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Abstract

A general NTU game is interpreted as a collection of pure bargaining games that can be played by individual coalitions. The threat points in these pure bargaining games reflect the players' opportunities outside a given coalition. We develop a solution concept for general NTU games that is consistent in the sense that the players' outside opportunities are determined by the solution to a suitably defined reduced game. For any general NTU game the solution predicts which coalitions are formed and how the payoffs are distributed among the players.

1 Introduction

There are many economic situations in which coalition formation and bargaining over the gains from cooperation play a central role. A prominent example is a coalition production economy where each coalition possesses a production technology and the agents have to decide which firms to form and how to distribute the profits among the owners. Another example is the provision of public goods in a local public good economy.

Such problems can be formulated as general Non Transferable Utility (NTU) games. Rather than predicting coalition formation the literature has mainly concentrated on finding solutions for general NTU games that assign to each game a utility distribution among the players which is feasible for the *grand* coalition. The Shapley NTU Value (Shapley [12]) and the Harsanyi solution (Harsanyi [5]) both consider as well what players can achieve when they form smaller coalitions. The idea is that a player who can get a high utility in coalitions different from the grand one will usually have a lot of bargaining power and can therefore demand a "large piece of the cake" when he joins the grand coalition. However, the threat to deviate to a subcoalition is only credible if there is some positive probability that this subcoalition really forms in case there is disagreement in the grand coalition. Neither value is based on such considerations. Also, since the final utility distribution has to be feasible for the grand coalition there are several counterintuitive results for a large class of games. This is true in particular for non-superadditive games, where only the formation of smaller coalitions seems to be rational for the players.[1] Thus, a solution concept should take into account not only the problem of distributing the gains among players but also the issue of coalition formation which is of course directly correlated to the former.

[1]See the Aumann–Roth–Shafer discussion in [1], [10], [11].

Defining solution concepts with respect to coalition structures ([2]) may be one way to solve the problem if some additional notion of stability on the set of partitions is introduced (see for example Shenoy [14]). However, it is difficult to justify the existence of a solution concept with respect to a coalition structure. Rational players will not commit themselves to form a coalition prior to bargaining about the payoff in this coalition. This is true in particular as, once a coalition structure is fixed, the players cannot use other coalitions as a (credible) threat any more when bargaining about their payoffs.

In this paper we therefore take a different approach. In each coalition the players face a pure bargaining game with a well defined bargaining region and a disagreement point that reflects the players' opportunities outside that coalition.[2] For each coalition we take as exogenously given a solution concept for pure bargaining games that is based on what is considered as being "fair" by the members of that coalition. Players then can determine the payoffs in the various coalitions and decide which coalitions to form. The main issue will be to determine the disagreement points for each coalition. The $\mathcal{C}$-solution we are going to define will be consistent in the sense that the disagreement point in each coalition S is — apart from feasibility constraints — given by the players' expected payoffs in the game that is reduced by coalition S, where these payoffs are those predicted by the $\mathcal{C}$-solution for the reduced game. We will apply the dynamic solution for an abstract game to determine the stable coalition structures given players' payoffs in each coalition. Since we only have a finite number of coalitions, the dynamic solution and therefore the $\mathcal{C}$-solution exist for all NTU games. This is a remarkable result.

[2]In contrast to general NTU games all bargaining regions in *pure bargaining games* are degenerate except those for the grand coalition and those for single player coalitions. General NTU games are sometimes also called *coalitional bargaining games*.

The paper is organized as follows. In Section 2 we review solution concepts for abstract games. The $\mathcal{C}$-solution is defined in Section 3 where we also discuss differences to other well known solution concepts. Section 4 contains some examples. We conclude in Section 5.

2 Solution Concepts for Abstract Games

Let X be an arbitrary set and $\text{dom} \subset X \times X$ a binary relation on X called *domination.* Then (X, dom) is called an *abstract game*. An element $x \in X$ is said to be *accessible* from $y \in X$, denoted $y \rightarrow x$, if there exist $z_0 = x, z_1, \ldots, z_m = y$ such that

$$x = z_0 \text{ dom } z_1 \text{ dom } z_2 \text{ dom } \ldots \ z_{m-1} \text{ dom } z_m = y.$$

If we also assume that $x \rightarrow x \ \forall x \in X$ then the binary accessibility relation is reflexive and transitive. Note that on the contrary dom is neither assumed to be reflexive nor to be transitive. The relation accessible is the transitive and reflexive closure of dom. For $x, y \in X$ we write $x \leftrightarrow y$ to indicate that $x \rightarrow y$ and $y \rightarrow x$.

$$\text{Dom}(x) = \{y \in X | x \text{ dom } y\} \text{ is called } \textit{dominion} \text{ of } x \in X.$$
$$\text{Dom}(A) = \bigcup_{x \in A} \text{Dom}(x) \text{ is called } \textit{dominion} \text{ of } A \subset X.$$

The set $C = X \setminus \text{Dom}(X)$ is the *Core* of the abstract game (X, dom). Since the Core is empty for a large class of games we look for a weaker solution concept.

Definition 2.1 *$S \subset X$ is an* **elementary dynamic solution** *of an abstract game* (X, dom) *if*

1. $x \in S, y \in X \setminus S \Rightarrow x \not\rightarrow y$.

2. $x, y \in S \Rightarrow x \leftrightarrow y$.

P is the **dynamic solution** *(d-solution) of an abstract game* (X, dom) *if*

$$P = \bigcup\{S \mid S \text{ is an elementary dynamic solution of } (X, \text{dom})\}.$$

Observe that the d-solution of an abstract game always exists and is unique, though it may be empty. The concept of the d-solution was developed by Shenoy ([13], [14]) and is closely related to the notion of an R-admissible set in the context of social choice correspondences (see [6]). It can easily be shown that the Core is a subset of the d-solution.

Proposition 2.2 *For all abstract games* $C \subset P$.

If X is finite the d-solution can be characterized as follows.

Lemma 2.3 *If X is finite, then P is the d-solution of an abstract game* (X, dom) *if and only if P satisfies*

1. **(Internal Stability)**
   $x, y \in P \Rightarrow [x \to y \iff y \to x]$.

2. **(External Stability)**

   (a) $x \in P,\ y \in X \setminus P \Rightarrow x \not\to y$.

   (b) $y \in X \setminus P \Rightarrow \exists x \in P$ *such that* $y \to x$.

