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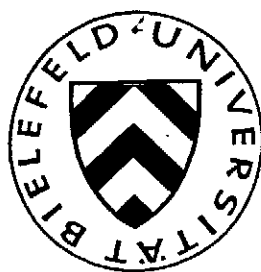
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**Separable Aggregation and the  
Existence of Nash Equilibrium**

by

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**Separable Aggregation  
and the Existence of Nash Equilibrium<sup>†</sup>**

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**Abstract**

Algebraic conditions for the existence of a Nash equilibrium are studied for two classes of strategic form games: games with public and private objectives and games with decreasing best replies. Considerable evidence is gathered to support the claim that in both cases the crucial factor is the presence of a separable ordering on the set of outcomes, connected with the players' utilities in a certain way. Some technical problems related to the description of separable orderings and to establishing separability are also discussed.

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## 1. Introduction

It is a usual practice in economic theory to assume everything (or at least as much as possible) smooth and convex. In many cases such assumptions are necessary for reaching clear-cut conclusions, but sometimes they just obscure the real reasons for the existence (or good behavior) of solutions for problems under investigation. Recently this point was raised forcefully by Milgrom and Roberts (1994), who, in particular, pointed out to the importance of monotonicity assumptions.

This paper concerns with purely combinatorial (or algebraic) causes for the existence of a Nash equilibrium, needing no convexity or smoothness. Two classes of strategic form games are considered; the first generalizes the usual model of the voluntary provision of a public good (or a public bad); the second is quite abstract: the strategy sets are ordered and the best replies are decreasing. Besides adding new results to those established earlier, the paper expounds the view that, in both cases, the crucial factor is the presence of separable orderings on  $\mathbb{R}^m$  (or its subsets). Unfortunately, the most distinct formulation of the view remains a hypothesis in either case.

It was not without hesitation that I decided to print the results in their current, rather preliminary state. I can put forward three reasons for this decision. First, even the already established results show an unexpected and unexplained similarity between two quite different game-theoretic contexts;

it is difficult not to think that separability can also play an essential role in other situations. Second, there is no grounds to believe that all the remaining open questions could be answered in the foreseeable future. Finally, I have no idea of when I could have another opportunity to relate my results.

Section 2 contains necessary definitions and notations.

Section 3 considers "games with public and private objectives"; the first example of a model of this kind was suggested by Germeier and Vatel' (1974). The most advanced results on the existence of Nash or strong equilibrium in such games are to be found in Kukushkin (1992, 1994b). Theorems 1 and 1' unify these results (in the part concerning Nash equilibrium) and explicitly show the crucial role of separability here. Unfortunately, the necessity of separability for the "persistent" existence of a Nash equilibrium, established under stronger assumptions in Kukushkin (1994b), remains a hypothesis.

Section 4 considers games where strategy sets are (partially) ordered and the best replies are decreasing. It should be noted that in many economic situations the assumption that the best replies are monotonic (i.e. either increasing or decreasing) is quite natural, see Fudenberg and Tirole (1984) and Bulow et al. (1985). However, there is a striking difference between the two cases. The existence of a Nash equilibrium in a game with increasing best replies is guaranteed by Tarski's (1955) theorem (under relatively mild restrictions on the strategy sets). A subclass of such games - games with strategic complementarities, also have nice comparative statics properties,

see Topkis (1979), Milgrom and Roberts (1990), Vives (1990), Milgrom and Shannon (1994). On the other hand, not only is the straightforward analogue of Tarski's theorem for decreasing mappings untrue, but the existence of an equilibrium in such a game is by no means a forgone conclusion. As to the comparative statics, I am unaware of any result in this direction (except for duopoly games) and cannot suggest anything myself.

The section is devoted to a sole problem: Under what additional assumptions do decreasing best replies guarantee the existence of a Nash equilibrium? Departing from the first ever result of this kind, Novshek (1985), refined in Kukushkin (1994a), several new existence theorems are obtained (without any attempt to assess the comparative importance of different sets of assumptions). In each case it turns out that the existence of an equilibrium is guaranteed when there is a separable ordering on the set of outcomes such that each player's preferences are affected by the partners' choices only to the extent that the latter affect the ranking of the outcome. An exact formulation is given to the idea that the existence of an equilibrium should not be expected otherwise, but this statement also remains a hypothesis, except for just one particular case (Theorem 8).

Section 5 may be regarded as an unnecessary technical appendix. The point is that the two necessity results obtained in the area (Kukushkin, 1994b, and Theorem 8 in Section 4) require the continuity assumption and establish additive separability; their proofs are strikingly similar to one another and resemble to some extent the technique used in the classical

work on separability, Debreu (1960). Debreu's result was later strengthened by Gorman (1968a), using quite a different approach. Theorem 1 of Section 5 shows that my technique (i.e. an extension of Debreu's one) is enough to advance considerably in the direction of Gorman's result. I believe the Debreu-Gorman theorem is so important that it justifies paying attention to purely technical aspects of feasible proofs.

Theorems, examples, and formulas are numbered in each section separately.

## 2. General Definitions

An ordering  $\vartheta$  on a set  $X$  is just a reflexive and transitive binary relation on  $X$ ; the ordering is complete if, for every  $x, y \in X$ , either  $x\vartheta y$  or  $y\vartheta x$ . both). The asymmetric part of an ordering  $\vartheta$  will be denoted  $\vartheta^\circ$ , i.e.  $x\vartheta^\circ y$  means  $x\vartheta y$  but not  $y\vartheta x$ .  $x \in X' \subseteq X$  is a maximal element of  $\vartheta$  on  $X'$  if no  $y \in X'$  satisfies  $y\vartheta^\circ x$ . An ordering  $\vartheta$  on  $\mathbb{R}^m$  (or on a subset of  $\mathbb{R}^m$ ) is called Pareto compatible if  $x' \vartheta x''$  whenever  $x'$  Pareto dominates  $x''$ .

A normal (or strategic) form game is defined by a finite set of players  $N$  and, for each  $i \in N$ , a set of strategies  $X_i$  and an ordering  $\mathcal{R}_i$  on the set of outcomes  $X = \prod_{i \in N} X_i$  ( $x \mathcal{R}_i y$  means that  $x$  is better for player  $i$  than  $y$ ). Denoting  $X_{-i} = \prod_{j \neq i} X_j$ , we define  $R_i(x_{-i})$  (for  $x_{-i} \in X_{-i}$ ) as the set  $\{x_i \in X_i \mid (x_i, x_{-i}) \text{ is a maximal element of } \mathcal{R}_i \text{ on } \{(x_i, x_{-i}) \mid x_i \in X_i\}\}$ . Viewed as a correspondence  $X_{-i} \rightarrow X_i$ ,  $R_i$  is called the best reply correspondence (for player  $i$ ); generally speaking, it may have empty values for some (or even all)  $x_{-i} \in X_{-i}$ .

An outcome  $x^\circ \in X$  is a Nash equilibrium if  $x_i^\circ \in R_i(x_{-i}^\circ)$  for

all  $i \in N$ .

It should be noted that we allow the players to have incomplete preference orderings (cf. Aumann, 1962, 1964; Rozen, 1976). This fact does not mean that I attach any particular importance to such models; just there is no need to assume the completeness. By the way, the above definition gives a rather weak interpretation of the concept of Nash equilibrium for this case.

Now I introduce an abstract definition playing a crucial part throughout all the paper. Suppose there is a finite set  $N$  and an ordering  $\vartheta$  on  $X = \prod_{i \in N} X_i$ ; suppose also that  $M \subseteq N$  and  $\vartheta_M$  is an ordering on  $X_M = \prod_{i \in M} X_i$ . We say that  $\vartheta_M$  is a separable projection of  $\vartheta$  on  $X_M$  if, for every  $x^M, y^M \in X_M$  and  $x^{-M} \in X_{-M}$ ,  $\langle x^M, x^{-M} \rangle \vartheta \langle y^M, x^{-M} \rangle$  if and only if  $x^M \vartheta_M y^M$ . Note that if  $\vartheta$  is complete, then  $\vartheta_M$  is also complete.

The separability property emerges naturally in a wide variety of situations and was studied by many authors. In particular, the most popular aggregation rules in social choice theory - utilitarian and leximin ones, see e.g. d'Aspremont and Gevers (1977) or Deschamps and Gevers (1978) - have separable projections on all coordinate subspaces. Leximax and fixed-order lexicographies have the same property.

