indeed, Moulin–Shenker (1993) showed that average cost pricing is the unique CSR satisfying Pareto optimality, separable costs, and monotonicity on sets of cost sharing problems (N,C,q), which contain $(N,\tilde{C}^q,q)$ and $(N,\tilde{C}^q {\scriptstyle \wedge} C,q)$, where $\tilde{C}^q$ is the "constant returns" cost function given by $\tilde{C}(y) \cdot q(N) = y \cdot C(q(N))$. Since Γ possesses this property by (2), in view of Theorem 5.1 the same result holds true for φ, ν, and ν^* on Γ. Moreover, if $|U| \geq 2$, the logical independence of PO, SC, and MON can be verified by considering the following cost sharing rules. The CSR, which assigns $\varphi_i(N,C,q)+(C(q_i)-\tilde{C}_i^q(q_i))_+$ to agent i for every CSP (N,C,q) of Γ satisfies SC and MON but is not Pareto optimal. The "equal split" solutions shows the independence of SC. Finally, the CSR which assigns

$$C(q_i)+(1/n) \cdot (C(q(N))- \sum_{j \in N} C(q_j))$$

to agent i for every CSP (N,C,q) of Γ satisfies PO and SC but is not monotonic.

A cost function C possesses the **global coincidence property (GCP)**, if C satisfies CP at every $\alpha \in \mathbb{R}_{\geq 0}$. The set of cost functions satisfying GCP can be characterized. I am indebted to A. Sobolev who found the elementary proof of the following

Theorem 5.3: A cost function C satisfies GCP, if and only if C is a parabola or if it is linear, i.e. there are nonnegative real numbers A,B such that

$$C(y) = A \cdot y^2 + B \cdot y \text{ for } y \geq 0. \tag{3}$$

Proof:

Every function $C: \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by (3) is a cost function. The proof that C satisfies (29 for every α and $0 \leq y \leq \alpha$ is straightforward and skipped. For the converse let C be a

cost function satisfying GCP. By (2) we have

$$y \cdot (C(z+x) - C(z-x)) = x \cdot C(2z) \tag{4}$$

for $0 \leq z \leq x$, which can be seen by applying (2) to $\alpha = 2z$ and $y = z+x$. Define

$$\Delta = \Delta_n = 1/2n, \ z = k \cdot \Delta_n, \ x = m \cdot \Delta_n, \ y_p = C(p \cdot \Delta_n)$$

for $n,p,k,m \in N$ with $m \leq k$. By (4) we come up with

$$k \cdot (y_{k+m} - y_{k-m}) = m \cdot y_{2k}. \tag{5}$$

The following recursive formulae

$$y_{2k} \qquad = k^2 \cdot (y_3 - y_2 - y_1) + k(2y_2 + y_1 - y_3) \tag{6}$$

and $\qquad y_{2k-1} \qquad = (k-1)^2 \cdot (y_3 - y_2 - y_1) + (k-1) \cdot y_2 + y_1 \tag{7}$

are obviously valid for $k = 1,2$. By induction on k it can be shown that (6),(7) are generally true. Indeed, for $k \geq 3$ first of all the application of (5) with $m = 1$ shows (6). Secondly (5) applied to $m = k-1$ shows (7). The necessary computations are straightforward and skipped.

With $\eta = 2y_2 + y_1 - y_3$ and $\rho = y_3 - y_2 - y_1$ we obtain

$$C(1) \qquad = C(2n \cdot \Delta) = y_{2n} = n^2 \cdot \rho + n \cdot \eta,$$

$$C(2) \qquad = C(4n \cdot \Delta) = y_{4n} = 4n^2 \cdot \rho + 2n \cdot \eta, \qquad \text{(both by (6))}$$

thus $\qquad n^2 \rho \qquad = (C(2) - 2 \cdot C(1))/2$

and $\qquad n\eta \qquad = (4 \cdot C(1) - C(2))/2.$

The observation

$$C(k/n) = C(2k/2n) = C(2k \cdot \Delta) = k^2 \cdot \rho + k \cdot \eta \qquad \text{(by (6))}$$

$$= (k/n)^2 \cdot n^2 \cdot \rho + (k/n) \cdot n \cdot \rho$$

$$= ((C(2) - 2 \cdot C(1))/2) \cdot (k/n)^2 + ((4 \cdot C(1) - C(2))/2) \cdot (k/n)$$

for $k,n \in \mathbb{N}$ shows that $C(y) = Ay^2 + By$ for any rational number $y \geq 0$, where

$$A = (C(2) - 2 \cdot C(1))/2 \ \text{and} \ B = (4 \cdot C(1) - C(2))/2.$$

Since C is a cost function (nondecreasing and nonnegative), A and B are nonnegative. Corollary 5.2, i.e. continuity of C, shows that C possesses the desired properties.   q.e.d.