Proof: We first show that the set P is uniquely determined by the conditions $1., 2.(a)$ and $2.(b)$. Assume that there exist $P, Q \subset X$ that fulfill the conditions $1., 2.(a), 2.(b)$ and let $x \in P \setminus Q$. By condition $2.(b)$ since $x \notin Q$ there exists $y \in Q$ such that $x \to y$. This implies $y \in P$ by condition $2.(a)$.

Therefore, $y \to x$ because of condition 1. which contradicts $x \notin Q$. This implies $P \subset Q$. Analogously $Q \subset P$, i.e. $P = Q$.

It remains to be shown that if P is the d-solution for (X, dom) then P fulfills the conditions 1., 2.(a) and 2.(b).

1. Let $x, y \in P$. Then either there exists an elementary dynamic solution S such that $x, y \in S$ in which case $x \leftrightarrow y$. Or $x \in S, y \in S'$, where S, S' are two disjoint elementary dynamic solutions. Then $x \not\to y$ and $y \not\to x$.

2. (a) Let $x \in P$ and $y \in X \setminus P$. Then there exists an elementary dynamic solution S such that $x \in S$ and $y \notin S$ which implies $x \not\to y$.

   (b) Let $A = \{y \in X \setminus P | y \not\to x \; \forall \, x \in P\}$. We will show that $A = \emptyset$. Assume $A \neq \emptyset$. A is finite since X is finite. Define a binary relation $\succ$ on X as follows. For $x, y \in X$ let $x \succ y$ if $y \to x$ and $x \not\to y$. $\succ$ is irreflexive and transitive. Define $B = \{y \in A | x \not\succ y \; \forall \, x \in A\} = \{y \in A | y \text{ is } \succ\text{-maximal in } A\}$. $B \neq \emptyset$ since $\succ$ is acyclic and A is finite. Let $y \in B$ and $S = \{z \in B | y \leftrightarrow z\}$. We will show that S is an elementary dynamic solution for (X, dom) which is a contradiction and proves that $A = \emptyset$. Let $x, z \in S$. Then $x \to y \to z$ and $z \to y \to x$, i.e. $x \leftrightarrow z$. Let $x \in S$. We have to show that for all $z \in X \setminus S$ it is true that $x \not\to z$.

      i. Let $z \in P$. Then $x \not\to z$ since $x \in S \subset A$.

      ii. Let $z \in A \setminus S$. Then $z \not\succ x$, i.e. either $x \not\to z$ or $x \leftrightarrow z$. The latter implies $z \in B$: Assume there exists $w \in A$ such that $w \succ z$. Then $w \succ x$ which is a contradiction to $x \in B$. However, $x \in S$, $z \in B$ and $x \leftrightarrow z$ imply $z \in S$. Contradiction. Therefore, $x \not\to z$.

iii. Let $z \in X \setminus (P \cup A)$. By the definition of A there exists $w \in P$ such that $z \to w$. This implies $x \not\to z$ because otherwise $x \to w$ which contradicts $x \in A$.

□

In contrast to the Core the d-solution is always nonempty if X is finite.

Theorem 2.4 *If X is finite then the d-solution is nonempty.*

Proof: The claim directly follows from Lemma 2.3. Since the d-solution always exists it cannot be empty because of condition 2.(b) in Lemma 2.3.

□

The definition of the d-solution for an abstract game is similar to the definition of the von Neumann–Morgenstern abstract stable set ([15]) for the relation $\to$.[3] Both solution concepts require some form of *internal* and *external* stability. For the d-solution and finite X this can be seen from Lemma 2.3. However, the definition of internal stability for the d-solution (1. in Lemma 2.3) is weaker than the one used for defining the von Neumann–Morgenstern abstract stable set. In contrast to the latter two elements of the d-solution may be accessible from each other. On the other hand the definition of external stability is weaker for the von Neumann–Morgenstern abstract stable set since no element of the d-solution is allowed to be accessible from an element outside (2.(a) in Lemma 2.3). These differences account for the fact that for finite X the von Neumann–Morgenstern abstract stable set may not exist and is not unique in general whereas the d-solution is unique and nonempty according to Theorem 2.4.

[3] $K \subset X$ is a *von Neumann–Morgenstern abstract stable set* for $(X, \to)$ if 1. $x, y \in K \Rightarrow x \not\to y$ and 2. $y \in X \setminus K \Rightarrow \exists x \in K$ such that $y \to x$.

It is straightforward to see that the d-solution is the von Neumann–Morgenstern abstract stable set for the game $(X, \succ)$, where $\succ$ is the irreflexive and transitive binary relation on X defined in the proof of Lemma 2.3.[4]

There is a clear dynamic interpretation of the stability notion inherent in the definition of the d-solution. If we assume that there exists a positive probability for moving from x to y if $x \to y$ and if P is the d-solution of the abstract game (X, dom) then the elements $x \in P$ are persistent whereas the elements $x \notin P$ are transient in the theory of Markov chains. Thus, for finite X the probability of staying forever outside the d-solution is zero which implies that any process that starts with an arbitrary element $x \in X$ will enter the d-solution after a finite number of steps with probability one. Therefore, the d-solution is stable in a very natural sense. In contrast the von Neumann–Morgenstern abstract stable set does not have this desirable property. The process defined above can leave and re-enter the stable set with positive probability at any time. The differences between both solution concepts are best illustrated with the following example.

Example 2.5 Let $X = \{x, y, z\}$ and dom such that x dom y, y dom z, z dom x, i.e. $x \leftrightarrow y \leftrightarrow z \leftrightarrow x$. The d-solution for (X, dom) is given by $P = X$ whereas any of the sets $\{x\}, \{y\}, \{z\}$ is a von Neumann–Morgenstern abstract stable set for $(X, \to)$. (The von Neumann–Morgenstern abstract stable set for (X, dom) does not exist.)

In this example it seems to be counterintuitive to single out one element $x \in X$ as a stable set $\{x\}$ since all elements of X exhibit the same stability properties. In contrast to the von Neumann–Morgenstern abstract stable set the d-solution reflects this symmetry of the game.

[4]This fact was already pointed out in [6].

3 A Solution Concept based on Endogenous Coalition Formation

We fix the set of players $N = \{1, \ldots, n\}$, where $n \in I\!N$. A set $S \subset N$, $S \neq \emptyset$, is called a *coalition*. Let $\mathcal{P}(N)$ denote the set of coalitions. For $x, y \in I\!R^N$ $x \geq y$ means $x_i \geq y_i \ \forall\, i \in N$, $x > y$ means $x \geq y$ and $x \neq y$, and $x \gg y$ means $x_i > y_i \ \forall\, i \in N$. $I\!R^N_+ = \{x \in I\!R^N | \, x \geq 0\}$, $I\!R^N_{++} = \{x \in I\!R^N | \, x \gg 0\}$. For $x \in I\!R^N$, $S \in \mathcal{P}(N)$, x_S denotes the projection of x to the subspace $I\!R^N_S$ that is spanned by the vectors $(e^i)_{i \in S}$ where $e^i \in I\!R^N$ denotes the ith unit vector. By $|A|$ we denote the cardinality of a set A.