In his seminal paper Debreu (1960) showed that every continuous, Pareto compatible ordering on  $\mathbb{R}^m$  ( $m > 2$ ) which has a separable projection on every two-dimensional subspace can be represented by the sum of rescaled coordinates (i.e. is utilitarian in a somewhat generalized sense). In fact, Debreu's result was formulated for orderings on more general sets but,

as he showed himself, the general case is easily reduced to that of a rectangular subset of  $\mathbb{R}^m$ . The same statement was earlier reported by Fleming (1952), but he assumed smoothness and treated infinitesimals rather freely. (The result of Harsanyi (1955), also establishing conditions for additive separability, apparently has no connection with Debreu's theorem either technically or conceptually). In another seminal paper, Gorman (1968a) strengthened the theorem considerably, weakening the separability assumption; see also the edifying discussion between Gorman (1971a,b) and Vind (1971a,b, 1974).

Debreu's approach was based on a geometric theorem due to W. Blaschke; later Fishburn (1970, Chapter 5) replaced the reference to Blaschke's theorem with a direct inductive reasoning. Curiously, the proof of additive separability (in a particular context) developed in Kukushkin (1994b, Appendix) (without any knowledge of the Debreu-Gorman theorem, I must confess) resembles that of Fishburn (1970, Theorem 5.4). From the geometrical viewpoint, it could be interpreted as extending a version of Blaschke's theorem to a bunch of planes (unfortunately, I am no expert in either geometry or German to say whether Blaschke himself ever considered such an extension). The proof is recalled in the following several times; in principle, the reader may substitute "as in Fishburn (1970)" for "as in Kukushkin (1994b)" in each case, although this would make the reference a bit less relevant.



### 3. Games with Public and Private Objectives

A game with public and private objectives is a strategic form game of the following type. There is a finite set of players  $N$  (without restricting generality, we assume  $0 \in N$ ) and, for each  $i \in N$ , a compact strategy set  $X_i$ , a continuous function  $f_i: X_i \rightarrow \mathbb{R}$ , and a Pareto compatible ordering  $\rho_i$  on  $\mathbb{R}^2 = \mathbb{R}_0 \times \mathbb{R}_i$ ; there is also a continuous function  $f_0: X \rightarrow \mathbb{R}$ . Now the preferences of each player  $i$  on the set of outcomes  $X$  are these:

$$x \mathcal{R}_i y \text{ if and only if } (f_0(x), f_i(x)) \rho_i (f_0(y), f_i(y)). \quad (1)$$

There is no real necessity to demand that  $\rho_i$  should be defined on the whole  $\mathbb{R}^2$ ; strictly speaking, it would be sufficient to have it on the set  $(f_0 \times f_i)(X)$ . However, for the interpretation of some results to follow, it is convenient to perceive  $\rho_i$  as a stable characteristic of player  $i$  who may find himself in different situations, playing different games.  $\mathbb{R}^2$  could be replaced with an open rectangular subset if only games with  $(f_0 \times f_i)(X)$  included in the subset are to be considered.

Throughout this section we will denote  $M$  the set  $\{0\} \cup N$ , i.e. the set of all (public and private) objective functions in the model under consideration.

**Theorem 1.** Suppose each strategy set  $X_i$  ( $i \in N$ ) is finite and there exists an ordering  $\vartheta$  on  $\mathbb{R}^M$  having separable projections  $\vartheta_i$  on all planes  $\mathbb{R}_0 \times \mathbb{R}_i$  such that  $a \rho_i b$  implies  $a \vartheta_i b$  for all  $i \in N$ ,  $(a, b) \in \mathbb{R}_0 \times \mathbb{R}_i$ . Then the game has a Nash equilibrium.

The proof is quite straightforward: consider the mapping  $f: X \rightarrow \mathbb{R}^M$  defined by  $f(x) = (f_0(x), \langle f_i(x_i) \rangle_{i \in N})$  and let  $f(x^0)$  be

a maximal point of  $\vartheta$  on  $f(X)$ . If  $x^\circ$  is not a Nash equilibrium, then there exist  $i \in N$  and  $y \in X$  such that  $y \mathcal{R}_i^\circ x^\circ$  and  $y_j = x_j^\circ$  for all  $j \neq i$ . Therefore,  $(f_0(y), f_1(y_1)) \vartheta_i^\circ (f_0(x^\circ), f_1(x_1^\circ))$ , hence (by separability)  $f(y) \vartheta f(x^\circ)$ , contradicting the choice of  $x^\circ$ .

We call an ordering  $\vartheta$  on  $\mathbb{R}^M$  regular if  $\vartheta$  has a maximal element on every compact subset of  $\mathbb{R}^M$ .

**Theorem 1'.** Assume everything as in Theorem 1 with the exception that each  $X_i$  is compact and ordering  $\vartheta$  is regular; then the game has a Nash equilibrium.

The proof is virtually the same.

As particular cases of Theorem 1' could be listed the main result of Kukushkin (1992) in its sufficiency part (and restricted to Nash equilibrium), and the theorem (sufficiency part) and Proposition 3 of Kukushkin (1994b). The last case may deserve an explanation; first of all, I remind that every ordering  $\rho_i$  on  $\mathbb{R}^2$  was there defined by the function

$$F^\circ(a, b) = \begin{cases} a \cdot b, & \text{if } a > 0 \text{ and } b > 0, \\ \min\{a, b\}, & \text{otherwise.} \end{cases}$$

Now for any  $\langle a_i \rangle_{i \in M} \in \mathbb{R}^M$ , define  $\langle b_i \rangle_{i \in M}$  as follows: if  $a_i \leq 0$ , then  $b_i = a_i$ , otherwise,  $b_i$  is the product of all positive  $a_i$ 's; now the leximin ordering on  $b$ 's induces an ordering  $\vartheta$  on  $a$ 's, which has a separable projection  $\vartheta_i$  on every  $X_i$  and the ordering defined by  $F^\circ$  is a coarsening of  $\vartheta_i$ .

Theorems 1 and 1' are also applicable to players with lexicographical preferences (leximin, leximax, of fixed-order lexicography). The result of Kukushkin (1992) does not cover these situations. It should be noted also that the proof of the

latter result is not completely analogous to that of Theorem 1('). Consider the following example.

**Example.** Let  $N=\{1,2\}$ ,  $X_1=X_2=\{1,2\}$ ,  $f_1(x_1)\equiv 2$ ,  $f_2(x_2)=x_2$ ,  $f_0(1,1)=1$ ,  $f_0(2,2)=2$ ,  $f_0(x_1,x_2)=0$  if  $x_1\neq x_2$ ,  $u_i(x) = \min\{f_0(x), f_1(x_1)\}$ . Thus we have the following bimatrix game:

(1,1)	(0,0)
(0,0)	(2,2)

Obviously, there are two Nash equilibria, only one of which is strong, i.e. Pareto optimal. The proof from Kukushkin (1992) selects the strong equilibrium (Leximin in the utility space), while the proof of Theorem 1 may select either of them because in the space  $\mathbb{R}^M$  they produce equivalent vectors (1,2,2).

Another (and rather peculiar) example is produced by the Pareto ordering as  $\vartheta$ : it is regular and its separable projection on any subspace is the Pareto ordering over that subspace.

**Hypothesis.** Suppose there is a finite set  $N$  and, for each  $i\in N$ , a Pareto compatible ordering  $\rho_i$  on  $\mathbb{R}^2=\mathbb{R}_0\times\mathbb{R}_1$ ; suppose also that for any collection of finite sets  $X_i$  ( $i\in N$ ) and any functions  $f_i: X_i \rightarrow \mathbb{R}$  and  $f_0: X \rightarrow \mathbb{R}$ , the game with strategy sets  $X_i$  and preference relations  $\mathcal{R}_i$  defined by (1) has a Nash equilibrium. Then there exists an ordering  $\vartheta$  on  $\mathbb{R}^M$  having separable projections  $\vartheta_i$  on all planes  $\mathbb{R}_0\times\mathbb{R}_1$  such that  $x\rho_i^0 y$  implies  $x\vartheta_i^0 y$  for all  $i\in N$ ,  $x,y\in\mathbb{R}_0\times\mathbb{R}_1$ .

**Remark.** The hypothesis is formulated as a converse to Theorem 1 rather than Theorem 1' because I have no plausible hypothesis concerning the regularity property (in fact, I even have no example of a separable ordering, say, on  $\mathbb{R}^3$  which is

not regular).