6. Summarizing diagrams

Table 1 shows the properties of the cost sharing rules discussed in this paper. A "−" means that the corresponding CSR does not satisfy this property at least on the set of all cost sharing problems with agent set contained in some universe U of more than two members. The symbols "+" and "⊕" mean that the corresponding property is satisfied or belongs to the axiomatization respectively.

	a	φ	ν	ν^*
PO	+	+	+	+
RAN	+	+	+	+
SC	+	+	+	+
ET	+	+	+	+
NP	+	+	+	+
AN	+	+	+	+

"common properties"

	a	φ	ν	ν^*
MON	+	−	−	−
ADD	+	+	−	−
MMADD	+	+	−	−
CON	+	+	−	−
CONS	+	−	+	−
SCONS	−	−	−	−
DCONS	+	−	−	+
COV	−	+	+	+

Table 1: "Properties"

With a finite subset N of U such that n $\geq$ 3 and $a > 0$ the following 3 tables summarize the presented axiomatizations on Γ_N, $\Gamma(U)$, $\Gamma^*(U)$, and on the set Γ of cost sharing problems (N,C,q) with cost functions C satisfying the coincidence property at a, where $a = q(N)$ is the aggregate demand of the agents. The abbreviation "M–S" indicates that this axiomatization is due to Moulin–Shenker (1993). The numbers refer to the name of the Theorem or to the section where this result is proved. Table 2 treats cost sharing rules on cost sharing problems Γ_N with a fixed agent set.

on Γ_N	φ		a
	2.1	2.2	M–S
ET	⊕	⊕	+
NP	⊕	⊕	+
ADD	⊕	+	+
MMADD	+	⊕	+
COV	⊕	⊕	−
SC	+	+	⊕
PO	+	+	⊕
MON	−	−	⊕

Table 2: "On Γ_N"

Though Theorems 2.1 and 2.2 also hold for $\Gamma(U)$, Table 3 neither mentions this fact nor repeats the characterization of average cost pricing. These axiomatizations are strongly based on "varying agent sets", i.e. on certain consistency principles (see Sections 2–4).

on $\Gamma(U)$	ν		ν^*	φ
	3.3	3.4	4.2	2.5
ET	⊕	⊕	⊕	⊕
CON	−	−	−	⊕
CONS	+	⊕	−	−
SCONS	⊕	−	−	−
DCONS	−	−	⊕	−
COV	⊕	⊕	⊕	⊕

on $\Gamma^*(U)$ U infinite

Table 3: "On $\Gamma(U)$"

Table 4 presents the common axiomatizations on the set Γ.

on Γ	φ	ν	ν^*	a
	Section 5			M–S
ET	⊕	⊕	⊕	⊕
CON	⊕	⊕	⊕	⊕
CONS	⊕	⊕	⊕	⊕

Table 4: "Coincidence"

References:

Aumann, R. and Shapley, L.S., 1974, Values of non atomic games. Princeton University press, princeto New Jersey

Davis, M. and Maschler, M., 1965, The kernel of a cooperative game. Naval Research Logist. Quarterly 12, 223–259

Dubey, P., 1975, On the uniqueness of the Shapley value. Int. Journal of Game Theory 4, 132–139

Hart, S. and Mas–Colell, A., 1989, Potential, value, and consistency. Econometrica 57, 589–614

Kalai, E. and Samet, D., 1988, Weighted Shapley values. Essays in Honor of L.S. Shapley, 83–99

Kohlberg, E., 1971, On the nucleolus of a characteristic function game. SIAM J. Appl. Math. 20, 62–66

Maschler, M., 1992, The bargaining set, kernel, and nucleolus: a survey. In: Handbook of Game Theory 1, 591–665