Definition 3.1 *Let $S \in \mathcal{P}(N)$. Then (A, t) is a* **(pure) bargaining game** *for coalition S if*

1. $t \in A \subset I\!R^N_S$.[5]

2. *A is convex and closed in the relative topology of $I\!R^N_S$.*

3. *$\{x \in A | \, x \geq t\}$ is bounded.*

4. *A is comprehensive, i.e. $[x \in A,\, y \in I\!R^N_S,\, y \leq x] \Rightarrow \ y \in A$.*

The set A is called the *bargaining region*. If the players in S cannot agree on a payoff vector $x \in A$ the outcome of the game will be t. Therefore, t is called *disagreement point* or *threat point*. For $S \in \mathcal{P}(N)$ let

$$H^S = \{(A, t) | \, (A, t) \text{ is a pure bargaining game for coalition } S\}.$$

A non transferable utility game is defined as follows.

[5]To simplify the presentation we define bargaining regions for all coalitions as subsets of $I\!R^N$.

Definition 3.2 $V : \mathcal{P}(N) \twoheadrightarrow I\!R^N$ *is called a* **non transferable utility (NTU) game** *if*

1. $V(S) \subset I\!R^N_S \quad (S \in \mathcal{P}(N))$.

2. $V(S) \neq \emptyset$, $V(S)$ *is convex and closed in the relative topology of* $I\!R^N_S$ $(S \in \mathcal{P}(N))$.

3. $V(S)$ *is comprehensive, i.e.*

$$[x \in V(S),\ y \in I\!R^N_S,\ y \leq x] \Rightarrow\ y \in V(S) \quad (S \in \mathcal{P}(N)).$$

4. $V(\{i\})$ *is bounded from above* $(i \in N)$.

5. $\{x \in V(S) |\, x \geq \underline{x}_S\} \neq \emptyset$ *and bounded from above* $(S \in \mathcal{P}(N),\ |S| \geq 2)$, *where* $\underline{x}_i = \sup\{t |\, te^i \in V(\{i\})\} \quad (i \in N)$.

Let $\mathcal{N}$ be the class of NTU games. Of course, all pure bargaining games and all TU games belong to $\mathcal{N}$. In the following we will develop a solution concept for $\mathcal{N}$. The main idea is to interpret an NTU game as a "menu" of pure bargaining games for individual coalitions. Each player can only be a member of *one* coalition at the same time. In contrast to a large part of the literature we take the view that players form coalitions that are in general different from the grand one and that they bargain over the utility distribution in the resulting pure bargaining games. Especially for non-superadditive NTU games, this approach seems to be more natural than assuming that the players will agree on a utility distribution which is feasible for the grand coalition.[6]

[6] $V \in \mathcal{N}$ is called *superadditive* if $V(S) + V(T) \subset V(S \cup T)$ for all $S, T \in \mathcal{P}(N)$ such that $S \cap T = \emptyset$.

The players are thus facing two decision problems: (i) which coalitions to form; and (ii) which payoff vector to choose from the bargaining region for the members of any coalition that has formed. Of course, both problems are interrelated since no player will commit himself to joining a coalition prior to knowing what payoff he can expect in this coalition. The choice of a payoff vector in a given coalition depends on the one hand on the notion of fairness that the members of this coalition have agreed upon and on the other hand on the opportunities the players have in other coalitions since the latter determines their bargaining power. For each coalition $S \in \mathcal{P}(N)$, $|S| \geq 2$, we take as exogenously given a *bargaining function* $\varphi^S : H^S \to \mathbb{R}^N_S$ that assigns a solution to each pure bargaining game for coalition S, i.e. a payoff vector that is considered "fair" by the members of S. We impose a minimum set of conditions upon φ^S. For all $(A, t) \in H^S$

1. **(Feasibility)** $\varphi^S(A, t) \in A$.
2. **(Individual Rationality)** $\varphi^S(A, t) \geq t$.
3. **(Pareto Optimality)** $x \in \mathbb{R}^N_S, x > \varphi^S(A, t) \Rightarrow x \notin A$.

Pareto Optimality of the bargaining functions φ^S is not needed in the following. Without this requirement, however, the rationality of our solution concept would be obscure. Examples of bargaining functions are the Nash solution ([9]), the Kalai-Smorodinsky solution ([7]) etc. Given an NTU game V and bargaining functions φ^S ($S \in \mathcal{P}(N)$, $|S| \geq 2$) the main issue will be to determine the threat point for the pure bargaining game with bargaining region $V(S)$ that can be played by coalition S.

Let Π be the set of all *coalition structures* on N, i.e.

$$\Pi = \left\{ \{S_1, \ldots, S_m\} \,|\, S_i \in \mathcal{P}(N)\ \forall\, i,\ S_i \cap S_j = \emptyset \text{ for } i \neq j,\ \bigcup_{i=1}^{m} S_i = N \right\}.$$

For $V \in \mathcal{N}$ and $P \in \Pi$ let $\mathcal{F}_V(P)$ be the set of payoff vectors that are feasible given coalition structure P, i.e.

$$\mathcal{F}_V(P) = \{x \in I\!R^N \,|\, x_S \in V(S) \; \forall \; S \in P\}.$$

An element $(Q, x) \in \bigcup_{P \in \Pi}(\{P\} \times \mathcal{F}_V(P))$, is called a *payoff configuration.* The $\mathcal{C}$-solution on $\mathcal{N}$ we are going to define maps any NTU game V to a set of payoff configurations thus predicting which coalitions will form and how the payoffs will be distributed within these coalitions. On $\bigcup_{P \in \Pi}(\{P\} \times \mathcal{F}_V(P))$ we define the following dominance relation.

Definition 3.3 *Let* $(P_1, x), (P_2, y) \in \bigcup_{P \in \Pi}(\{P\} \times \mathcal{F}_V(P))$. *Then* (P_1, x) **dominates** (P_2, y), *denoted by* (P_1, x) dom (P_2, y), *iff there exists* $R \in P_1$, *such that* $x_i > y_i \; \forall i \in R$.

