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Airport Problems and Consistent Solution Rules

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Abstract. A class of solution rules for airport problems are considered. The common properties of these solution rules are single valuedness (SIVA), Pareto optimality (PO), reasonability (REAS) and a weak form of consistency (WCONS). These solution rules are automatically "core selectors" for the associated airport game. The dual weighted Shapley values, the nucleolus, the modified nucleolus and the prenucleolus of the dual game turn out to belong to this class of solution rules. As a side result we prove that, for airport games, the modified nucleolus and the prenucleolus of the dual game are the same. The τ-value, however, turns out not to belong to this class of solution rules. Furthermore, we investigate monotonicity properties of the solution rules and in the last section we characterize the Shapley value, nucleolus and modified nucleolus on the class of airport games.

Key words: Airport problem, airport game, dual weighted Shapley value, nucleolus, modified nucleolus

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Introduction

Airport problems and airport games were introduced by Littlechild and Owen (1973). The problem they pose is how the cost of a landing strip should be distributed among users who need runways of different lengths. Further discussion of the model can be found e.g. in Littlechild and Thompson (1977) and Dubey (1982). These authors studied solution concepts like the Shapley value and the nucleolus for this kind of cost sharing problems.

In this paper we consider solution rules that can consistently be derived from a generating function and try to give some additional insight in the structure of the various solution rules with this characteristic. Classical solution rules like the Shapley value and the nucleolus are solution concepts of this type as well as more recently introduced solution concepts as the weighted Shapley value (Kalai and Samet (1988)) and the modified nucleolus (Sudhölter (1995a,b)).

In Section 1 we recall the definition of an airport problem, introduce the idea of weak consistency and of a generating function. A generating function gives, for each airport problem, the payment to be asked from the "first" player i.e., the/a player with the lowest needs. We show that, under mild conditions a solution rule consistently derived from a generating function g is a "core selector".

Section 2 presents solution rules like the Shapley value, the nucleolus, the modified nucleolus and the prenucleolus of the dual game as weakly consistent solutions derived from a generating function. In fact the modified nucleolus and the prenucleolus of the dual cost game coincide on the class of airport games. The τ-value, on the class of airport games closely related to the proportional rule, is presented as a solution rule *not* consistently derivable from a generating function.

In Section 3 we investigate monotonicity properties of these solution concepts. The monotonicity properties we will discuss are strong monotonicity, (Young (1985)), coalitional monotonicity, population monotonicity (Sprumont (1990)) and fair ranking. Fair ranking is a common property of the Shapley value, the nucleolus and the modified nucleolus and the same is true for population monotonicity. The weighted Shapley values are known to be strongly monotonic (Young (1985)), both the nucleolus-like solution concepts turn out to be coalitional monotonic but not strongly monotonic.

In the last section we axiomatize the Shapley value, the nucleolus and the modified nucleolus on the class of airport games by common axioms like Pareto optimality (PO), the equal treatment property (ETP) and covariance (COV) and specific axioms like strong monotonicity, ν–consistency and μ–consistency, respectively.

1. Definition of an airport problem and airport game.

Let us assume that in a society a list of public projects is under discussion and that the projects can be ordered $p_1 \prec p_2 \prec \ldots \prec p_q$ where $p_i \prec p_j$ means that project p_j is a real extension of project p_i. The members of the community have given their opinion about the different plans and n_i members voted for project p_i. The coalition of people voting for project p_i is denoted by N_i. We assume that $n_i \geq 1$ for $i = 1, \ldots, q$. The realization of project p_i generates cost equal to $\gamma(i)$ and we assume that $\gamma(i) < \gamma(j)$ if $p_i \prec p_j$ and that $\gamma(i) \geq 0$ for every project p_i. The community authorities decided to realize the largest project p_q and to charge the community members with the cost $\gamma(q)$ to be made. They like to take the voting behavior of the community members in consideration. The problem is to find 'nice' allocation rules for problems of this kind. We call such problems *airport cost sharing problems*, as in the literature a model like this is known where the projects p_i are landing strips of different lengths. We can summarize the situation by

$$\langle \gamma(1), N_1; \gamma(2), N_2; \ldots; \gamma(q), N_q \rangle \text{ with } n_i = |N_i| \geq 1 \text{ and } 0 \leq \gamma(i) < \gamma(j) \text{ if } i < j.$$

To be able to deal with non-symmetric solutions and to have an environment in which 'reduced airport problems' make sense, we choose for a slightly more general frame work. Let $\Omega \subseteq \mathbf{N}$ be a universe of players. The ordering structure inherited by Ω from $\mathbf{N}$ will only be used as a 'universal tie-breaking rule' (see next paragraph). An *airport problem* consists of a finite nonvoid player set $N \subset \Omega$ and a map $\gamma : N \rightarrow \mathbf{R}$. The map γ will be used to introduce an ordering $\prec$ on N. We say: player i precedes player j ($i \prec j$) if $\gamma(i) < \gamma(j)$. In order to extend the ordering $\prec$ to a linear ordering we have to break ties between players i and j with $\gamma(i) = \gamma(j)$. We do this by using the ordering inherited from $\mathbf{N}$ and define $i \prec j$ if $\gamma(i) < \gamma(j)$ or $\gamma(i) = \gamma(j)$ and $i < j$. Given this ordering we can speak about the first player, the second player up to the n-th player of N. Notice that the ordering depends mainly on the values of γ and in the second place on the ordering of N inherited from $\mathbf{N}$. To avoid a too clumsy notation we mostly will denote the first, second up to n-th player of N by [1], [2] up to $[n]$, if this will not cause confusion.

In the subsequent part of the paper we are mainly interested in airport problems with only nonnegative values but for the moment we also allow negative values. The class of airport problems will be denoted by $\mathcal{A}_\Omega$ or $\mathcal{A}$ for short. The subclass of *nonnegative airport problems* is denoted by $\mathcal{A}_+$.

A *solution rule* σ on a set $\mathcal{B}$ of airport problems in $\mathcal{A}$ assigns a payment vector $\sigma(N,\gamma) \in \mathbf{R}^N$ to each airport problem (N,γ) in $\mathcal{B}$. So we will be talking about *single valued* solutions (SIVA) all the time with exception of the last remark of Section 4.

A solution rule σ is called *Pareto optimal* (PO) if $\sum_{i\in N} \sigma(N,\gamma)_i = \max_{i\in N} \gamma(i)$. Defining Pareto optimality in this way implies that we understand the cost $\gamma(i)$ as a 'part' of the cost $\gamma(j)$ if $\gamma(i) \leq \gamma(j)$ (cf. the story we started with).