Particular cases of this hypothesis are proven in Kukushkin (1992, 1994b): necessity parts of the main theorems. If the hypothesis were proved in its full generality, then the latter result would follow directly from the Debreu-Gorman theorem; interestingly, its homogeneous version can be proved with the same technique as used by Debreu (1960). I formulate this version as a separate result here.

**Theorem 2.** Suppose there is a continuous, strictly increasing (in both arguments) function  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for any finite set  $N$ , any collection of finite sets  $X_i$  ( $i \in N$ ), and any functions  $f_i: X_i \rightarrow \mathbb{R}$  and  $f_0: X \rightarrow \mathbb{R}$ , the game with strategy sets  $X_i$  and utilities  $u_i(x) = F(f_0(x), f_i(x_i))$  has a Nash equilibrium. Then there exist continuous, strictly increasing functions  $\lambda, \mu_0, \mu: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$F(a, b) = \lambda(\mu_0(a) + \mu(b)) \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

Lemma 4 (b) from Kukushkin (1994b, Appendix), in projection to the case under consideration effectively says that if we have  $a_0 < b_0 < c_0$  and  $a_1 < b_1 < c_1$  such that  $F(a_0, b_1) = F(b_0, a_1)$  and  $F(a_0, c_1) = F(b_0, b_1) = F(c_0, a_1)$ , then  $F(b_0, c_1) = F(c_0, b_1)$ . And this is Blaschke's condition for the existence of a homeomorphic transformation of the plane preserving the lines parallel to the coordinate axes and converting the indifference curves of  $F$  into straight lines (parallel to one another) - exactly the same condition as used by Debreu (1960) for similar purposes.

It would be interesting to obtain a general formulation (with separable orderings) for the main result of Kukushkin

(1992), showing the existence of a strong equilibrium among players with minimum- or maximum-like utilities in a game with public and private objectives. Unfortunately, my thoughts on the subject are too preliminary to be written here; the above example shows some of the emerging difficulties. I can only state that no connection with the properties of similar aggregation rules studied in the literature (see e.g. Moulin 1988; Thomson and Lensberg, 1989) has been established as yet.

#### 4. Games with Decreasing Best Replies

##### 4.1. Preliminaries

We will call a function  $f$  increasing if  $x \leq y$  implies  $f(x) \leq f(y)$  and decreasing if  $x \leq y$  implies  $f(x) \geq f(y)$ .

Throughout the section, we assume that in the game under consideration each best reply correspondence has non-empty values for all  $x_{-i}$ . In fact we will study fixed points rather than Nash equilibria; the basic model may be described as follows: There is a finite set (of players)  $N$ , and for each  $i \in N$  there is a (partially) ordered set  $X_i$  and a correspondence  $R_i: X_{-i} \rightarrow X_i$ , allowing a decreasing single-valued selection  $r_i: X_{-i} \rightarrow X_i$ . A fixed point for the system of  $\langle R_i \rangle_{i \in N}$  is a point  $x^\circ \in X = \prod_{i \in N} X_i$  such that

$$x_i^\circ \in R_i(x_{-i}^\circ) \quad \text{for all } i \in N.$$

In many cases the correspondences  $R_i$  may be replaced with their (arbitrary) decreasing single-valued selections  $r_i$ , in which case the definition of a fixed point becomes this:

$$x_i^\circ = r_i(x_{-i}^\circ) \quad \text{for all } i \in N. \quad (1)$$

When one is interested in fixed points of a system of increasing mappings, Tarski's (1955) theorem gives a general existence result: if all  $X_i$ 's are complete lattices and  $r_i$ 's increasing, then (1) is always satisfied for some  $x^0$ . However, the situation with decreasing mappings is more complicated as the straightforward analogue of Tarski's theorem for such mappings is not true (consider e.g. the set  $X=\{0,1\}$  and the mapping  $r(0)=1, r(1)=0$ ). Nonetheless, Tarski's theorem could sometimes be applied to systems of decreasing mappings too.

The first result of this kind was obtained by Vives (1990): If  $\#N=2$  and each  $X_i$  is a complete lattice then a fixed point always exists. To see that, let us turn one of the sets, say  $X_2$ , upside down; now both mappings are increasing and Tarski's theorem implies the existence of a fixed point (since (1) contains no inequality, our trick with turning  $X_2$  does no harm). There is no obvious way to extend the trick (attributed by Milgrom and Roberts, 1990, to X. Vives) to  $\#N>2$ ; moreover, the corresponding statement is just not true.

**Example 1.** Let  $N=\{1,2,3\}$ ,  $X_i=\{0,1\}$  ( $i \in N$ ) and  $r_1(x_2, x_3)=1-x_2$ ,  $r_2(x_1, x_3)=1-x_3$ ,  $r_3(x_1, x_2)=1-x_1$ . It is easy to see that no  $x^0$  satisfies (1): we should have  $x_1^0=1-x_2^0=x_3^0=1-x_1^0$ .

#### 4.2. "Novshek-style" Results

The first ever result on the existence of a Nash equilibrium with decreasing best replies is apparently due to Novshek (1985). He showed (although his proof is open to criticism concerning its mathematical rigor) that decreasing best replies in the Cournot model imply the existence of an

equilibrium without any further assumptions (it should be noted that this property holds in a "great majority" of models considered in the literature). Novshek's proof was refined in Kukushkin (1994a), the main result of which paper sounds as follows:

**Theorem 1.** Suppose  $N$  is a finite set and for each  $i \in N$  there is a compact set  $X_i$  of reals and an upper hemi-continuous correspondence  $R_i: S_i \rightarrow X_i$ , where  $S_i = \sum_{j \in N \setminus \{i\}} X_j$ , allowing of a decreasing single-valued selection. Then there exists a point  $x^0 \in X = \prod_{i \in N} X_i$  such that

$$x_i^0 \in R_i(\sum_{j \neq i} x_j^0) \quad \text{for all } i \in N.$$

The key point in the proof is the lemma stating that the theorem is valid if every  $X_i$  is the set of integers satisfying the inequalities  $0 \leq x_i \leq m_i$ ; let us denote such a set  $\text{Int}[0, m_i]$ . The lemma is proved by a backward induction process, which in fact somewhat simplifies that developed by Novshek for continuous variables. Then it can easily be seen that the theorem remains true if every  $X_i$  is a subset of  $\text{Int}[0, m_i]$ ; finally, an integer approximation proves the theorem for arbitrary compact  $X_i$ 's (here the continuity of addition and assumed upper hemi-continuity of each  $R_i$  are essential).

If all  $X_i$ 's are finite, every mapping has a closed graph, so there is no need to consider correspondences (however, the continuity of addition remains essential).

**Theorem 1'.** Suppose  $N$  is a finite set and for each  $i \in N$  there is a finite set  $X_i$  of reals and a decreasing function  $r_i: S_i \rightarrow X_i$ , where  $S_i = \sum_{j \in N \setminus \{i\}} X_j$ . Then there exists a point

$x^0 \in X = \prod_{i \in N} X_i$  such that

$$x_i^0 = r_i(\sum_{j \neq i} x_j^0) \quad \text{for all } i \in N. \quad (2)$$

If the finiteness assumption in Theorem 1' is replaced with compactness (or compactness in order interval topology), the proof through approximation as in Kukushkin (1994a) becomes obviously invalid. However, I know no example disproving such a statement and even strongly suspect that in fact it is true (I had spent some time under the impression that I had a proof to the effect before recognizing its invalidity).

As was noted in Kukushkin (1994a), the difference between the situations covered by Theorems 1 and 1' and the general case lies in the "limited interdependence" property: the partners' choices enter every player's utility only through their aggregate, the sum. It is important for deeper understanding of the theorems to note that their straightforward extension to the multi-dimensional case is untrue.

**Example 2.** Consider the three mappings  $r_i: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  ( $i=1,2,3$ ):

$$r_1(s_1, s_2, s_3) = \begin{cases} (1, 0, 0), & s_2 \leq 1/2, \\ (0, 0, 0), & s_2 > 1/2, \end{cases}$$

$$r_2(s_1, s_2, s_3) = \begin{cases} (0, 1, 0), & s_3 \leq 1/2, \\ (0, 0, 0), & s_3 > 1/2, \end{cases}$$

$$r_3(s_1, s_2, s_3) = \begin{cases} (0, 0, 1), & s_1 \leq 1/2, \\ (0, 0, 0), & s_1 > 1/2. \end{cases}$$

The ranges of the mappings are  $X_1 = \{(0, 0, 0), (1, 0, 0)\}$ ,  $X_2 =$



$\{(0,0,0), (0,1,0)\}, X_3 = \{(0,0,0), (0,0,1)\}$ .