Thus, (P_1, x) dominates (P_2, y) if and only if some coalition R in coalition structure P_1 strictly prefers x to y. We require $R \in P_1$ so that R can enforce the payoff x_R without the cooperation of any player $i \notin R$ who might be worse off with x. A natural condition for a coalition to be formed is that it can guarantee individually rational payoffs for its members such that at least one player can be made strictly better off than if he remained alone. We call these coalitions *decisive.* Let $V \in \mathcal{N}$ and recall that $\underline{x}_i = \sup\{t \,|\, te^i \in V(\{i\})\}$ for $i \in N$.

Definition 3.4 *A coalition* $S \in \mathcal{P}(N), |S| \geq 2$, *is called* **decisive** *in game* $V \in \mathcal{N}$ *if there exists* $y \in V(S)$ *such that* $y > \underline{x}_S$.

Rational players will either form decisive coalitions or they will stay on their own. If for each $V \in \mathcal{N}$ and each decisive coalition S we can determine the threat point for the (pure) bargaining game played by coalition S then the definition of a solution concept for V is straightforward. In this case, given the bargaining functions for (pure) bargaining games, the players can determine their payoffs in each of the decisive coalitions. The dynamic solution then picks the stable coalition structures among those that are generated by the decisive coalitions.

The main problem will therefore be to determine the threat points. If a threat point lacks credibility, i.e. if it does not properly reflect the players' outside opportunities it probably will not be accepted by the members of a coalition. In the context of pure bargaining games it is often assumed that the threat point is given by the Nash equilibrium outcome of the underlying non-cooperative game. For an NTU game credibility requires the threat point to reflect the players' opportunities outside a given coalition, i.e. player i's threat point in the decisive coalition S should be given by his expected payoff if negotiations in S break down and i settles for his alternatives in the remaining decisive coalitions. The $\mathcal{C}$-solution is consistent in the sense that the players' outside opportunities in coalition S are determined by the $\mathcal{C}$-solution for the NTU game that results if the set of decisive coalitions is reduced by S. Formally, for $V \in \mathcal{N}$ and $S \in \mathcal{P}(N)$ the *reduced game* $V^{-S} : \mathcal{P}(N) \twoheadrightarrow I\!R^N$ is defined as follows:

$$V^{-S}(T) = \begin{cases} V(T) & , T \neq S \\ \{y \in I\!R^N_T \mid y \leq \underline{x}_T\}, & T = S \end{cases}.$$

In general, however, there is no guarantee that the payoffs for the members of S that are computed for V^{-S} are feasible. In order to deal with outside opportunities that are not feasible for any $V \in \mathcal{N}$ and any coalition S, $|S| \geq 2$, we take as exogenously given a function $t^S_V : \{x \in I\!R^N_S \mid x \geq \underline{x}_S\} \rightarrow V(S)$ with the following properties: For $x \in I\!R^N_S$, $x \geq \underline{x}_S$,

1. $t^S_V(x) \geq \underline{x}_S$.

2. $t_V^S(x) = x$ if $x \in V(S)$.

The function t_V^S assigns to each individually rational outside option vector a payoff that is feasible and individually rational for coalition S. For $x \in I\!R_S^N$, $x \geq \underline{x}_S$, $t_V^S(x)$ will be the threat point in the pure bargaining game played by coalition S. t_V^S can be interpreted as an agreement within coalition S about which threats of the players will be regarded as credible given the players' outside opportunities. Since at this stage we want to be as general as possible we refer to section 4 for an example of the functions t_V^S.

Consistency requires a recursive definition of the $\mathcal{C}$-solution.

Definition 3.5 *For $V \in \mathcal{N}$ let $\mathcal{E}^V$ denote the set of decisive coalitions. The* **$\mathcal{C}$-solution** *for V is a set of payoff configurations which is inductively defined over $|\mathcal{E}^V|$ as follows:*

1. $|\mathcal{E}^V| = 0:$
   For $V \in \mathcal{N}, \mathcal{E}^V = \emptyset$, the $\mathcal{C}$-solution is given by

$$(\{\{1\},\{2\},\ldots,\{n\}\},(\underline{x}_1,\ldots,\underline{x}_n)).$$

2. $|\mathcal{E}^V| = m, m \geq 1:$
   Let the $\mathcal{C}$-solution be defined, nonempty and individually rational for $V \in \mathcal{N}$ with $|\mathcal{E}^V| \leq m-1$, $m \geq 1$.[7] *Let $V \in \mathcal{N}$ with $|\mathcal{E}^V| = m$. For $S \in \mathcal{E}^V$ let $\{(P_1,x^1),(P_2,x^2),\ldots,(P_{k(S)},x^{k(S)})\} \subset \bigcup_{P\in\Pi}(\{P\}\times\mathcal{F}_{V^{-S}}(P))$ be the $\mathcal{C}$-solution for V^{-S} and $y_j^S = \frac{1}{k(S)}\sum_{l=1}^{k(S)} x_j^l$ be the average payoff player $j \in S$ can achieve in V^{-S}. Set $y_j^S = 0$ for $j \notin S$. The $\mathcal{C}$-solution for V is given by the dynamic solution to the abstract game (X, dom) where $X = \{(P,x) \in \Pi \times I\!R^N \mid S \in P \Rightarrow [S \in \mathcal{E}^V$ and $x_S = \varphi^S(V(S), t_V^S(y^S))]$ or $[S = \{i\}$ and $x_i = \underline{x}_i]\}$.*

[7] $A \subset \bigcup_{P\in\Pi}(\{P\}\times\mathcal{F}_V(P))$ is called *individually rational* if $x \geq \underline{x}$ $\forall (P,x) \in A$.

Remark 3.6 It is straightforward to show that the $\mathcal{C}$-solution is well defined. By the induction hypothesis $y^S \geq \underline{x}_S$ for the outside option vector y^S defined in 2. of Definition 3.5. Therefore, $(V(S), t_V^S(y^S)) \in H^S$ and the set $X \subset \bigcup_{P\in\Pi}(\{P\} \times \mathcal{F}_V(P))$ is well defined. Since X is finite, the dynamic solution for (X, dom) is nonempty by Theorem 2.4. Since φ^S is individually rational, X and therefore the $\mathcal{C}$-solution are individually rational.

One can criticize the fact that we take the average of the payoffs in the $\mathcal{C}$-solution as a threat point in the reduced games. However, we have not modelled any bargaining process that leads to the dynamic solution for (X, dom) once the payoffs in all decisive coalitions have been determined and therefore there is no point in discriminating between the elements of the $\mathcal{C}$-solution. The latter all exhibit the same stability property which justifies the assignment of equal weights.