Maschler, M., and Peleg, B., 1966, A characterization, existence proof and dimension bounds for the kernel of a game. Pacific J. Math. 18, 289–328

Maschler, M., Peleg, B., and Shapley, L.S., 1972, The kernel and bargaining set for convex games. Int. Journal of Game Theory 1, 73–93

Maschler, M., Peleg, B., and Shapley, L.S., 1979, Geometric properties of the kernel, nucleolus, and related solution concepts. Math. of Operations Research 4, 303–338

Moulin, H., and Shenker, S., 1992, Serial cost sharing. Econometrica 60, 1009–1037

Moulin, H., and Shenker, S., 1993, Average cost pricing versus serial cost sharing: an axiomatic comparison. GREQE Working Paper 93a03, Marseille

Orshan, G., 1993, The prenucleolus and the reduced game property: equal treatment replaces anonymity. Int. Journal of Game Theory 22, 241–248

Peleg, B., 1989, Introduction to the theory of cooperative games. Research Memoranda No. 81 – 88, Center for Research in Math. Economics and Game Theory, The Hebrew University, Jerusalem, Israel

Potters, J. and Sudhölter, P., 1995, Airport problems and consistent solution rules. Working Paper 246, Inst. of Math. Ec., University of Bielefeld

Schmeidler, D., 1969, The nucleolus of a characteristic function game. SIAM J. Appl. Math. 17, 1163–1179

Shapley, L.S., 1953, A value for n–person games. Contributions to the Theory of Games II, H. W. Kuhn and W. Tucker (eds.), Annals of Mathematical Studies 28, Princeton University Press, princeton, 307–317

Shenker, S., 1989, Making greed work in networks: a game–theoretic analysis of gateway service disciplines. mimeo, Xerox Palo Alto Research Center

Shenker, S., 1990, Efficient network allocations with selfish users. In "Performance '90", P. King, I. Mitrani, R. Pooley (eds.), North Holland, New York

Sobolev, A.I., 1975, The characterization of optimality principles in cooperative games by functional equations. In: N.N. Vorobiev, ed., Math. Methods Social Sci. 6, Academy of Sciences of the Lithunian SSR, Vilnius, 95–151 (in Russian)

Sudhölter, P., 1993, Independence of characterizing axioms of the pre–nucleolus. Working Paper 220, Inst. of Math. Ec., University of Bielefeld