All the three functions are decreasing, but no point  $x^\circ = \langle x_1^\circ, x_2^\circ, x_3^\circ \rangle$  from the product of  $X_i$ 's satisfies (2): if e.g.  $x_1^\circ = (0,0,0)$ , we would have  $x_3^\circ = (0,0,1)$ , hence  $x_2^\circ = (0,0,0)$ , hence  $x_1^\circ = (1,0,0)$  (essentially, this example is equivalent to the previous one). No fixed point will emerge if we replace each  $X_i$  e.g. with its convex hull; the functions can be slightly modified so that they become strictly decreasing in every argument.

In fact, there is no "limited interdependence" in the example since knowing the sum of  $x_i$ 's allows one to know each of them separately.

There may be different opinions about the exact message of Example 2. I will interpret it as showing that the sum in Theorems 1 and 1' should be understood as a means to define a complete ordering on  $X_i$  rather than just an algebraic operation applicable to multi-dimensional objects. This interpretation is supported by the following simple result.

**Theorem 2.** Suppose  $N$  is a finite set and for each  $i \in N$  there is a partially ordered, finite set  $X_i$ , an increasing real-valued function  $f_i$  on  $X_i$ , and a decreasing function  $r_i: S_i \rightarrow X_i$ , where  $S_i = \sum_{j \in N \setminus \{i\}} f_j(X_j) \subseteq \mathbb{R}$ . Then there exists a point  $x^\circ \in X = \prod_{i \in N} X_i$  such that

$$x_i^\circ = r_i(\sum_{j \neq i} f_j(x_j^\circ)) \quad \text{for all } i \in N.$$

It is sufficient to apply Theorem 1' to the sets  $Y_i = f_i(X_i) \subseteq \mathbb{R}$  and the mappings  $q_i = f_i \circ r_i$ . Having a fixed point  $y^\circ$ ,  $y_i^\circ = f_i \circ r_i(\sum_{j \neq i} y_j^\circ)$ , we denote  $x_i^\circ = r_i(\sum_{j \neq i} y_j^\circ)$ ; now  $f_i(x_i^\circ) = y_i^\circ$ ,

so  $r_1(\sum_{j \neq i} f_j(x_j^\circ)) = r_1(\sum_{j \neq i} y_j^\circ) = x_i^\circ$ .

The leximax ordering turns out to be almost as good, from our current viewpoint, as the utilitarian one, defined by the sum. We will write  $x' \mathcal{L} x''$  (for  $x', x'' \in \mathbb{R}^m$ ) whenever  $x'$  dominates  $x''$  in the leximax sense or both are equivalent in the same sense. A mapping  $r: \mathbb{R}^m \rightarrow \mathbb{R}$  is called leximax decreasing if  $x' \mathcal{L} x''$  implies  $r_1(x') \leq r_1(x'')$ .

**Theorem 3.** Suppose  $N$  is a finite set and for each  $i \in N$  there is a finite set  $X_i \subseteq \mathbb{R}$  and a leximax decreasing mapping  $r_i: X_i \rightarrow \mathbb{R}$ . Then there exists a point  $x^\circ \in X = \prod_{i \in N} X_i$  such that

$$x_i^\circ = r_i(x_{-i}^\circ) \quad \text{for all } i \in N. \quad (1)$$

Denoting  $\mathbb{N}$  the set of all non-negative integers, we define  $f: \mathbb{N} \rightarrow \mathbb{N}$  by  $f(k) = (n-1)^k - 1$ , where  $n = \#N$ ,  $k \in \mathbb{N}$ , and  $g: \cup_{i \in N} X_i \rightarrow \mathbb{N}$  as the rank function ( $g(x) = 0$  for the smallest of all  $x_i$  (all  $i \in N$ ),  $g(x) = 1$  for the second smallest, etc.). The following equivalence holds for all  $x', x'' \in \mathbb{N}^{n-1}$  because  $f(k) > (n-1) \cdot f(k-1)$  for any  $k \geq 1$ :

$$\sum_{j=1}^{n-1} f(x_j') \geq \sum_{j=1}^{n-1} f(x_j'') \quad \text{if and only if } x' \mathcal{L} x''. \quad (3)$$

For each  $i \in N$  we define  $Y_i = f \circ g(X_i) \subseteq \mathbb{N}$  and  $S_i = \sum_{j \neq i} Y_j (\subseteq \mathbb{N})$ ; since  $r_i$  is leximax decreasing, it easily follows from (3) that there exists a decreasing function  $h_i: S_i \rightarrow X_i$  such that

$$r_i(x_{-i}) = h_i(\sum_{j \neq i} (f \circ g(x_j))) \quad \text{for all } x_{-i} \in X_{-i}. \quad (4)$$

Now it follows from Theorem 1' that the system of sets  $Y_i$  and functions  $q_i = f \circ g \circ h_i: S_i \rightarrow Y_i$  has a fixed point  $y^\circ$  such that  $y_i^\circ = q_i(\sum_{j \neq i} y_j^\circ)$  ( $i \in N$ ). Denoting  $x_i^\circ = g^{-1} \circ f^{-1}(y_i^\circ)$  for each  $i \in N$ , we have  $f \circ g(x_i^\circ) = f \circ g \circ h_i(\sum_{j \neq i} f \circ g(x_j^\circ))$ . Since both  $f$  and

$g$  are monomorphisms, from (4) we obtain  $x_1^0 = r_1(x_{-1}^0)$ .

**Remark 1.** A particular case of leximax decreasing function is a function depending only on  $\max_{j \in N \setminus \{i\}} x_j$ . Since the maximum function is continuous, the complete analogue of Theorem 1 for this case is valid. Interestingly, I have another proof of this statement, based directly on Tarski's theorem rather than on Novshek's induction process, but I can see no point in actually writing it down here. For general leximax decreasing functions, the validity of a similar statement (even if the closed graph is assumed) remains unclear.

**Remark 2.** By replacing each  $X_i$  with  $-X_i$ , we obtain the exact analogue of Theorem 3 for mappings decreasing w.r.t. leximin ordering and the analogue of Theorem 1 for mappings depending on the minimum (as in the above remark).

**Remark 3.** The extensions of Theorem 3 described in previous remarks are obviously applicable to games with utilities of the form  $u_i(x_i, \max_{j \neq i} x_j)$  or of the form  $u_i(x_i, \min_{j \neq i} x_j)$  (provided the best replies exist, have closed graphs, and are decreasing). Such utilities really emerge in some public good models, see e.g. Hirshleifer (1983).

Quite a similar approach works for the function  $F^0$  considered in Kukushkin (1994b):

$$F^0(\langle x_i \rangle_{i \in N}) = \begin{cases} \prod x_i, & \text{if all } x_i > 0, \\ \min x_i, & \text{otherwise} \end{cases}$$

(note that the definition is equally meaningful for any number of arguments).

**Theorem 4.** Suppose  $N$  is a finite set and for each  $i \in N$  there is a compact set  $X_i$  of reals and an upper hemi-continuous correspondence  $R_i: S_i \rightarrow X_i$ , where  $S_i = F^0(\prod_{j \in N \setminus \{i\}} X_j)$ , allowing of a decreasing single-valued selection. Then there exists a point  $x^0 \in X = \prod_{i \in N} X_i$  such that

$$x_i^0 \in R_i(F^0(\langle x_j^0 \rangle_{j \neq i})) \quad \text{for all } i \in N.$$

The proof follows the same lines as in Theorem 3 and in Remark 1 after it. First we consider finite models where positive  $x_i$ 's form a geometric progression (with the same multiplier for all  $i \in N$ ) while negative  $x_i$ 's may be assumed integer. Every such model is isomorphic to a model satisfying the assumptions of Theorem 1': the isomorphism is established by applying a logarithmic function to positive  $x_i$ 's and a function similar to  $f(\cdot)$  from the proof of Theorem 3 to negative  $x_i$ 's; therefore, it has a fixed point. Every model with compact  $X_i$ 's can be approximated with such models just as in Proposition 3 of Kukushkin (1994a).

Certainly, analogues of Theorem 2 for dependencies considered in Theorems 3 and 4 (and in the remarks following the former) are also valid; in the following, I will restrict myself to the case  $X_i \subseteq \mathbb{R}$  without explicitly mentioning the possibility of such an extension every time.

#### 4.3. On Sufficiency in General

Now let us consider the question: what unites all the results of the previous subsection? In my view, it is the separability property.