On the other hand — as mentioned in Section 2 — there is a Markov process whose persistent states are the elements of the dynamic solution of the abstract game. Given a set X of payoff configurations we can think of the following bargaining process that leads to the dynamic solution of the game (X, dom). Nature chooses — with equal probabilities — the payoff configuration with which we start the process. Assuming that there are positive probabilities for moving from x to y if $x \to y$ $(x, y \in X)$ the Markov process that is defined by these probabilities has the property that it will enter the dynamic solution after a finite number of steps with probability one. Given the probabilities with which the bargaining process is started we can compute the long run probabilities for the persistent states of the Markov process which are the elements of the dynamic solution.[8] It would be natural to use these probabilities for the determination of the players' expected payoffs out-

[8]This is possible since the set of persistent states in our case consists of irreducible, aperiodic, closed sets.

side a given coalition. Since the definition of the $\mathcal{C}$-solution, however, is free from any dynamic considerations like the one illustrated above taking equal weights seems to be appropriate.

The $\mathcal{C}$-solution has several interesting features. It provides a model of endogenous coalition formation in which the players bargain about their payoffs in each coalition taking into account their outside opportunities and then, given these payoffs, decide which coalition to form. The threats players use are credible since they are given by (or, to ensure feasibility, directly correlated with) players' expected payoffs if the current negotiations break down.

Before presenting some examples we briefly discuss the differences between the $\mathcal{C}$-solution and other well known solution concepts for general NTU games. Neither the Shapley NTU Value ([12]) nor the Harsanyi solution ([5]) capture the aspect of coalition formation as an important part of a solution for an NTU game. The Shapley NTU Value, an extension of the Shapley TU Value to NTU games, assigns to each game a utility distribution that is feasible for the grand coalition and that takes into account what players can achieve in all possible coalitions since this obviously influences their bargaining power within the grand coalition. The fact that the utility distribution has to be feasible for the grand coalition easily leads to counterintuitive results, especially for games that lack some form of superadditivity i.e., games without "increasing returns to cooperation". The same is true for the Harsanyi solution, although it assigns a feasible utility distribution to each coalition. However, since coalition formation is not modelled explicitely, the Harsanyi solution also suggests that the grand coalition will form in the end. Thus, both values answer the question of how to distribute utilities among players *given* that the grand coalition has formed.

In [14] Shenoy assumes that there is a rule for the allocation of payoffs within each coalition structure (e.g. the Core) and uses the dynamic solution concept to determine the stable coalition structures. However, the meaning of a solution concept with respect to a coalition structure can be questioned, in particular if the payoffs in one coalition depend on which other coalitions form. Once such a structure is fixed there is no way in which players could use other coalitions as a (credible) threat when bargaining about their payoffs. Thus, a rational player will never commit himself to forming a coalition without having information about the payoffs he can expect there.

In contrast we do not allow payoffs to depend on coalition structures and we compute the payoffs in each coalition prior to knowing which coalitions will form. In a second step the stable coalition structures are determined. From an abstract point of view this approach fits into Shenoy's framework since the payoffs within coalitions naturally lead to a payoff solution concept with respect to a coalition structure. However, because of the consistency property of the $\mathcal{C}$-solution, unlike in [14] it is in fact not possible to define this solution concept without simultaneously introducing a notion of stability.

Yet another approach to the solution of a general NTU game with a special focus on coalition formation is the one by Bennett and Zame ([4]) where they consider the concept of bargaining aspirations. An aspiration is defined as an n-vector of "prices" or reservation payoffs that players demand for their participation in any coalition. This vector is maximal in the sense that no coalition can improve upon it, and it is achievable in the sense that for each player there exists a coalition which can afford the prices of its members. A bargaining aspiration x has the additional property that no player i is *vulnerable* at x meaning that there exists no player j such that the coalitions $S \ni i$ which can afford x form a strict subset of the coalitions $T \ni j$ which can afford x. It turns out (see [3]) that aspirations are consistent conjectures about the players' payoffs in each coalition in the following sense. Given that each coalition has a conjecture about the agreements in other coalitions the threat point in coalition S is given by the maximum amount each member can achieve outside (apart from feasibility constraints). The payoffs in coalition S are then computed according to a bargaining function φ^S for S. These payoffs in turn serve as a conjecture for coalitions different from S and so on. A consistent conjecture then is a fixed point of the mapping described above and it can be proved that any consistent conjecture defines an aspiration.

What is critical about this so called multilateral bargaining approach is that the members of each coalition take as a threat the *maximum* amount they can achieve outside regardless of the fact that the coalition which guarantees this payoff might not form. A general problem with bargaining aspirations is that they only predict the outcome of a game if there exists a *partition* of the set of players into coalitions who can afford the prices of their members. Otherwise it is not clear what the remaining players will do once a coalition that is predicted by a bargaining aspiration has formed. In case there exists a partition of the form described above the payoff configuration defined by the bargaining aspiration is called a bargaining aspiration outcome. While Bennett and Zame prove that the aspiration bargaining set is nonempty for every strongly comprehensive NTU game, they do not analyse the existence of a bargaining aspiration outcome in general. In fact the maximality requirement is so strong (and similar to that for the Core) that in general one expects the set of bargaining aspiration outcomes to be empty.

4 Examples

As we have seen the existence of the $\mathcal{C}$-solution does not depend on the nature of the bargaining functions φ^S or on the functions t_V^S. In the examples we present in the following we use the Nash solution as a bargaining function for all coalitions and define $t_V^S : \{x \in \mathbb{R}_S^N \,|\, x \geq \underline{x}_S\} \to V(S)$ as follows. For

$x \in I\!R^N_S,\ x \geq \underline{x}_S,$

$$\left(t^S_V(x)\right)_i = \begin{cases} x_i & \text{, if } x \in V(S) \\ \max\{\underline{x}_i, \max\{t \mid (t, x_{-i}) \in V(S)\}\} & \text{, otherwise} \end{cases}$$

for $i \in S$ and $\left(t^S_V(x)\right)_i = 0$ for $i \notin S$ where by definition $\max(\emptyset) = -\infty$.[9] Obviously, $t^S_V(x) \in V(S)$ for all $x \in I\!R^N_S,\ x \geq \underline{x}_S$, since $V(S)$ is comprehensive. This definition reflects the following idea. If the players' outside opportunities give rise to a payoff vector which is not feasible for S then the largest amount player i can use as a threat is given by the maximum payoff so that the outside opportunities are feasible for all $j \in S,\ j \neq i$. Any larger amount would make it impossible for the other players to achieve their outside payoffs so that they would object to i's claim. If there is no such payoff, or if the payoff would require player i to settle for less than what he could achieve on his own, it seems reasonable to take $\underline{x}_i$ as i's threat.