The next property of solution rules is the main topic of this paper.

Suppose we have a solution rule σ. Then $\sigma(N,\gamma)_{[1]}$ is the payment of the first player of N, when the solution rule σ is applied. If only this payment has been done, the remaining players are faced with a cost function $\bar{\gamma}[j] := \gamma[j] - x$, if $j \neq 1$ and $x = \sigma(N,\gamma)_{[1]}$. In case the player set contains at least two persons we define the *reduced airport problem (with respect to x)* as the airport problem $(\bar{N},\bar{\gamma})$ with $\bar{N} := N\backslash[1]$ and $\bar{\gamma}[j] = \gamma[j] - x$ for $[j] \neq [i]$. If this new airport problem belongs to $\mathcal{B}$ we apply the solution rule σ to $(\bar{N},\bar{\gamma})$ and find a new payment vector for the players in $\bar{N}$, namely $\sigma(\bar{N},\bar{\gamma})$. The solution rule σ is called *weakly consistent* (WCONS) if $(N,\gamma) \in \mathcal{B}$ implies $(\bar{N},\bar{\gamma}) \in \mathcal{B}$ and $\sigma(N,\gamma)_{\bar{N}}$ and $\sigma(\bar{N},\bar{\gamma})$ are the same for each airport problem (N,γ). Weak consistency implies that the solution rule σ is completely determined by the payment $\sigma(N,\gamma)_{[1]}$ done by the first player in each problem.

Conversely, given a function $g: \mathcal{A} \to \mathbf{R}$ we can define a weakly consistent solution rule

σ^g on $\mathcal{A}$ by recursion: $\sigma^g(N,\gamma)_{[1]} := g(N,\gamma)$ and, if $|N| > 1$, $\sigma^g(N,\gamma)_{N\setminus[1]} := \sigma^g(\bar{N},\bar{\gamma})$

wherein $\bar{N} = N\setminus[1]$ and $\bar{\gamma}[j] = \gamma[j] - g(N,\gamma)$. Notice that the first player $\overline{[1]}$ in $\bar{N}$ is the second player [2] in the original problem.

In this way every function g defines a weakly consistent solution σ^g on $\mathcal{A}$ and given a weakly consistent solution σ we can define $g(N,\gamma) := \sigma(N,\gamma)_{[1]}$ and find $\sigma = \sigma^g$.

A weakly consistent solution σ *generated by the function* g is Pareto optimal if $g(N,\gamma) = \gamma[1]$ if $|N| = 1$ ('the last player pays the remaining costs').

We call a solution rule σ *reasonable (on both sides)* (cf. Milnor (1952) and Sudhölter (1995a)) if $0 \le \sigma(N,\gamma)_j \le \gamma(j)$ for all airport problems with $\gamma \ge 0$.

A weakly consistent rule σ^g on $\mathcal{A}_+$ is reasonable iff $0 \le g(N,\gamma) \le \gamma[1]$ for every nonnegative problem (N,γ). Notice, that the reduced airport problem with respect to a reasonable solution rule is nonnegative if the original game is nonnegative.

Discussion: The condition $\sigma(N,\gamma)_j \le \gamma(j)$ is also 'reasonable' in the non-technical meaning of the word. No player pays more than the total cost of the plan he voted for. The other condition $\sigma(N,\gamma)_j \ge 0$ is less 'reasonable' than it seems to be. The following rather convincing solution rule does not satisfy the second condition: $\rho(N,\gamma)_j := \gamma(j) - \lambda$ where λ is chosen such that the rule is Pareto optimal. So every player j pays the total costs of his plan $\gamma(j)$ but get some 'discount' λ to make the solution efficient. In fact $\lambda := \frac{\sum_{i<n} \gamma[i]}{n}$. It is easy to check that this solution rule is weakly consistent, Pareto optimal and satisfies the first reasonability condition. We use this solution rule as a counter example at the end of the paper.

If we discuss the modified nucleolus in the next section we will meet a variant of this solution rule.

Weakly consistent (WCONS), Pareto optimal (PO), and reasonable (REAS) solutions of airport problems will be the topic of this paper. To say it differently, we will investigate the solution rules generated by a function $g: \mathcal{A} \to \mathbf{R}$ satisfying

$$g(N,\gamma) = \gamma[1] \quad \text{if } |N| = 1 \quad \text{and} \quad 0 \le g(N,\gamma) \le \gamma[1], \qquad \text{if } \gamma \ge 0.$$

If (N,γ) is a airport problem, we define the associated *airport game* as the cooperative cost game (N,c) with coalition values $c(S) := \max_{i\in S} \gamma(i)$. Notice that the values of γ can be rediscovered from the game (N,c), since $\gamma[i] = c[i]$. Therefore the airport problem and the associated cost game are frequently identified.

If the game is nonnegative, then it is clearly a concave game and has therefore, a nonempty core. The first theorem states that solution rules satisfying WCONS, PO, and REAS are core selectors.

Theorem 1. *If σ is a solution rule defined on $\mathcal{A}_+$ and satisfies weak consistency (WCONS), Pareto optimality (PO) and reasonability (REAS), then $\sigma(N,\gamma) \in$ Core(N,c) for every airport problem $(N,\gamma) \in \mathcal{A}_+$ and associated cost game (N,c).*

Proof: The proof is by induction to $|N|$, the number of players. If there is one player, the theorem follows from PO. Suppose that the theorem has been proved for problems with less than k players and suppose that (N,γ) is a nonnegative airport problem with k players. Let $x := \sigma(N,\gamma)$. By weak consistency (WCONS) we have $\sigma(\bar{N},\bar{\gamma}) = \sigma(N,\gamma)_{\bar{N}}$ and this is a core allocation of the game $(\bar{N},\bar{c})$ associated with the reduced airport problem $(\bar{N},\bar{\gamma})$ by the induction hypothesis. If $S \subset \bar{N} = N\backslash[1]$,

$$\bar{c}(S) = \max_{i\in S} \bar{\gamma}(i) = \max_{i\in S} \gamma(i) - x[1] = c(S) - x[1].$$

Because $x \geq 0$, REAS, and $x(S) \leq \bar{c}(S)$ we have $x(S) \leq c(S)$. If S is a coalition containing player [1], we have $x(S\backslash[1]) \leq \bar{c}(S\backslash[1]) = c(S\backslash[1]) - x[1]$ and therefore, also in this case $x(S) \leq c(S)$. ◁

The following proposition will only be used in Section 4.