**Hypothesis 1.** Suppose there is a finite set  $N$  and, for each  $i \in N$ , there is a finite set  $X_i$  of reals and a mapping  $r_i: X_{-i} \rightarrow X_i$ ; suppose also that there is a complete, Pareto compatible ordering  $\vartheta$  on  $X (\subseteq \mathbb{R}^N)$ , having separable projections on all sets  $X_i$ ; suppose finally that each  $r_i$  is decreasing w.r.t. the projection  $\vartheta_{-i}$  (i.e.  $x' \vartheta_{-i} x''$  implies  $r_i(x') \leq r_i(x'')$ ). Then there exists a fixed point satisfying (1).

**Remark.** The assumption that  $\vartheta$  is complete is essential: otherwise Pareto dominance would be acceptable and the hypothesis would be disproved by Example 1 of this section (Subsection 4.1). Thus we see a technical difference with Theorem 1 of Section 3.

It is easy to see that Theorems 1', 3, and 4 (reduced to finite sets) are indeed particular cases of this statement. Unfortunately, I am unable to prove Hypothesis 1 in its full generality. The main difficulty lies in my inability to describe all separable orderings on finite sets. As a kind of inductive support for the hypothesis, I end up this subsection with a couple of examples not covered by Novshek's scheme. For simplicity, we will restrict ourselves to the case  $\#N=3$ .

**Theorem 5.** Suppose there are given three sets  $X_i \subseteq \mathbb{R}$  ( $i=1,2,3$ ) compact in the order interval topology; suppose there are three functions  $r_i: X_{-i} \rightarrow X_i$  such that  $r_3(x_1, x_2)$  is decreasing in both arguments (not necessarily strictly),  $r_1(x_2, x_3)$  is lexicographically decreasing in the sense that  $r_1(x_2', x_3') \leq r_1(x_2'', x_3'')$  if  $x_2' > x_2''$ , or  $x_2' = x_2''$  and  $x_3' > x_3''$ , and  $r_2(x_1, x_3)$  is lexicographically decreasing in the same sense (first  $x_1$  matters and only then  $x_3$ ). Then there exists a fixed

point satisfying

$$x_i^0 = r_i(x_{-i}^0) \quad \text{for all } i \in N. \quad (1)$$

Given  $x_2$ , we have two decreasing mappings,  $r_1(x_2, \cdot)$  and  $r_3(\cdot, x_2)$ , between  $X_1$  and  $X_3$ . By Tarski's theorem (with turning one of the sets upside down), there exists a fixed point  $x_1 = q_1(x_2)$ ,  $x_3 = q_3^{(2)}(x_2)$  such that  $q_1(x_2) = r_1(x_2, q_3^{(2)}(x_2))$ ,  $q_3^{(2)}(x_2) = r_3(q_1(x_2), x_2)$ . Now  $x_2' < x_2''$  implies  $r_1(x_2', x_3') \geq r_1(x_2'', x_3'')$  for any  $x_3', x_3''$ ; therefore,  $q_1(\cdot)$  is decreasing. Similarly, there exist (for each  $x_1 \in X_1$ )  $q_2(x_1) \in X_2$  and  $q_3^{(1)}(x_1)$  such that  $q_2(x_1) = r_2(x_1, q_3^{(1)}(x_1))$  and  $q_3^{(1)}(x_1) = r_3(x_1, q_2(x_1))$ ;  $q_2(\cdot)$  is also decreasing. Applying Tarski's theorem to the pair  $q_1(\cdot)$ ,  $q_2(\cdot)$ , we obtain  $x_1^0$ ,  $x_2^0$  such that  $x_1^0 = q_1(x_2^0)$  and  $x_2^0 = q_2(x_1^0)$ . Define  $x_3^0 = q_3^{(1)}(x_1^0) = r_3(x_1^0, x_2^0) = q_3^{(2)}(x_2^0)$ . Obviously,  $x_1^0$ ,  $x_2^0$ ,  $x_3^0$  constitute the fixed point needed.

**Theorem 6.** All the assumptions are the same as in Theorem 5 with the following exception:  $r_1(\cdot, \cdot)$  and  $r_2(\cdot, \cdot)$  are lexicographically decreasing in the reversed sense, i.e.  $r_1$  reacts to  $x_3$  first and then to  $x_2$ , and  $r_2$  reacts to  $x_3$  and then to  $x_1$ . (There seems to be no formal equivalence between the two situations). Then there exists a fixed point satisfying (1).

For each  $x_3$ , there exist  $q_1(x_3) \in X_1$  and  $q_2(x_3) \in X_2$  such that  $q_1(x_3) = r_1(q_2(x_3), x_3)$  and  $q_2(x_3) = r_2(q_1(x_3), x_3)$  (by Tarski's theorem applied to a duopoly with decreasing best replies). Since  $x_3' < x_3''$  implies  $r_1(x_2', x_3') \geq r_1(x_2'', x_3'')$  for all  $x_2', x_2'' \in X_2$ ,  $q_1(\cdot)$  is decreasing; similarly,  $q_2(\cdot)$  is decreasing. Now we have two decreasing mappings  $r_3: X_{-3} \rightarrow X_3$  and  $q_1 \times q_2: X_3 \rightarrow X_{-3}$ ; Tarski's theorem implies the existence of  $x^0$

such that  $x_3^0 = r_3(x_1^0, x_2^0)$ ,  $x_1^0 = q_1(x_3^0)$ , and  $x_2^0 = q_2(x_3^0)$ ; obviously, this is just what we need.

To connect Theorems 5 and 6 with Hypothesis 1, appropriate separable orderings should be produced. In the case of Theorem 5, I suggest lexicographical maximization of the following functions (listed in the order of decreasing importance):  $\langle -r_3(x_1, x_2), x_1, x_2, x_3 \rangle$ . For Theorem 6, the same functions should be lexicographically maximized, but in a different order of importance:  $\langle x_3, -r_3(x_1, x_2), x_1, x_2 \rangle$ . (Were the function  $r_3(\cdot, \cdot)$  supposed strictly decreasing, the terms  $x_1$  and  $x_2$  could be omitted in both cases).

**Theorem 7.** All the assumptions are the same as in Theorem 5 with the following exception: there exists a level  $x_3^*$  such that, whenever  $x_3 \geq x_3^*$ ,  $r_1(x_2, x_3)$  and  $r_2(x_1, x_3)$  are lexicographically decreasing in the sense of Theorem 6, while for  $x_3 < x_3^*$ , they are decreasing in the sense of Theorem 5 (to preserve monotonicity, we also require  $r_1(x_j', x_3') \geq r_1(x_j'', x_3'')$ , whenever  $x_3' < x_3^* \leq x_3''$  for all  $i, j = 1, 2, i \neq j, x_j', x_j'' \in X_j$ ). Then there exists a fixed point satisfying (1).

Denote  $x_1^* = r_1(\min X_2, x_3^*)$ ,  $x_2^* = r_2(\min X_1, x_3^*)$ ; by the assumptions,  $x_3 \geq x_3^*$  implies  $r_1(x_j, x_3) \leq x_1^*$ , and  $x_3 < x_3^*$  implies  $r_1(x_j, x_3) \geq x_1^*$  (for all  $i, j \in \{1, 2\}, i \neq j, x_j \in X_j$ ).

Suppose  $x_3^* \leq r_3(x_1^*, x_2^*)$  and denote  $X_3^* = \{x_3 \mid x_3 \geq x_3^*\}$ ,  $X_1^* = \{x_1 \mid x_1 \leq x_1^*\}$  ( $i=1, 2$ ). It can easily be checked that  $r_3$  maps  $X_1^* \times X_2^*$  into  $X_3^*$ , while each  $r_i$  ( $i=1, 2$ ) maps  $X_j^* \times X_3^*$  into  $X_i^*$ . Now for  $r_i$ 's ( $i \in N$ ) restricted to  $X_i^*$ 's the assumptions of Theorem 6 are fulfilled, hence the existence of a fixed point.

Now suppose  $x_3^* > r_3(x_1^*, x_2^*) = x_3^{**}$  and denote  $X_3^* = \{x_3 \mid$

$x_3 \leq x_3^{**}$ ),  $X_i^* = \{x_i \mid x_i \geq x_i^*\}$  ( $i=1,2$ ); again each  $r_i$  maps  $X_{-i}^*$  into  $X_i$  ( $i \in N$ ). This time for  $r_i$ 's ( $i \in N$ ) restricted to  $X_i^*$ 's, the assumptions of Theorem 5 are fulfilled; the theorem is proved.