In the following φ^S denotes the Nash solution on H^S and $t^S = t^S_V$ denotes the function defined above.[10] We start with a simple example of a 3-person TU game.

Example 4.1 Let $N = \{1,2,3\}$ and $V : \mathcal{P}(N) \twoheadrightarrow I\!R^N$ be given by

$$V(\{1,2\}) = \{x \in I\!R^N_{\{1,2\}} \mid x_1 + x_2 \leq 50\},$$
$$V(\{1,3\}) = \{x \in I\!R^N_{\{1,3\}} \mid x_1 + x_3 \leq 50\},$$
$$V(S) = \{x \in I\!R^N_S \mid x_S \leq \mathbf{0}\} \text{ else.}$$

The set of decisive coalitions is given by $\mathcal{E}^V = \{\{1,2\},\{1,3\}\}$. We first consider the reduced game $V^{-\{1,2\}}$ where coalition $\{1,3\}$ is the only decisive

[9] For $x \in I\!R^N$, $i \in N$ and $t \in I\!R$ we write $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(t, x_{-i}) = (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$.

[10] $\varphi^S : H^S \to I\!R^N_S$ is called *Nash solution* if for $(A,t) \in H^S$ $\varphi^S(A,t) = \text{argmax}$ $\{\prod_{i \in R_{(A,t)}} \mid x \in A, x \geq t\}$ where $R_{(A,t)} = \{i \in S \mid \exists x \in A, x \geq t, \text{ such that } x_i > t_i\}$.

coalition. It is straightforward to see that the $\mathcal{C}$-solution for $V^{-\{1,2\}}$ is given by $(\{\{1,3\},\{2\}\},(25,0,25))$. Thus, the average payoff for players 1 and 2 outside coalition $\{1,2\}$ is given by $y_1^{\{1,2\}} = 25$ and $y_2^{\{1,2\}} = 0$, respectively. Obviously, $t^{\{1,2\}}((25,0,0)) = (25,0,0)$ so that we get

$$\varphi^{\{1,2\}}\left(V(\{1,2\}), t^{\{1,2\}}((25,0,0))\right) = (37.5, 12.5, 0).$$

Similarly we compute

$$\varphi^{\{1,3\}}\left(V(\{1,3\}), t^{\{1,3\}}((25,0,0))\right) = (37.5, 0, 12.5,).$$

There are two possible payoff configurations that do not dominate each other. Therefore, the $\mathcal{C}$-solution for V is given by

$$\{(\{\{1,2\},\{3\}\},(37.5,12.5,0)),(\{\{1,3\},\{2\}\},(37.5,0,12.5))\}.$$

By contrast intuition suggests that player 1 should be able to get the whole surplus i.e., one could expect a final payoff distribution of $(50,0,0)$, which is also the payoff distribution predicted by the bargaining aspiration outcomes. However, empirical results show that for the majority of cases the outcome of the game is very close to what is predicted by the $\mathcal{C}$-solution (see [8]).

Since the game is not superadditive the Shapley NTU Value and the Harsanyi solution (both coincide with the Shapley TU Value in this case) completely fail to predict a reasonable utility distribution. The Shapley Value is given by $(50/3, -25/3, -25/3)$.

The second example illustrates that the $\mathcal{C}$-solution predicts the grand coalition to form if the game is sufficiently superadditive.

Example 4.2 Let $N = \{1,2,3\}, c \in I\!R_+$, and $V : \mathcal{P}(N) \twoheadrightarrow I\!R^N$ be defined by

$$\begin{aligned} V(\{i\}) &= \{x \in I\!R^N_{\{i\}} |\ x_i \leq 0\}\ (i \in N), \\ V(\{i,j\}) &= \{x \in I\!R^N_{\{i,j\}} |\ x_i + x_j \leq 10\}\ (i,j \in N), \\ V(N) &= \{x \in I\!R^N |\ \sum_{i=1}^{3} x_i \leq c\}. \end{aligned}$$

The set of decisive coalitions is given by $\mathcal{E}^V = \{\{1,2\},\{1,3\},\{2,3\},N\}$. Since in any reduced game V^{-S} all players $i \in S$ are symmetric the average payoffs of the members of S in the $\mathcal{C}$-solution for V^{-S} are the same by symmetry of the Nash solution. Thus, the members of any coalition $S \in \mathcal{E}^V$ have the same threat points and again by symmetry of the Nash solution and the functions t^S receive the same payoffs. Therefore, we can compute the payoffs in the decisive coalitions as follows ($*$ indicates equal payoffs for the respective players):

$$\begin{aligned} \varphi^{\{1,2\}}\left(V(\{1,2\}), t^{\{1,2\}}((*,*,0))\right) &= (5,5,0), \\ \varphi^{\{1,3\}}\left(V(\{1,3\}), t^{\{1,3\}}((*,0,*))\right) &= (5,0,5), \\ \varphi^{\{2,3\}}\left(V(\{2,3\}), t^{\{2,3\}}((0,*,*))\right) &= (0,5,5), \\ \varphi^{N}\left(V(N), t^{N}((*,*,*))\right) &= (c/3,c/3,c/3), \end{aligned}$$

Now it is easy to see that the $\mathcal{C}$-solution is given as follows:

$c < 15$: All permutations[11] of $(\{\{1,2\},\{3\}\},(5,5,0))$,

$c = 15$: All permutations of $(\{\{1,2\},\{3\}\},(5,5,0))$ and $(\{N\},(5,5,5))$,

$c > 15$: $(\{N\},(c/3,c/3,c/3))$.

For $c > 15$ the game V is sufficiently superadditive to induce the formation

[11] It should be clear what is meant by the permutation of a payoff configuration.

of the grand coalition and the payoff predicted by the $\mathcal{C}$-solution coincides with the Shapley NTU Value and the Harsanyi Value, which are given by $(c/3, c/3, c/3)$ for all c. For $c \leq 15$ the set of bargaining aspiration outcomes is identical to the $\mathcal{C}$-solution. However, for $c > 15$ there is a whole continuum of bargaining aspiration outcomes, all predicting the grand coalition to form: $(\{N\}, (x_1, x_2, c - x_1 - x_2))$, where $0 < x_1, x_2 < c - 10$ and $10 < x_1 + x_2 < c$. Thus, except for the payoff $(c/3, c/3, c/3)$ the bargaining aspiration outcomes predict asymmetric payoffs for the (symmetric!) players.