Proposition 2. *Let σ be a weakly consistent (WCONS) solution rule on $\mathcal{A}_+$ generated by the generating function g. For an airport problem (N,γ) and a number t with $0 \leq t \leq \gamma[1]$, let (N,γ_t) be the airport problem with cost function $\gamma_t[i] := \gamma[i] - t$, $([i] \in N)$. Then $\sigma^g(N,\gamma_t) \leq \sigma^g(N,\gamma)$, if the generating function satisfies the inequalities $g(N,\gamma) - t \leq g(N,\gamma_t) \leq g(N,\gamma)$ for every nonnegative airport problem (N,γ) and $0 \leq t \leq \gamma[1]$.*

Proof: The proof is by induction. We only prove the induction step. We start with $\sigma(N,\gamma_t)_{[1]} = g(N,\gamma_t) \leq g(N,\gamma) = \sigma(N,\gamma)_{[1]}$

by one side of the inequality and the reduced airport problems have cost functions $\gamma_t[i] - g(N,\gamma_t)$ and $\gamma[i] - g(N,\gamma)$ for $i \geq 2$. By the other part of the inequality $\gamma[i] - g(N,\gamma) \geq (\gamma[i] - t) - g(N,\gamma_t)$ for $[i] \neq [1]$ and we are in the same situation as before: weak consistency and the induction hypothesis give $\sigma(N,\gamma_t)_{\bar{N}} \leq \sigma(N,\gamma)_{\bar{N}}$.

◁

2. Weak consistency of solution concepts for airport games

In this section we start with the airport game associated with nonnegative airport problems and investigate which single valued solution concepts for TU-games are weakly consistent. The solution concepts we will consider are the nucleolus, the prenucleolus of the dual game (also called the antinucleolus of the game), the dual weighted Shapley value, the modified nucleolus and the τ-value. All these solution rules will turn out to be weakly consistent except the τ-value. For each of the weakly consistent rules we will give the generating function.

The dual weighted Shapley value

We start with the dual weighted Shapley value. Let us briefly repeat the definition (see Kalai and Samet (1988) for more details). Like the classical Shapley value, the dual weighted Shapley value is a linear function of the game. So it is enough to define the solution for the elements of a basis of the vector space G^N of all TU-games with player set N. For this basis the set of *representation games* $\{(N, u_S^*)\}_{S \subset N}$, the set of duals of the unanimity games (N, u_S), $(S \subset N)$ is chosen. Remember that the dual game (N, c^*) of a cost game (N, c) is the *cost game* defined by $c^*(S) := c(N) - c(N \backslash S)$. So the dual game assigns to a coalition S *the marginal cost* of coalition S, if S joins the coalition $N \backslash S$. If $S \subset N$, the game (N, u_S^*) is the simple game with $u_S^*(T) = 1$ if and only if $S \cap T \neq \emptyset$. To define the (positively) weighted Shapley value we need a positive weight function on the universe Ω i.e. $w: \Omega \to \mathbf{R}_{++}$.
We define the *positively dual w-weighted Shapley value* ϕ_w to be the linear solution rule given by $\phi_w(N, u_S^*)_i := \dfrac{w(i)}{w(S)}$ if $i \in S$ and $\phi_w(N, u_S^*)_i := 0$ if $i \notin S$.
Hence the symmetric Shapley value (see Shapley (1953)) is obtained as a weighted Shapley value if all weights coincide. The first theorem of this section states the weak consistency of the weighted Shapley value.

Theorem 3. *The w-weighted Shapley value ϕ_w is weakly consistent (on nonnegative airport games) and its generating function is* $g(N, \gamma) := \dfrac{w[1]}{w(N)} \gamma[1]$.

Proof: First we compute the generating function, under the assumption that ϕ_w is weakly consistent. If (N, γ) is a nonnegative airport problem, the game (N, c) is
$$c = \gamma[1](N, u_N^*) + \sum_{1 \le i < n} (\gamma[i+1] - \gamma[i])\,(N, u^*_{([i+1], \ldots, [n])}).$$
By the definition of the weighted Shapley value,

$$\phi_w(N,c)_{[1]} = \gamma\,[1]\,\phi_w(N,u^*_N)_{[1]} = \gamma\,[1]\,\frac{w\,[1]}{w\,(N)}.$$

Hence, if the weighted Shapley value has a generating function, it is the function mentioned in the theorem. Let (N,γ) be a nonnegative airport problem and $x := \phi_w(N,c)_{[1]} = \gamma\,[1]\,\frac{w\,[1]}{w\,(N)}$. The game $(\bar{N},\bar{c})$ associated with the reduced problem $(\bar{N},\bar{\gamma})$ is

$$(\bar{N},\bar{c}) = (\gamma\,[2]-x)\,(\bar{N},u^*_{([2],\ldots,[n])}) + \textstyle\sum_{2\leq i<n}\,(\gamma\,[i+1]-\gamma\,[i])\,(\bar{N},u^*_{([i+1],\ldots,[n])}).$$

It is clear from the definition of the weighted Shapley value that

$$\phi_w\,(\bar{N},u^*_S) = \phi_w\,(N,u^*_S)_{\bar{N}} \text{ if } S\subset\bar{N}\subset N.$$

So we are left to prove that,

$$(\gamma\,[2]-x)\;\phi_w(\bar{N},u^*_{\bar{N}}) = \gamma\,[1]\,\phi_w(N,u^*_N)_{\bar{N}} + (\gamma\,[2]-\gamma\,[1])\;\phi_w(N,u^*_{([2],\ldots,[n])})_{\bar{N}}.$$

i.e. we have to prove that, for $i\geq 2$,

$$(\gamma\,[2]-x)\;\frac{w\,[i]}{w\,(\bar{N})} \text{ and } \gamma\,[1]\,\frac{w\,[i]}{w\,(N)} + (\gamma\,[2]-\gamma\,[1])\;\frac{w\,[i]}{w\,(\bar{N})} \text{ are the same.}$$

This means $\quad \gamma\,[1]\,\frac{w\,[i]}{w\,(N)} = (\gamma\,[1]-x)\;\frac{w\,[i]}{w\,(\bar{N})}.$

By substituting the expression for x we see that this equality is correct. ◁

The nucleolus

The next result is about the weak consistency of the nucleolus of airport games. For the definition of the nucleolus of an airport game we need the *excess map* $E\colon\ I(N,c) := \{x\in\mathbf{R}^N \mid x\,(N) = c\,(N),\, x_i\leq c\,(i)\,(i\in N)\}\to\mathbf{R}^{2^N\setminus\{N,\emptyset\}}$ with $E\,(x)_S := c\,(S)-x\,(S)$ and the *coordinate ordering map* $\theta\colon\mathbf{R}^{2^N\setminus\{N,\emptyset\}}\to\mathbf{R}^{2^n-2}$ that orders the coordinates of a vector in a weakly *increasing* order.