Sudhölter, P., 1996, The modified nucleolus: properties and axiomatizations. To appear in the Int. Journal of Game Theory
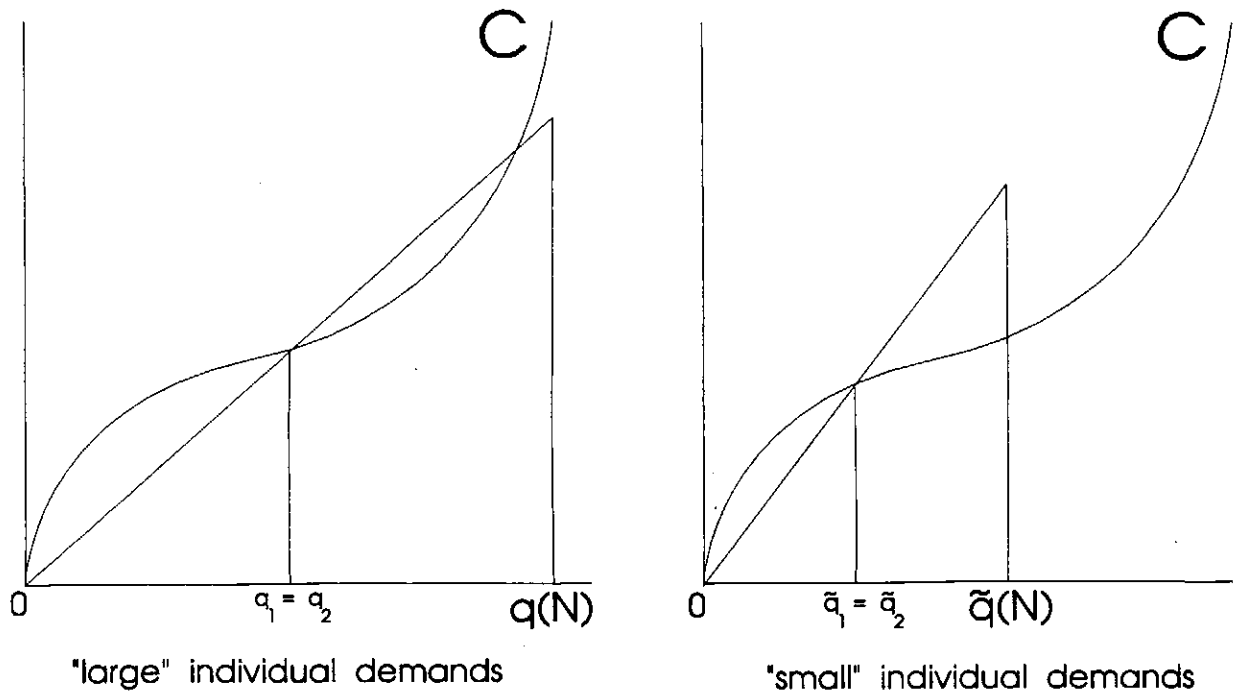
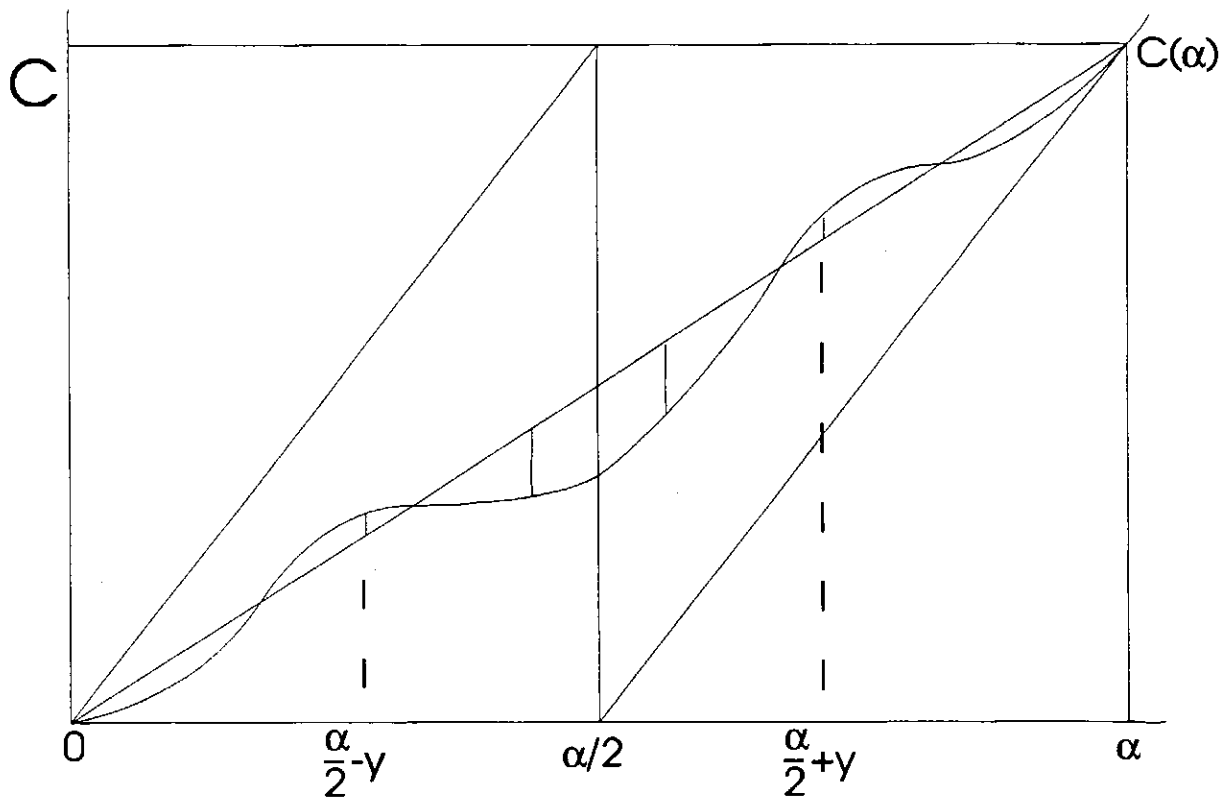"large" individual demands   "small" individual demands

Figure 1: "S-shaped" cost function

Figure 2: C with the CP at α