The third example is similar to one given in Bennett and Zame [4].

Example 4.3 Let $N = \{1, 2, 3\}$ and let a vector of "productivities" be given by $w = (10, 20, 30)$. $V : \mathcal{P}(N) \twoheadrightarrow I\!R^N$ is defined by

$$V(S) = \left\{x \in I\!R^N_S \mid \sum_{i \in S} x_i \leq 0\right\}, \; S \in \mathcal{P}(N), |S| \neq 2,$$
$$V(\{i,j\}) = \{x \in I\!R^N_{\{i,j\}} \mid x_i + x_j \leq w_i + w_j\}, \; (i, j \in N, \, i \neq j).$$

The set of decisive coalitions is given by $\mathcal{E}^V = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. Consider the reduced game, in which $\{1,2\}$ is the only decisive coalition. The $\mathcal{C}$-solution for this subgame is given by $(\{\{1,2\},\{3\}\}, (15, 15, 0))$. Similarly, the $\mathcal{C}$-solution for the subgame in which $\{1,3\}$ is the only decisive coalition is given by $(\{1,3\},\{2\}\}, (20, 0, 20))$. Consider now the reduced game $V^{-\{2,3\}}$ with decisive coalitions $\{1,2\}, \{1,3\}$. From the computation above it follows that in this game the average payoffs for players 1 and 2 outside coalition $\{1,2\}$ are given by $y_1^{\{1,2\}} = 20$ and $y_2^{\{1,2\}} = 0$, respectively. We have $t^{\{1,2\}}((20,0,0)) = (20,0,0)$ and get

$$\varphi^{\{1,2\}}\left(V(\{1,2\}), t^{\{1,2\}}((20,0,0))\right) = (25, 5, 0).$$

Similarly,

$$\varphi^{\{1,3\}}\left(V(\{1,3\}), t^{\{1,3\}}((15,0,0))\right) = (27.5, 0, 12.5).$$

Thus, the $\mathcal{C}$-solution for $V^{-\{2,3\}}$ is given by $(\{\{1,3\},\{2\}\},(27.5,0,12.5))$. Similarly we compute the $\mathcal{C}$-solution for $V^{-\{1,2\}}$ as $(\{\{1\},\{2,3\}\},(0,15,35))$ and for $V^{-\{1,3\}}$ as $(\{\{1\},\{2,3\}\},(0,32.5,17.5))$.

Now we can compute the payoffs in the decisive coalitions of V. The average payoffs for players 2 and 3 outside coalition $\{2,3\}$ are given by $y_2^{\{2,3\}}=0$ and $y_3^{\{2,3\}}=12.5$, respectively. We have $t^{\{2,3\}}((0,0,12.5))=(0,0,12.5)$ and get

$$\varphi^{\{2,3\}}\left(V(\{2,3\}),t^{\{2,3\}}((0,0,12.5))\right)=(0,18.75,31.25).$$

Similarly we compute

$$\varphi^{\{1,2\}}\left(V(\{1,2\}),t^{\{1,2\}}((0,15,0))\right)=(7.5,22.5,0).$$

$$\varphi^{\{1,3\}}\left(V(\{1,3\}),t^{\{1,3\}}((0,0,17.5))\right)=(11.25,0,28.75).$$

The payoff configurations generated by $\mathcal{E}^V$ and by the payoffs computed above are all accessible from each other so that the $\mathcal{C}$-solution for V is given by

$$\begin{aligned}\{&(\{\{1,2\},\{3\}\},(7.5,22.5,0)),\\ &(\{\{1,3\},\{2\}\},(11.25,0,28.75)),\\ &(\{\{2,3\},\{1\}\},(0,18.75,31.25))\}.\end{aligned}$$

These payoff configurations are quite different from the bargaining aspiration outcomes which also predict that one of the 2-player coalitions will form but with utility distributions given by the corresponding components of w. In contrast the payoffs in the $\mathcal{C}$-solution reflect what players can achieve in alternative coalitions. For example, when bargaining with player 1 player 2 is relatively strong since he has the possibility to form a coalition with player 3 whereas player 1 will remain alone if bargaining in $\{1,2\}$ breaks down. Thus, player 2 achieves more than his own productivity in the coalition with player 1. When bargaining with player 3, however, player 2 is relatively weak since he will stay on his own and player 3 will form a coalition with player 1 if bargaining in coalition $\{2,3\}$ breaks down.

Since the game is not superadditive, the Shapley NTU Value and the Harsanyi solution again fail to predict a reasonable utility distribution: the Shapley Value is given by $(-5, 0, 5)$.

The fourth example is due to Roth [10]. Having presented the computation of the $\mathcal{C}$-solution in detail in the previous examples we just give the results in the following.

Example 4.4 Let $N = \{1,2,3\}$ and $0 \le p < 1/2$. $V : \mathcal{P}(N) \twoheadrightarrow \mathbb{R}^N$ is given by

$$\begin{aligned}
&V(\{i\}) = \{x \in \mathbb{R}^N_{\{i\}} \,|\, x_i \le 0\} \ (i \in N),\\
&V(\{1,2\}) = \{x \in \mathbb{R}^N_{\{1,2\}} \,|\, (x_1, x_2) \le (1/2, 1/2)\},\\
&V(\{1,3\}) = \{x \in \mathbb{R}^N_{\{1,3\}} \,|\, (x_1, x_3) \le (p, 1-p)\},\\
&V(\{2,3\}) = \{x \in \mathbb{R}^N_{\{2,3\}} \,|\, (x_2, x_3) \le (p, 1-p)\},\\
&V(N) = \{x \in \mathbb{R}^N \,|\, x \le y \text{ for some } y \text{ in the convex hull of } \{(1/2,1/2,0),\\
&\qquad\qquad (p,0,1-p),(0,p,1-p)\}\ \}.
\end{aligned}$$

The set of decisive coalitions is given by $\mathcal{E}^V = \{\{1,2\},\{1,3\},\{2,3\},N\}$. The $\mathcal{C}$-solution for V is given by

$$\{(\{\{1,2\},\{3\}\},(1/2,1/2,0)),(\{N\},(1/2,1/2,0))\}.$$

As Roth argues, the coalition structure $\{\{1,2\},\{3\}\}$ together with the payoff vector $(1/2, 1/2, 0)$ is the unique outcome of the game that is consistent with the hypothesis that the players are rational utility maximizers. The payoff configurations predicted by the $\mathcal{C}$-solution are identical to the set of bargaining aspiration outcomes of V (see [4]). By contrast the utility distributions predicted by the Shapley NTU Value are given by $(1/3, 1/3, 1/3)$ if $p > 0$ and additionally $(1/2, 1/2, 0)$ if $p = 0$. The Harsanyi solutions are given by $(1/2 - p/3, 1/2 - p/3, 2p/3)$ and $(1/2, 1/2, 0)$ if $p < 1/4$ and additionally $(p, 0, 1-p), (0, p, 1-p)$ if $p \geq 1/4$. If p goes to zero the Harsanyi solution converges to the utility distribution predicted by the $\mathcal{C}$-solution.