The *nucleolus* $Nu\,(N,c)$ is the (set of) imputation(s) x with $\theta\circ E\,(x)\succeq_{\text{lex}}\theta\circ E\,(y)$ for all cost allocations $y\in I\,(N,c)$ (Schmeidler (1969)).

The nucleolus of a cooperative (cost) game consists of one point and if the game has a nonempty core it coincides with the *prenucleolus* defined by

$$Nu^*(N,c) = x^* \text{ if } \theta\circ E\,(x)\succeq_{\text{lex}}\theta\circ E\,(y) \text{ for all pre-imputations } y.$$

By a well known theorem (Sobolev (1975)) the prenucleolus satisfies the *reduced game property* i.e. if $x = Nu^*(N,c)$ and $(S,\bar{c})$ is the *reduced game of* (N,c) defined by $\quad \bar{c}\,(S) = c\,(N)-x\,(N\backslash S) \qquad \bar{c}\,(\emptyset) = 0$

$$\bar{c}\,(T) = \min_{Q\subset N\backslash S}\,[c\,(T\cup Q)-x\,(Q)] \text{ if } T\subset S \text{ and } T\neq S,\emptyset,$$

then $Nu^*\,(S,\bar{c}) = Nu^*\,(N,c)_S$ for all coalitions $S\neq\emptyset$.

In Littlechild and Thompson (1977) the nucleolus of airport games has been determined. From the results of this paper one can deduce (see also Sönmez (1993)) that $Nu(N,c)_{[1]} = \min_{i<n} \dfrac{\gamma[i]}{i+1}$ in case $|N| > 1$.

Hence, if the nucleolus is weakly consistent, it is generated by the function

$$g(N,c) := \min_{i<n} \frac{\gamma[i]}{i+1}.$$

Theorem 4. *The nucleolus is weakly consistent on the class of nonnegative airport games and the generating function is* $g(N,c) := \min_{i<n} \dfrac{\gamma[i]}{i+1}$ for $|N| > 1$ and $g(N,c) := \gamma[1]$ if $|N| = 1$.

Proof: The weak consistency follows from the fact that the cost game associated with a reduced airport problem is the reduced game of the game associated with the original airport problem. Let (N,γ) be a nonnegative airport problem with $|N| > 1$ and $x \in \mathbf{R}_+$ with $x \leq \gamma[1]$. The cost game associated with the reduced airport problem $(\bar{N},\bar{\gamma})$ with $\bar{N} = N\backslash[1]$ and $\bar{\gamma}[j] := \gamma[j] - x$ for $j > 1$ assigns to a coalition $\bar{S} \subset \bar{N}$, $(\bar{S} \neq \bar{N}, \emptyset)$ the coalition value $\max_{j\in\bar{S}} (\gamma[j] - x) = c(\bar{S}) - x = c(\bar{S} \cup [1]) - x$. In the reduced game of (N,c) the same coalition $\bar{S}$ has the coalition value

$$\min(c(\bar{S}), c(\bar{S} \cup [1]) - x).$$

So we must prove that $c(\bar{S}\cup[1]) - x \leq c(\bar{S})$. This holds true because $c(\bar{S}\cup[1]) = c(\bar{S})$ if $\bar{S} \neq \emptyset$ and $x \geq 0$. Applying this result with $x = Nu(N,c)_{[1]}$ we find the weak consistency of the nucleolus of airport games. ◁

Remark: If $Nu(N,c)_{[1]} = \dfrac{\gamma[i]}{i+1}$, then $Nu(N,c)_{[j]} = Nu(N,c)_{[1]}$ for $1 \leq j \leq i$.

The τ-value and the proportional rule.

For the τ-value we obtain a negative result: the τ-value is not weakly consistent for airport games. First we repeat the definition of the τ-value (Tijs (1981)). If (N,c) is a cost game, we define the *marginal vector* $M(N,c)$ by $M(N,c)_i := c(N) - c(N\backslash i)$, $(i \in N)$. If the game (N,c) has a nonempty core, $\sum_{i\in N} M(N,c)_i \leq c(N)$.

The remaining costs are measured by the vector $m(N,c)$ with coordinates

$$m(N,c)_i := \min_{S:i\in S} \left[c(S) - \textstyle\sum_{j\in S, j\neq i} M(N,c)_j\right].$$

For cost games with a nonempty core $\sum_{i\in N} m(N,c)_i \geq c(N)$. The *$\tau$-value of a cost game* is the unique efficient point on the line segment between $M(N,c)$ and $m(N,c)$. For *concave cost games* we have $m(N,c)_i = c(i)$ for all players $i \in N$. The next example shows that the τ-value is not weakly consistent.

Example: $N = \{1,2,\ldots,4\}$ and $\gamma(1) = \gamma(2) = 1$ and $\gamma(3) = \gamma(4) = 2$. Notice that the marginal vector $M(N,c) = 0$ and the τ-value assigns to the players a payment proportional to their cost: $\tau(N,c) = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

If the τ-value would be weakly consistent, the generating function would be

$$g(N,\gamma) := \lambda\,\gamma(1) \text{ with } \lambda := \frac{\gamma[n]}{\gamma(N)}$$

(at least for airport problems with $\gamma[n] = \gamma[n-1]$ to be sure that all marginals are zero).

But the weakly consistent rule σ^g generated by this function is not symmetric. Accordingly, let us compute $\sigma^g(N,c)_{[2]}$. The reduced airport problem has three players, one with costs $\frac{2}{3}$ and two with costs $\frac{5}{3}$.

The value of $g(\bar{N},\bar{\gamma})$ is $\dfrac{\frac{2}{3} \cdot \frac{5}{3}}{\frac{12}{3}} = \dfrac{5}{18} < \dfrac{1}{3}$.

Notice that, for airport problems with $\gamma[n] = \gamma[n-1]$, the τ-value is equal to the proportional rule. Therefore, also the proportional rule is not weakly consistent

The modified nucleolus and the prenucleolus of the dual game

For the modified nucleolus we proçeed differently. We give a formula for a solution rule and prove that it is a weakly consistent rule. The next step will be that we prove that this solution rule gives the modified nucleolus and the prenucleolus of the dual cost game.