A similar treatment can be given to the reversed case where (for  $i=1,2$ ) the ordering from Theorem 5 is used for big  $x_3$ 's and that from Theorem 6 for small ones. The underlying separable orderings on  $\mathbb{R}^3$  are constructed in a straightforward way.

Most likely, the "right" proof of Hypothesis 1 should consist in applying Tarski's theorem to auxiliary mappings (somewhat similar to the proofs of Theorems 5 - 7 but in a more complicated way), in which case Novshek's scheme, with all its mysterious beauty, will become superfluous; alas, the progress of knowledge often demands sacrifices! On the other hand, it is also possible that some orderings  $\vartheta$  may be treated with Tarski's theorem, while others really require Novshek's scheme (Remark 1 after Theorem 3 hints that some overlapping is possible). In the latter case the problem of how to relax the closed graph assumption in Theorem 1 (and in its analogues) becomes quite intriguing.

#### 4.4. On Necessity

**Hypothesis 2.** Suppose there is a finite set  $N$  and, for each  $i \in N$ , a Pareto compatible ordering  $\eta_i$  on  $\mathbb{R}^{N \setminus \{i\}}$  such that a fixed point (1) exists for any collection of finite sets  $X_i$  of reals and mappings  $r_i: X_{-i} \rightarrow X_i$  whenever each  $r_i$  is decreasing w.r.t. ordering  $\eta_i$ . Then there must exist a complete ordering  $\vartheta$



on  $\mathbb{R}^N$  having separable projection  $\vartheta_{-i}$  on each  $\mathbb{R}^{N \setminus \{i\}}$  such that  $x \vartheta_{-i} y$  implies  $x \eta_i y$  for all  $i \in N$ ,  $x, y \in \mathbb{R}^{N \setminus \{i\}}$ .

The statement is trivially true for  $\#N=2$ . I can only prove it for  $\#N=3$  under stronger assumptions on  $\eta_i$ 's.

**Theorem 8.** Suppose  $\#N=3$  and there are three continuous, strictly increasing functions  $s_i: \mathbb{R}^{N \setminus \{i\}} \rightarrow \mathbb{R}$  having the following property: For any finite sets  $X_i \subseteq \mathbb{R}$  and any mappings  $r_i: X_{-i} \rightarrow X_i$  such that  $r_i = q_i \circ s_i$ , where  $q_i: \mathbb{R} \rightarrow \mathbb{R}$  is decreasing, there exists a fixed point satisfying (1). Then there exist strictly increasing continuous functions  $\lambda_i, \mu_i: \mathbb{R} \rightarrow \mathbb{R}$  ( $i \in N$ ) such that  $s_i(x_j, x_k) = \lambda_i(\mu_j(x_j) + \mu_k(x_k))$  (here and later on we adopt the convention  $i, j, k \in N$ ,  $i \neq j \neq k \neq i$ ; the arguments of the functions  $s_i$  will always be accompanied with subscripts, so we may not bother about their correct order).

The proof is somewhat similar to that of Kukushkin (1994b, Appendix), but the current situation is much easier to handle (not only because we assumed  $n=3$ ). We will consider three copies of the real line,  $\mathbb{R}_i$  ( $i \in N$ ) simultaneously; we say that a closed interval  $I_i = [a_i, b_i] \subseteq \mathbb{R}_i$  matches an interval  $I_j = [a_j, b_j] \subseteq \mathbb{R}_j$  iff  $s_k(a_i, b_j) = s_k(b_i, a_j)$ , and denote this  $I_i \approx I_j$ .

**Lemma 1.** If  $I_i \approx I_j$  and  $I_j \approx I_k$ , then  $I_i \approx I_k$ .

Suppose the contrary; let, for example,

$$s_j(a_i, b_k) > s_j(b_i, a_k). \quad (5)$$

Taking into account the monotonicity and continuity, we may slightly decrease first  $b_k$  and then  $b_j$  so that (5) continue to hold and besides

$$s_i(a_j, b_k) < s_i(b_j, a_k).$$

$$s_k(a_i, b_j) < s_k(b_i, a_j).$$

Define  $X_i = \{a_i, b_i\}$ ,  $X_j = \{a_j, b_j\}$ ,  $X_k = \{a_k, b_k\}$ ,

$$q_i(t) = \begin{cases} b_i, & \text{if } t \leq s_i(a_j, b_k), \\ a_i, & \text{otherwise,} \end{cases}$$

$$q_j(t) = \begin{cases} b_j, & \text{if } t \leq s_j(a_k, b_i), \\ a_j, & \text{otherwise,} \end{cases}$$

$$q_k(t) = \begin{cases} b_k, & \text{if } t \leq s_k(a_i, b_j), \\ a_k, & \text{otherwise;} \end{cases}$$

then we have

$$r_i(x_j, x_k) = q_i \circ s_i(x_j, x_k) = \begin{cases} b_i, & x_j = a_j, \\ a_i, & x_j = b_j, \end{cases}$$

$$r_j(x_i, x_k) = \begin{cases} b_j, & x_k = a_k, \\ a_j, & x_k = b_k, \end{cases}$$

$$r_k(x_i, x_j) = \begin{cases} b_k, & x_i = a_i, \\ a_k, & x_i = b_i. \end{cases}$$

It is easy to see that the situation is essentially equivalent to that of Example 1 and has no fixed point for the same reason. The lemma is proved.

We call an interval  $I_i$  short if there exist  $I_j, I_k$  such that  $I_j \approx I_i \approx I_k$ ; short intervals obviously exist. If  $I_1', I_1'' \subset \mathbb{R}_1$  are short, we say that  $I_1' \approx I_1''$  if there exists  $I_j$  ( $j \neq i$ ) such that  $I_1' \approx I_j \approx I_1''$ .

**Lemma 2.** The relation  $\approx$  is an equivalence on the set of all short intervals (from all the three axes put together).

The only thing worth proving is transitivity. If the three intervals belong to different axes, Lemma 1 immediately applies. Suppose  $I_1' \approx I_1'' \approx I_1'''$ ; we have to show  $I_1' \approx I_1'''$ . By definition, we have

$$I_1' \approx I_j \approx I_1'' \text{ and } I_1'' \approx I_k \approx I_1'''; \quad (6)$$

note that  $j$  and  $k$  here may coincide, in which case we have to move a bit longer. Let us start with this case:  $I_1' \approx I_j \approx I_1''$  and  $I_1'' \approx I_j \approx I_1'''$ . Since  $I_j''$  is short, there must exist  $I_k$  (this time certainly  $i \neq j \neq k \neq i!$ ) such that  $I_j'' \approx I_k$ ; by Lemma 1 we have  $I_1''' \approx I_k$  and so we return to (6) with  $j \neq k$ . Now Lemma 1 implies  $I_j \approx I_k$ , hence  $I_j \approx I_1'''$ , hence  $I_1' \approx I_1'''$ . The proof for the case when two intervals belong to the same axis and the third to a different one is even simple.

The rest of the proof literally follows the proof from the Appendix of Kukushkin (1994b) with several simplifications: There is no need to distinguish between  $\psi$  and  $\varphi_i$ ; all  $i \in \mathbb{N}$  are perfectly symmetric. The analogue of Lemma 1 (of Kukushkin, 1994b) remains obviously valid; Lemmas 3 and 4 transform into an obvious statement of the existence of a unique middle for every short interval ( $c_i$  is a middle for  $[a_i, b_i]$  if  $[a_i, c_i] \approx [c_i, b_i]$ ); Lemma 5 becomes completely superfluous. The main induction process of the proof goes virtually the same way.

As was noted in the previous section, this approach can be interpreted as an extension of Blaschke's theorem to a bunch of planes and is thus comparable with that of Debreu (1960). Most likely, the necessity to assume  $\#N=3$  in Theorem 8 follows from the fact that my technique is essentially two-dimensional (though it can be applied to objects of a more complex geometrical nature than a bunch of planes, see the next section). It should be noted that Debreu's problem was quite multi-dimensional but he used Blaschke's theorem only to start an induction process; I have been unable to do anything similar in

the current context. I am also unable to apply in this case Gorman's technique, which seems more powerful.

### 5. A Version of the Debreu-Gorman Theorem

The following result is, in fact, an intermediate version of the theorem: stronger than that of Debreu (1960) but apparently weaker than that of Gorman (1968a). As noted in the Introduction, it is related here since it shows the limits to which the approach of Kukushkin (1994b) can be stretched.