The last example deals with symmetric Apex games.

Example 4.5 We investigate symmetric Apex games of the following form. There is one distinguished player called the Apex and $n-1$ symmetric Base players. The only decisive coalitions are two-person coalitions that contain the Apex and the coalition of all Base players. Thus, an n-person Apex game, $n \geq 4$, is a TU game $v^n : \mathcal{P}(N) \rightarrow I\!R$ defined by

$$v^n(S) = \begin{cases} 1 \text{ , if } |S| = 2 \text{ and } 1 \in S \text{ or if } S = \{2, \ldots, n\} \\ 0 \text{ , else} \end{cases} .$$

Of course, v^n can be represented as an NTU game. The symmetry of the Base players together with the symmetry of the Nash solution implies the equality of their threat points when bargaining in the Base coalition $\{2, \ldots, n\}$. By symmetry of the Nash solution each Base player therefore receives a payoff of $\frac{1}{n-1}$ in the Base coalition. Thus, again by symmetry there are only three possibilities for the coalition structures predicted by the $\mathcal{C}$-solution: Either the Base coalition forms and the Apex player remains alone and/or any of the Apex coalitions $\{1, i\}, i \in \{2, \ldots, n\}$, forms and the remaining Base players stay on their own.

Table 1 shows the payoffs for the Apex and the Base players in an Apex coalition, the payoffs for Base players in the Base coalition and the coalition structures predicted by the $\mathcal{C}$-solution based on these payoffs for $n = 4, \ldots, 12$. We observe a regular switching between the formation of

n	Payoff in Apex Coalition		Payoff in Base Coalition	Coalition Structure
	Apex	Base		
4	$\frac{37}{48}$	$\frac{11}{48}$	$\frac{1}{3}$	Base Coalition
5	$\frac{25}{64}$	$\frac{39}{64}$	$\frac{1}{4}$	Apex Coalitions
6	$\frac{113}{160}$	$\frac{47}{160}$	$\frac{1}{5}$	Apex Coalitions
7	$\frac{91}{128}$	$\frac{37}{128}$	$\frac{1}{6}$	Apex Coalitions
8	$\frac{769}{896}$	$\frac{127}{896}$	$\frac{1}{7}$	Base Coalition
9	$\frac{1905}{2048}$	$\frac{143}{2048}$	$\frac{1}{8}$	Base Coalition
10	$\frac{683}{1536}$	$\frac{853}{1536}$	$\frac{1}{9}$	Apex Coalitions
11	$\frac{1485}{2048}$	$\frac{563}{2048}$	$\frac{1}{10}$	Apex Coalitions
12	$\frac{19457}{22528}$	$\frac{3071}{22528}$	$\frac{1}{11}$	Apex Coalitions

Table 1: **The $\mathcal{C}$-solution for Apex Games**

Apex and Base coalitions. This can be explained as follows. On the one hand increasing the number of Base players reduces the payoffs in the Base coalition and thus makes the Apex coalition more attractive for the Base players. On the other hand the relative bargaining power of the Apex player increases with a growing number of Base players, so that (for $n = 8$) the Base players prefer to cooperate. Now, if n is large v^n is a good approximation for the game that results if v^{n+1} is reduced by one Apex coalition.[12] Thus, the formation of the Base coalition in v^n lowers the bargaining power of the Apex in v^{n+1} so that finally the Base players prefer the Apex coalition again.

[12]As an example, taking the $\mathcal{C}$-solution for v^9 as given we would predict a payoff of 9/16 for a Base player in an Apex coalition of v^{10}. Since $9/16 > 1/9$ the Base players prefer the Apex coalition over the Base coalition and we expect the $\mathcal{C}$-solution to predict the formation of the Apex coalitions in v^{10}. Table 1 shows that this conjecture is correct and moreover, that 9/16 is a good approximation for the payoff of the Base player in an Apex coalition of v^{10}.

This causes the Apex to become more powerful and so on. Therefore, we conjecture that the coalition structures predicted by the $\mathcal{C}$-solution follow the pattern exhibited in Table 1 for $n \rightarrow \infty$.

5 Conclusion

The questions of coalition formation and payoff distribution among the players are central to the theory of general NTU games. Nevertheless there are only few approaches that simultaneously address both points. Often it is assumed that players will form the grand coalition and distribute the payoffs they can achieve there taking into account their possibilities in subcoalitions. It is obvious that this approach to a solution for general NTU games is not appropriate in general, especially for games that are not superadditive.

We have provided a model of coalition formation based on which payoffs the players achieve in the individual coalitions. When bargaining about the payoffs in a coalition the players are facing a pure bargaining game. The threat point in this game is given by the players' expected payoffs if bargaining in the given coalition breaks down. The $\mathcal{C}$-solution is consistent in the sense that these payoffs are determined by the solution to an appropriately reduced game. We proved that the $\mathcal{C}$-solution is always nonempty which is not the case for any other solution concept for general NTU games. Moreover, because of the recursive definition, computation is relatively easy.

It has been argued that there exists a bargaining process leading to the stable payoff configurations which make up the $\mathcal{C}$-solution. Thus, there is also a dynamic interpretation of our solution concept.

As most of the examples indicate the payoff configurations predicted by the $\mathcal{C}$-solution are close to what intuition suggests to be the outcome of an NTU game, at least if we take the Nash solution as a bargaining function for all coalitions.[13] Unfortunately, experiments mostly deal with small TU games, where the number of players often does not exceed 4, so that we cannot make a general statement about the goodness of the C-solution with respect to "real" outcomes of NTU games. Nevertheless we believe that our solution concept captures many important aspects that determine which coalitions are formed and how the payoffs are distributed if a general NTU game is played by rational players.

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