If (N,γ) is a nonnegative airport problem, we can do the following: every player pays the total cost of the plan he is supporting and later on he obtains a discount λ in order to make the solution efficient. If the discount makes a player's payment negative, he pays nothing. So more formally, $\sigma^m(N,\gamma)_i := (\gamma(i)-\lambda)_+ = \max(0, \gamma(i)-\lambda)$ and λ is chosen in such a way that σ^m is Pareto optimal (PO). The number λ is uniquely determined by this condition. To see this, consider the function $q: [0,\gamma[n]] \to \mathbf{R}$ defined by $q(t) = \sum_{i\in N}(\gamma(i)-t)_+$. This is a *strictly decreasing* function on the segment $[0,\gamma[n]]$. For $t = 0$ we have $q(t) \geq \gamma[n]$ and for $t = \gamma[n]$ we have $q(t) = 0 \leq \gamma[n]$. Then there is exactly one number λ with $q(\lambda) = \gamma[n]$.

Theorem 5. *The solution rule σ^m on nonnegative airport problems is weakly consistent.*

Proof: Let (N,γ) be a nonnegative airport problem, let $x = \sigma^m(N,\gamma)_{[1]} = (\gamma[1]-\lambda)_+$ be the payment by the first player [1] and consider the reduced problem $(\bar{N},\bar{\gamma})$. Let $\bar{\lambda}$ be the solution of the equation $\sum_{j\geq 2}(\gamma[j]-x-t)_+ = \gamma[n]-x$.

If we substitute $t := \lambda - x$ in the function $\bar{q}: t \to \sum_{j\geq 2}(\bar{\gamma}[j] - t)_+$, we obtain $\sum_{j\geq 2}(\gamma[j] - \lambda)_+ = \gamma[n] - (\gamma[1] - \lambda)_+ = \gamma[n] - x$. The first equality follows from the definition of λ. So $t = \lambda - x$ is the unique solution of $\bar{q}(t) = \bar{\gamma}[n]$, i.e. $\bar{\lambda}$. Then the weak consistency of σ^m follows. ◁

The next step is to prove that the weakly consistent rule σ^m assigns to each nonnegative airport problem the modified nucleolus of the associated airport game. First we repeat the definition of the modified nucleolus (Sudhölter (1995a,b)).

When the nucleolus is an attempt to make the excesses of cost games as large as possible, the modified nucleolus tries to make the differences between the excesses as small as possible.

To define the modified nucleolus we need the *bi-excess map*

$^{\beta}E\colon I^*(N,c) \to \mathbf{R}^{2^N} \times \mathbf{R}^{2^N}$ with coordinates

$^{\beta}E(x)_{S,T} := (c(S) - x(S)) - (c(T) - x(T)) \qquad (S \subset N, T \subset N).$

The second ingredient to define the modified nucleolus is the coordinate ordering map Θ that orders the coordinates of a vector from $\mathbf{R}^{2^N} \times \mathbf{R}^{2^N}$ in a weakly *decreasing* order.

The *modified nucleolus of a cost game* $^{\mu}Nu(N,c)$ is the set of pre-imputation x with the property that $\Theta \circ {}^{\beta}E(x) \preceq_{\text{lex}} \Theta \circ {}^{\beta}E(y)$ for all pre-imputations y. The modified nucleolus is an example of a general nucleolus studied in Maschler, Potters and Tijs (1992).

The modified nucleolus consists of one point and shares several properties with the (pre)nucleolus (cf. Sudhölter (1993)). But it has also two beautiful properties in common with the Shapley value. The first property is *self-duality* i.e. the modified nucleolus of the dual cost game is equal to the modified nucleolus of the original game. The reason is that $^{\beta}E(x \mid c^*)_{S,T} = {}^{\beta}E(x \mid c)_{N\setminus T, N\setminus S}$ and therefore, $\Theta \circ {}^{\beta}E(x \mid c^*) = \Theta \circ {}^{\beta}E(x \mid c)$ for all pre-imputations x. This property shares the modified nucleolus with the Shapley value. Another related property that the modified nucleolus shares with the Shapley value is the independence of the *interpretation of the game as profit game or cost game.* This time the reason is that $^{\beta}E(-x \mid -c)_{S,T} = {}^{\beta}E(x \mid c)_{T,S}$ and therefore, $\Theta \circ {}^{\beta}E(-x \mid -c) = \Theta \circ {}^{\beta}E(x \mid c)$ for every pre-imputation x.

The next theorem states that the modified nucleolus of a nonnegative airport game is the solution σ^m we introduced before. This means that the modified nucleolus is weakly consistent. As a side result we will see that the modified nucleolus coincides with the prenucleolus of the dual cost game (also called the antinucleolus of the original game).

Theorem 6. *For nonnegative airport problems we have* $^{\mu}Nu\,(N,c) = \sigma^m\,(N,\gamma)$.

Proof: Let (N,γ) be a nonnegative airport problem and $x = \sigma^m\,(N,\gamma)$. We prove that $^{\mu}Nu(N,c) = x$. Let λ be the number defined by $q\,(\lambda) = \gamma\,[n]$.

As σ^m is a single valued, weakly consistent, Pareto optimal and reasonable solution, $x \in \text{Core}\,(N,c)$ by Theorem 1. Let K be the coalition of players with $x_j = 0$ and G the coalition of players with $x_j > 0$. There is a ranking number t such that $K = \{[1],\ldots,[t]\}$ and $G = \{[t+1],\ldots,[n]\}$. Notice that $[n] \in G$ if $\gamma \neq 0$.

For the excesses $E(x\,|\,c)$ we prove the following inequalities:

(a) $E(x\,|\,c)_S = 0$ if $G \subset S$, (b) $E(x\,|\,c)_S \leq \lambda$ for all $S \subset N$

(c) $E(x\,|\,c)_S = \lambda$ if $|S \cap G| = 1$.

For a coalition S we denote the 'last' player of S by $\iota(S)$.

(a) If $G \subset S$, then $c\,(S) = \gamma\,[n] = c\,(N)$ and $x\,(S) = x\,(N)$. This gives (a).