**Theorem 1.** Suppose  $N$  is a finite set and  $\vartheta$  is an ordering on  $\mathbb{R}^N$  defined by a continuous, strictly increasing in each argument, function  $F$  (thus  $\vartheta$  is Pareto compatible). Let  $G$  be a graph with vertices  $i \in N$  such that  $i$  and  $j$  are connected with an edge if and only if  $\vartheta$  has a separable projection on  $\mathbb{R}_i \times \mathbb{R}_j$ . Then the equality:

$$F(\langle x_i \rangle_{i \in N}) = \lambda(\sum_{i \in N} (\mu_i(x_i)))$$

(for some continuous and strictly increasing functions  $\lambda, \mu_i: \mathbb{R} \rightarrow \mathbb{R}$ ) holds if and only if  $G$  is connected.

**Remark 1.** A resemblance with the formulation of the main result of Gorman (1968b) is obvious; I cannot explain the fact quite convincingly.

**Remark 2.** The theorem implies that if  $G$  is connected, it is complete.

The necessity is quite obvious. I start with a three-dimensional version of the sufficiency part of the result, which requires less cumbersome notations.

**Theorem 2.** Let  $N = \{1, 2, 3\}$ ; suppose that an ordering  $\vartheta$  on  $\mathbb{R}^N$  is defined by a continuous, strictly increasing in each

argument, function  $F$  and has separable projections on  $\mathbb{R}_1 \times \mathbb{R}_2$  and  $\mathbb{R}_2 \times \mathbb{R}_3$ . Then the equality:

$$F(x_1, x_2, x_3) = \lambda(\mu_1(x) + \mu_2(x) + \mu_3(x))$$

holds, for some appropriate continuous and strictly increasing functions  $\lambda, \mu_i: \mathbb{R} \rightarrow \mathbb{R}$ , for all  $(x_1, x_2, x_3) \in \mathbb{R}^N$ .

The proof goes along the same lines as in Kukushkin (1994b). For  $I_1 = [a_1, b_1] \subseteq \mathbb{R}_1$  and  $I_2 = [a_2, b_2] \subseteq \mathbb{R}_2$ , we say that  $I_1 \approx I_2$  if  $F(a_1, b_2, x_3) = F(b_1, a_2, x_3)$  ( $x_3$  is unessential, according to our assumptions); similarly is defined  $I_2 \approx I_3$ . For  $I_1$  and  $I_3$ , we will only use the equivalence relation  $I_1 \approx (x_2)I_3$ , given  $x_2 \in \mathbb{R}_2$ , meaning  $F(a_1, x_2, b_3) = F(b_1, x_2, a_3)$ .

Lemma 1. If  $I_1 = [a_1, b_1] \approx I_2 = [a_2, b_2]$ , and  $I_2 \approx I_3 = [a_3, b_3]$ , then  $I_1 \approx (a_2)I_3$  and  $I_1 \approx (b_2)I_3$ .

Indeed,  $I_1 \approx I_2$  implies  $F(b_1, a_2, a_3) = F(a_1, b_2, a_3)$ , while  $I_2 \approx I_3$  implies  $F(a_1, b_2, a_3) = F(a_1, a_2, b_3)$ , which means just  $I_1 \approx (a_2)I_3$ . To prove the second statement, we write  $I_1 \approx I_2$  as  $F(a_1, b_2, b_3) = F(b_1, a_2, b_3)$ , and  $I_2 \approx I_3$  as  $F(b_1, a_2, b_3) = F(b_1, b_2, a_3)$ .

Lemma 2. If  $I_1 \approx I_2$  and  $\{I_1 \approx (a_2)I_3$  or  $I_1 \approx (b_2)I_3\}$ , then  $I_2 \approx I_3$ .

$I_1 \approx I_2$  implies  $F(a_1, b_2, a_3) = F(b_1, a_2, a_3)$ , while  $I_1 \approx (a_2)I_3$  implies  $F(b_1, a_2, a_3) = F(a_1, a_2, b_3)$ . The second statement is proved quite similarly.

Now we choose  $I_1^\circ \approx I_2^\circ \approx I_3^\circ$  and denote their ends  $\mu_1^{-1}(0)$ ,  $\mu_1^{-1}(1)$  ( $i \in N$ ). Then the usual process takes place: if there exists  $x_2 \in \mathbb{R}_2$  such that  $I_1^\circ \approx [\mu_2^{-1}(1), x_2]$ , we define  $\mu_2^{-1}(2) = x_2$ ; similarly are defined  $\mu_2^{-1}(\pm n)$  for natural  $n$ . Finally, we define  $\mu_1^{-1}(\pm n)$  and  $\mu_3^{-1}(\pm n)$ , using  $I_2^\circ$  as a yardstick.

Lemma 3. For both  $i=1,3$  and all integers  $n$  and  $m$ ,  $[\mu_1^{-1}(n), \mu_1^{-1}(n+1)] \approx [\mu_2^{-1}(m), \mu_2^{-1}(m+1)]$ , provided both sides

are well defined. Similarly,  $[\mu_1^{-1}(n), \mu_1^{-1}(n+1)] \approx (\mu_2^{-1}(k))$   
 $[\mu_3^{-1}(m), \mu_3^{-1}(m+1)]$  for all integers  $n, m$ , and  $k$ .

The proof goes by induction, using Lemmas 1 and 2. I demonstrate just the first step:  $I_1^0 \approx [\mu_2^{-1}(1), \mu_2^{-1}(2)]$  by definition,  $I_1^0 \approx (\mu_2^{-1}(1))I_3^0$  by Lemma 1; therefore, by Lemma 2,  $[\mu_2^{-1}(1), \mu_2^{-1}(2)] \approx I_3^0$  and so on.

Lemma 4. If both sides are well defined, then

$$F(\mu_1^{-1}(m_1), \mu_2^{-1}(m_2), \mu_3^{-1}(m_3)) = F(\mu_1^{-1}(n_1), \mu_2^{-1}(n_2), \mu_3^{-1}(n_3))$$

if and only if  $m_1+m_2+m_3 = n_1+n_2+n_3$ .

Is derived from Lemma 3 by induction.

To complete the process of defining  $\mu_1^{-1}(\cdot)$ , we only need the existence of the middles for short intervals on the axes.

Lemma 5. For each  $I_1=[a_1, b_1]$  such that  $I_1 \approx I_2 \approx I_3$ , there exist a unique collection of  $c_i \in I_1$  such that:

- |                                       |                                       |
|---------------------------------------|---------------------------------------|
| (1) $[a_1, c_1] \approx [a_2, c_2]$ , | (5) $[a_3, c_3] \approx [a_2, c_2]$ , |
| (2) $[c_1, b_1] \approx [a_2, c_2]$ , | (6) $[c_3, b_3] \approx [a_2, c_2]$ , |
| (3) $[a_1, c_1] \approx [c_2, b_2]$ , | (7) $[a_3, c_3] \approx [c_2, b_2]$ , |
| (4) $[c_1, b_1] \approx [c_2, b_2]$ , | (8) $[c_3, b_3] \approx [c_2, b_2]$ . |

First of all, choose  $c_1$  and  $c_2$  so that (1) - (3) be satisfied; this is quite similar to the proof of Lemma 3 in Kukushkin (1994b): for each  $x_1 \in I_1$  we define  $e(x_1)$  by the equivalence  $[a_1, x_1] \approx [a_2, e(x_1)]$ , find a solution for the equation  $F(x_1, e(x_1), a_3) = F(a_1, b_2, a_3)$  and denote  $c_2 = e(c_1)$ . Then we choose  $c_3$  so that (5) be satisfied. Now, (1), (5), and Lemma 1 imply  $[a_1, c_1] \approx (c_2) [a_3, c_3]$ , and this with (3) and Lemma 2 imply (7), hence (6) too. Further, (2), (5), and Lemma 1 imply  $[c_1, b_1] \approx (c_2) [a_3, c_3]$ , and this together with (7) and Lemma 2 produce (4). Finally, (1), (6), and Lemma 1 imply  $[a_1, c_1] \approx (c_2)$

$[c_3, b_3]$ , hence (using (3) and Lemma 2), we have (8).

It should be noted that Lemmas 5 and 1 immediately imply that  $I_1' \approx (x_2)I_3'$ , where one may substitute either  $[a_1, c_1]$  or  $[c_1, b_1]$  for  $I_1'$ , either  $[a_3, c_3]$  or  $[c_3, b_3]$  for  $I_3'$  and any of  $a_2, c_2, \text{ or } b_2$  for  $x_2$ .