(b) and (c) If $S \subset K$, we have $c\,(S) = \gamma(\iota(S)) \leq \lambda$ and $x\,(S) = 0$. Let S be a coalition with $S \cap G \neq \emptyset$. For $j \in S \cap G$ with $j \neq \iota(S)$, we have $\gamma\,(j) > \lambda$ because $j \in G$. So $\sum_{j \in S\cap G\setminus\iota(S)} (\lambda - \gamma\,(j)) \leq 0$ and a strict inequality holds if the sum is nonempty i.e. $|S \cap G| > 1$. For the excess $E(x\,|\,c)_S$ we get

$$E(x\,|\,c)_S = c\,(S) - \textstyle\sum_{j\in S} (\gamma\,(j) - \lambda)_+ = \gamma\,(\iota(S)) - \sum_{j \in S\cap G} (\gamma\,(j) - \lambda) =$$

$$|S \cap G|\,\lambda - \textstyle\sum_{j\in S\cap G\setminus\iota(S)} \gamma\,(j) \leq \lambda \qquad \text{and there is inequality if } |S \cap G| > 1.$$

For the pre-imputation x we have $\max_{S,T}\, {}^{\beta}E\,(x\,|\,c)_{S,T} = \lambda$.

To prove that x is the modified nucleolus we will show that a pre-imputation y with maximal bi-excess λ must be x. In fact we will prove that if a pre-imputation has all excesses $\leq \lambda$ it will be x.

Suppose, y is a pre-imputation with $\text{E}\,(y\,|\,c)_S \leq \lambda$ for all coalitions S. We assume, for the moment, that $\lambda > 0$.

First we take $j \in G$ and find $\gamma(j) - y_j \leq \lambda$ and therefore, $y_j \geq \gamma(j) - \lambda = x_j$.

We take the sum over $j \in G$ and find $y\,(G) \geq x\,(G) = x\,(N) = y\,(N)$ and therefore, $y\,(K) \leq 0$. If $y\,(K) < 0$, we take, secondly, the coalition $K \cup j$ with $j \in G$. Then $\gamma(j) - y_j - y\,(K) \leq \lambda$ and therefore, $x_j \leq y_j + y\,(K)$. Again we take the sum over $j \in$

G and find $x(N) = x(G) \leq y(G) + |G|\, y(K) = y(N) + (|G|-1)\, y(K)$. Therefore, if $|G| \geq 2$, this cannot be true and we have $y(K) = 0$ and $y_j = x_j$ for $j \in G$. Suppose the coalition G contains less than two players. Then $x[n] = \gamma[n] = \gamma[n] - \lambda$ and $\lambda = 0$. We assumed, however, that $\lambda > 0$.

Finally, if some coordinate $y_i < 0 \ \ (i \in K)$ we take $S := (i,j)$ with $j \in G$ and find the excess $\gamma(j) - y_j - y_i \leq \lambda$ and so $y_j > x_j$. Contradiction!

If $\lambda = 0$, we have $\sum_{i=1}^{n} \gamma[i] = \gamma[n]$ and $\gamma[i] = 0$ for $i < n$. Then the game (N, c) is an additive game and the theorem is easy to check. ◁

Corollary: *The modified nucleolus of a nonnegative airport game is also the nucleolus of the dual of the airport game.*

Proof: In the preceding proof we showed that the point x is the unique pre-imputation with all excesses $\leq \lambda$. So the point x is also the prenucleolus of (N, c) 'if we understand it as a profit game' (i.e. with the wrong interpretation). But the antinucleolus of a cost game is also the prenucleolus of the dual cost game, as $\mathrm{E}(x \mid c^*)_S = -\mathrm{E}(x \mid c)_{N \setminus S}$ and $\theta \circ E(x \mid c^*) = \theta(-\mathrm{E}(x \mid c))$ shows. ◁

Remark: Theorem 6 gives also an easy way to compute the modified nucleolus of an airport game. One starts by computing $\lambda_1 := \frac{\sum_{i \leq n} \gamma[i]}{n}$. If $\lambda_1 \leq \gamma[1]$, the modified nucleolus is $(\gamma[i] - \lambda_1)_{[i] \in N}$. If not, the players with $\gamma[i] \leq \lambda_1$ get a payment zero under the modified nucleolus and we proceed with the remaining players to compute $\lambda_k := \frac{\sum_{k \leq i < n} \gamma[i]}{n+1-k}$ if k is the lowest index with $\gamma[k] > \lambda_1$. Because $\lambda_k > \lambda_1$ it may happen that $\lambda_k \geq \gamma[k]$. In that case more players get a payment zero and we proceed by computing the next relevant λ_l. To be more precise, let $\lambda_k := \frac{\sum_{k \leq i < n} \gamma[i]}{n+1-k}$ for all k and determine $j = \min\{k \in \{1, \cdots, n-1\} | \lambda_k \leq \gamma[k]\}$. Then the modified nucleolus assigns payments zero to players $[i]$ satisfying $\gamma[i] \leq \lambda_j$ and $\gamma[i] - \lambda_j$ to all other players $[i]$.

3. Monotonicity properties

For solution rules σ for airport problems there are three natural monotonicity conditions (cf., e.g., Sprumont (1990), Thomson (1993), Young (1985)):

(a) *Fair ranking*: it requires that $\sigma(N, \gamma)_i \leq \sigma(N, \gamma)_j$ if $\gamma(i) \leq \gamma(j)$.

(b) *Monotonicity in costs*: if the costs of player i is increased and the costs of the other players remain the same, then player i is not going to pay less i.e. if $\gamma'(i) > \gamma(i)$ and $\gamma'(j) = \gamma(j)$ for $j \neq i$, then $\sigma(N,\gamma')_i \geq \sigma(N,\gamma)_i$.

(c) *Population monotonicity*: if some players are deleted from the player set, the remaining players do not have advantage of this change: $\sigma(N,\gamma)_S \leq \sigma(S,\gamma_S)$, if S is the set of remaining players and γ_S is the restriction of γ to S.

For cooperative games we have also three kinds of monotonicity for single valued solution rules σ:

(d) *Coalitional monotonicity*: if (N,c) and (N,c') are cost games and $i \in N$, if $c(S) = c'(S)$ if $i \notin S$ and $c(S) \leq c'(S)$ if $i \in S$, then $\sigma(N,c)_i \leq \sigma(N,c')_i$.

(e) *Strong monotonicity*: if (N,c) and (N,c') are cost games and $i \in N$, if for all coalitions $S \subset N$ the marginals satisfy the inequalities $c(S \cup i) - c(S) \leq c'(S \cup i) - c'(S)$, then $\sigma(N,c)_i \leq \sigma(N,c')_i$.

(f) *Population monotonicity*: if (N,c) is a cost game, then $\sigma(N,c)_S \leq \sigma(S,c_S)$.

Discussion: Fair ranking can be translated into the game theoretical context. Indeed, if a payoff vector satisfies fair ranking for an airport problem, then it preserves desirability in the associated cost game in the sense of Maschler and Peleg (1966) or Sudhölter (1995(a)) and vice versa.

It is clear that the population monotonicity property (c) is the same as population monotonicity (f), when restricted to airport games.

For airport games coalitional monotonicity (d) is the same as monotonicity in costs (b). This can be understood as follows. If the games (N,c) and (N,c') are cost games associated with airport problems (N,γ) and (N,γ') and $i \in N$ satisfies the conditions of coalitional monotonicity, the 1-coalitions (j), $j \neq i$ have the same value and therefore, $\gamma(j) = \gamma'(j)$ if $j \neq i$. Furthermore, $\gamma'(i) \geq \gamma(i)$. The converse statement is also true. If only the cost of player i is increased, only coalitions containing player i get a weakly larger coalition value. The conclusions in (b) and (d) are the same.

Strong monotonicity is a stronger condition than coalitional monotonicity.

The fair ranking property (a) implies the equal treatment property (ETP) saying that if $\gamma(i) = \gamma(j)$, then $\sigma(N,\gamma)_i = \sigma(N,\gamma)_j$. This implies that asymmetric solutions like the weighted Shapley values with unequal weights do not satisfy fair ranking.

In the first theorem of this section we investigate the fair ranking property.

Theorem 7. *The symmetric Shapley value, the nucleolus and the modified nucleolus have the fair ranking property.*

Proof: It is well-known that the Shapley value preserves desirability for each cooperative game. Maschler and Peleg (1966) and Sudhölter (1995) respectively showed the same property for the prenucleolus, hence for the antinucleolus, and for the modified nucleolus respectively. Therefore all of these solution rules satisfy the fair ranking property. ◁

In the next theorem we study strong monotonicity and coalitional monotonicity (monotonicity in costs).

Theorem 8. *For nonnegative airport problems the weighted Shapley values have the strong monotonicity property, the nucleolus and the modified nucleolus are coalitionally monotonic.*

Proof: Weighted Shapley values are strongly monotonic on the class of all games (see Young (1985)). In fact, the coordinate of player i in the weighted Shapley value is a weighted sum of player i's marginals i.e. there are positive weights $W(S, i \mid w)$ (only dependent on S, i and w) such that

$$\phi_w(N, c)_i = \sum_{S \subset N \setminus i} W(S, i \mid w)\,(c(S \cup i) - c(S)).$$

This implies strong monotonicity of weighted Shapley values.

To prove the coalitional monotonicity of the nucleolus and the modified nucleolus let (N, γ) and $(N, \bar{\gamma})$ be two airport problems with $N = \{1, 2, \ldots, n\}$, $\bar{\gamma}[j] = \gamma[j]$ for $[j] \neq [i]$ and $\bar{\gamma}[i] > \gamma[i]$.

We start with the coalitional monotonicity of the nucleolus. Let x and $\bar{x}$ be the nucleolus of (N, c) and $(N, \bar{c})$, respectively. We assume that the required inequality holds for airport problems with a plaver set that is a proper subset of N and contains i. For the nucleolus the generating function is $g(N, \gamma) := \min_{[j] < [n]} \frac{\gamma[j]}{j+1}$.

First we suppose that $g(N, \gamma) = \frac{\gamma[j]}{j+1}$ for some player $[j]$ with $j < i$. We will prove that $g(N, \bar{\gamma}) = g(N, \gamma)$. Suppose that $\gamma[k] \leq \bar{\gamma}[i] \leq \gamma[k+1]$. Then $k \geq i - 1$ and

$$g(N, \bar{\gamma}) = \min_{t < i \text{ or } k+1 \leq t < n} \frac{\gamma[t]}{t+1} \wedge \min_{i < t \leq k} \frac{\gamma[t]}{t} \wedge \frac{\bar{\gamma}[i]}{k+1}.$$

Each of the fractions is at least $g(N, \gamma)$ and one is equal. Namely, for $t < i$ or

$k+1 \leq t < n$ we have $\frac{\gamma[t]}{t+1} \geq \frac{\gamma[j]}{j+1}$ (and one equality) by the definition of j, for $i < t \leq k$ we have $\frac{\gamma[t]}{t} \geq \frac{\gamma[t]}{t+1} \geq \frac{\gamma[j]}{j+1}$ and finally $\frac{\bar{\gamma}[i]}{k+1} \geq \frac{\gamma[k]}{k+1} \geq \frac{\gamma[j]}{j+1}$.

By weak consistency (WCONS) we have that the reduced airport games have the restrictions of x and $\bar{x}$ to $N \backslash [1]$ as nucleolus. Because $g(N,\gamma) = g(N,\bar{\gamma})$, the reduced airport problems also satisfy the condition for coalitional monotonicity and therefore, by induction $\bar{x}_i \geq x_i$.

If $j \geq i$, we find $x_i = x_1 \leq \bar{x}_1 \leq \bar{x}_i$. The first equality follows from the remark following Theorem 4 and the last inequality comes from fair ranking.

Let y and $\bar{y}$ be the modified nucleolus of (N,c) and $(N,\bar{c})$, respectively. We will prove that $\lambda \geq \bar{\lambda}$ if

λ is the solution of the equation $\quad \sum_{i \in N} (\gamma[i] - t)_+ = \max_{j \in N} \gamma[j] = \gamma[n]$

and $\bar{\lambda}$ solves the equation $\quad \sum_{[i] \in N} (\bar{\gamma}[i] - t)_+ = \max_{[j] \in N} \bar{\gamma}[j] = \gamma[n] \vee \bar{\gamma}[i]$.

If we substitute $t = \lambda$ in the second equation, we find

$$\textstyle\sum_{[j] \neq [i]} (\gamma[j] - \lambda)_+ + (\bar{\gamma}[i] - \lambda)_+ = \gamma[n] + (\bar{\gamma}[i] - \lambda)_+ - (\gamma[i] - \lambda)_+.$$

If $\gamma[n] \geq \bar{\gamma}[i]$, we find $\sum_{[j] \in N} (\bar{\gamma}[j] - \lambda)_+ \geq \gamma[n]$ because $\bar{\gamma}[i] > \gamma[i]$. If $\gamma[n] < \bar{\gamma}[i]$, we have $\sum_{[j] \in N} (\bar{\gamma}[j] - \lambda)_+ \geq \bar{\gamma}[i]$ but this time because of $\bar{\gamma}(i) \geq \gamma[n] \geq \lambda$ and $\gamma[n] \geq \gamma(i)$. Then $\bar{\lambda} \geq \lambda$ and $\bar{y}_j \leq y_j$ for $j \neq i$. By Pareto optimality and $\bar{c}(N) \geq c(N)$ we find $\bar{y}_i \geq y_i$. ◁

Population monotonicity is another monotonicity property that the Shapley value (and even the weighted Shapley values) shares with the nucleolus and the modified nucleolus (cf. Sprumont (1990), Rosenthal (1990), Sönmez (1993)).