Now Lemma 5 allows us to define  $\mu_1^{-1}(1/2)$  and, using Lemmas 1 and 2, extend each  $\mu_1^{-1}(\cdot)$  to all the "halves"; then we proceed to quarters, etc. without any difference with Kukushkin (1994b). Theorem 2 may be regarded as proven.

Turning to Theorem 1, we first delete some edges in  $G$  so that the remaining graph constitute a tree (the connectedness assumption allows us to do so). The simple equivalence relation  $I_i \approx I_j$ , meaning  $F(a_i, b_j, x_{-ij}) = F(b_i, a_j, x_{-ij})$  for all  $x_{-ij}$ , is well defined for  $(i, j) \in G$ ; in the general case, we will consider the relation  $I_i \approx (x_K)I_j$ , where  $K \subseteq N \setminus \{i, j\}$  and  $x_K \in X^K$  are given, meaning  $F(a_i, b_j, x_K, x_{-i,j,K}) = F(b_i, a_j, x_K, x_{-i,j,K})$  for all  $x_{-i,j,K}$  (in fact, each pair  $i, j$  requires just one set  $K$  to be considered but it depends on the choice of the initial vertex).

**Lemma 1.** If  $(i, j) \in G$ ,  $(j, k) \in G$ , and  $I_i \approx I_j \approx I_k$  ( $I_j = [a_j, b_j]$ ), then  $I_i \approx (a_j)I_k$  and  $I_i \approx (b_j)I_k$ .

**Lemma 2.** If  $(i, j) \in G$ ,  $I_i \approx I_j'$ , and  $I_j \approx (x_K)I_k$ , then  $I_i \approx (\langle x_K, a_j \rangle)I_k$  and  $I_i \approx (\langle x_K, b_j \rangle)I_k$ .

Both proofs are quite similar to those of Lemmas 1 and 2 from Theorem 2.

To start the induction process, we need a collection of  $I_i^\circ$ ,  $i \in N$ , such that  $I_i^\circ \approx I_j^\circ$  whenever  $(i, j) \in G$ ; such a collection really exists: we may go along the tree "backwards", from the terminal vertices, finding, for each  $i \in N$ , an interval  $I_i$  that

matches some intervals for all succeeding vertices. Lemmas 1 and 2 allow us to define points  $\mu_1^{-1}(m)$  for  $i \in N$ ,  $m$  integer such that  $F(\langle \mu_1^{-1}(m_1) \rangle_{i \in N}) = F(\langle \mu_1^{-1}(n_1) \rangle_{i \in N})$  if and only if  $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$  (provided both sides are well defined).

Then we prove the exact analogue of Lemma 5 from Theorem 2 and finish the proof in the same way (each induction process should start at the initial vertex and go along the tree).

Gorman (1968a) derived additive separability from Aczel's (1966) theorem on associative functions. Hopefully, there should exist a purely geometrical interpretation of Aczel's results - cf. Artin (1957) where e.g. the validity of Pappus' diagram is shown to be equivalent to the commutative law in the underlying field (p. 73-74); such an interpretation would finally clarify the relationship between Debreu's and Gorman's approaches.

On the other hand, to advance towards proving the above hypothesis in full generality, one has to abandon the continuity assumption, thus leaving a very small chance that even Gorman's technique could be of much help. Note that the ordering underlying the main result of Kukushkin (1992) has separable projections on each  $\mathbb{R}_0 \times \mathbb{R}_1$  ( $i \in N$ ) but not on  $\mathbb{R}_1 \times \mathbb{R}_j$ , in sharp contrast with the principal feature of Gorman's (1968a) result, captured in Theorems 1 and 2 above. The usual argument that everything is continuous in economics is not convincing in our context: the minimum- or maximum-like functions are continuous, but their relations with the separability property can only be established through leximin or leximax orderings, which are utterly discontinuous.



## References

Aczel J. (1966) Lectures on Functional Equations and their Applications. Academic Press, New York.

Artin E. (1957) Geometric Algebra. Interscience Publishers, New York.

d'Aspremont C. and L. Gevers (1977) Equity and the Informational Basis of Collective Choice. Review Econ. Stud. 44, 199-209.

Aumann R.J. (1962) Utility Theory without the Completeness Axiom. Econometrica 30, 445-462.

Aumann R.J. (1964) Utility Theory without the Completeness Axiom: A Correction. Econometrica 32, 210-212.

Bulow J.I., J.D. Geanakoplos, and P.D. Klemperer (1985) Multimarket Oligopoly: Strategic Substitutes and Complements. J. Polit. Economy 93, 488-511.

Debreu G. (1960) Topological Methods in Cardinal Utility. In: Mathematical Methods in Social Sciences. Stanford University Press. P.16-26.

Deschamps R. and L. Gevers (1978) Leximin and Utilitarian Rules: A Joint Characterization. J. Econ. Theory 17, 143-163.

Fishburn P.C. (1970) Utility Theory for Decision Making. J. Wiley & Sons, New York.

Fleming M. (1952) A Cardinal Concept of Welfare. Quart. J. Econ. 46, 366-384.

Fudenberg D. and J. Tirole (1984) The Fat-Cat Effect, The Puppy-Dog Ploy and the Lean and Hungry Look. American Economic Review 74, 361-366.

Germeier Yu.B., I.A. Vatel' (1974) On Games with a Hierarchical Vector of Interests. *Izv. Akad. Nauk SSSR. Tekhn. Kibernet.*, N.3, P.54-69 (in Russian; English transl. in *Engrg. Cybernetics*, V.12).

Gorman W.M. (1968a) The Structure of Utility Functions. *Rev. Econ. Studies* 35, 367-390.

Gorman W.M. (1968b) Conditions for Additive Separability. *Econometrica* 36, 605-609.

Gorman W.M. (1971a) Apologia for a Lemma. *Rev. Econ. Studies* 38, 114.

Gorman W.M. (1971b) Clontarf Revisited. *Rev. Econ. Studies* 38, 116.

Harsanyi J.C. (1955) Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. *J. Polit. Econ.* 63, 309-321.

Hirshleifer J. (1983) From Weakest-Link to Best-Shot: The Voluntary Provision of Public Goods. *Public Choice* 41, 371-386.

Kukushkin N.S. (1992) On Existence of Stable and Efficient Outcomes in Games with Public and Private Objectives. *Int. J. Game Theory*, V.20, N.3, P.295-303.

Kukushkin N.S. (1994a) A Fixed-Point Theorem for Decreasing Mappings. *Economics Letters* 46, 23-26.

Kukushkin N.S. (1994b) A Condition for the Existence of a Nash Equilibrium in Games with Public and Private Objectives. *Games Econ. Behavior* 7, 177-192.

Milgrom P. and J. Roberts (1990) Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* 58, 1255-1277.

Milgrom P. and J. Roberts (1994) Comparing Equilibria. American Economic Review 84, 441-459.

Milgrom P. and C. Shannon (1994) Monotone Comparative Statics. Econometrica 62, 157-180.

Moulin H. (1988) Axiomatic of Cooperative Decision Making. Cambridge University Press, Cambridge - New York - Sydney.

Novshek W. (1985) On the Existence of Cournot Equilibrium. Rev. Econ. Studies 52, 85-98.

Rozen V.V. (1976) Mixed Extensions of Games with Ordered Outcomes. Zh. Vychisl. Mat. i Mat. Fiz. 16, 1436-1450 (in Russian; English transl. in USSR Comput. Math. and Math. Phys. V.16, N6 52-67).

Tarski A. (1955) A Lattice-Theoretical Fixpoint Theorem and its applications. Pacific Journal of Mathematics 5, 285-309.

Thomson W. and Lensberg T. (1989) Axiomatic Theory of Bargaining with a Variable Number of Agents. Cambridge University Press, Cambridge - New York - Sydney.

Topkis D.M. (1979) Equilibrium Points in Nonzero-Sum  $n$ -Person Submodular Games. SIAM J. Contr. Optimization 17, 773-787.

Vind K. (1971a) Note on "The Structure of Utility Functions". Rev. Econ. Studies 38, 113.

Vind K. (1971b) Comment. Rev. Econ. Studies 38, 115.

Vind K. (1974) A Note on "A Four-Flagged Lemma". Rev. Econ. Studies 41, 571.

Vives X. (1990) Nash Equilibrium with Strategic Complementarities. J. Mathem. Econ. 19, 305-321.