Theorem 9. *For nonnegative airport games the weighted Shapley value, the nucleolus and the modified nucleolus are population monotonic.*

Proof: Let (N,γ) be a nonnegative airport problem and let $[i]$ be the i-th player in N. We consider the airport problem $(N \backslash [i], \gamma_{-[i]})$ after deletion of player $[i]$.

We start with the weighted Shapley value. The airport games belonging to the airport problems are: $\quad (N,c) := \sum_{k=1}^{n} (\gamma[k] - \gamma[k-1])(N, u^*_{[k],\ldots,[n]})$ and

$$(N \backslash [i], c_{-[i]}) := \sum_{k=1}^{n} (\gamma[k] - \gamma[k-1])(N \backslash [i], u^*_{[k],\ldots,[n] \backslash [i]}).$$

If $[j] \neq [i]$ is a player, his payment under the weighted Shapley value is the sum of j terms, namely $\quad (\gamma[k] - \gamma[k-1]) \frac{w[j]}{w(\{[k],\ldots,[n]\})}$ and

$$(\gamma[k]-\gamma[k-1])\frac{w[j]}{w(\{[k],\ldots,[n]\}\backslash[i])} \text{ for } k \leq j.$$

Each of these terms is in the first case smaller or equal to the same term in the second case, because $w(\{[k],\ldots,[n]\}) \geq w(\{[k],\ldots,[n]\}\backslash[i])$. This gives population monotonicity of the weighted Shapley value.

The population monotonicity of the nucleolus of airport games is the main result of Sönmez (1993).

For the modified nucleolus the proof goes like this: Suppose that ${}^{\mu}Nu(N,c)_{[j]} = (\gamma[j]-\lambda)_{+}$. Then $\sum_{j=1}^{n}(\gamma[j]-\lambda)_{+} = \gamma[n]$ and $\sum_{j=1,j\neq i}^{n}(\gamma[j]-\lambda)_{+} = \gamma[n]-(\gamma[i]-\lambda)_{+} \leq \gamma[n]$.

First we assume that $[i] \neq [n]$ or $[i] = [n]$ and $\gamma[n] = \gamma[n-1]$. If player $[i]$ is deleted, the value of λ is too large or just fine. This means that ${}^{\mu}Nu(N\backslash[i],\gamma_{-[i]}) \geq {}^{\mu}Nu(N,\gamma)_{N\backslash[i]}$.

If $[i] = [n]$ and $\gamma[n] > \gamma[n-1]$, the equation above reads like:

$$\sum_{j=1}^{n-1}(\gamma[j]-\lambda)_{+} = \gamma[n]-(\gamma[n]-\lambda)_{+} = \lambda$$

and we have to prove that $\lambda \leq \gamma[n-1]$. If not, then $(\gamma[n]-\lambda)_{+} < \gamma[n]-\gamma[n-1]$ and therefore, the coalition $\{[1],\ldots[n-1]\}$ pays more than $\gamma[n-1]$. Then ${}^{\mu}Nu(N,c)$ is not a core allocation but we know by Theorem 1 that ${}^{\mu}Nu$ is a core selector. ◁

In the following example we show that the nucleolus and the modified nucleolus do not satisfy strong monotonicity.

Example: $N = \{1,2,3\}$, $\gamma[1] = a$ and $\gamma[2] = \gamma[3] = b$ We take three values for a and b. As we can see, the values of a increase and therefore, the marginals of player [1]. The payments of player [1], however, do not (always) increase

a	b	$\phi_{[1]}$	$Nu_{[1]}$	${}^{\mu}Nu_{[1]}$
24	36	8	12	4
27	27	9	9	9
36	72	12	18	0

4. Axiomatizations

Up to now we only considered reduced airport problems, if the first player [1] is removed. In this section we define reduced airport problems in case the payment of

any player $[i]$ is known and this player is removed. We can do this in two essentially different ways. If the payment of player $[i]$ is known to be x, we may consider the *ν-reduced airport problem* with player set $\bar{N} := N \backslash [i]$ and cost function $\bar{\gamma}[j] = \min(\gamma[j], \gamma[i] - x)$ if $j < i$ and $\bar{\gamma}[j] := \gamma[j] - x$ if $j > i$. The payment is used to pay the 'last part' of $\gamma[i]$ and players with small γ-values keep the same value.
We may also consider the *μ-reduced airport problem* by taking $\bar{N} := N \backslash [i]$ and $\bar{\gamma}[j] := (\gamma[j] - x)_+$ if $j < i$ and $\bar{\gamma}[j] := \gamma[j] - x$ if $j > i$. This time the payment of player $[i]$ is used to pay the 'first part' of $\gamma[i]$ and all players get a lower γ-value. If $[i] = [1]$, both definitions give the old definition of reduced airport problem.
For the moment we only consider μ-reduced airport problems for $[i] \neq [n]$ or $[i] = [n]$ and $\gamma[n] = \gamma[n-1]$ (see the remark later on).
A solution rule σ on a subset $\mathcal{B}$ of $\mathcal{A}$ is called *ν-consistent* (ν-CONS) or *μ-consistent* (μ-CONS) if the ν-reduced or μ-reduced airport problem belongs to $\mathcal{B}$ and the solution σ assigns the restriction to $\bar{N}$ of the solution of the original airport problem to the ν-reduced or μ-reduced airport problem, respectively.
Proposition 10. *For nonnegative airport problems the nucleolus has the ν-consistency property (ν-CONS) and the modified nucleolus the μ-consistency property (μ-CONS).*
Proof: (cf. Granot and Maschler (1994) and Granot, Maschler, Owen and Zhu (1995))
We prove that the ν-reduced airport problem is associated with the reduced game in the sense of Davis–Maschler of the game associated with the original airport problem. Then ν-consistency of the nucleolus follows from the reduced game property for the (pre)nucleolus.
If (N, γ) is an airport game and the payment of the i-th player is x $(0 \leq x \leq \gamma[i])$. The TU-game generated by the ν-reduced airport problem has the coalition values for proper coalitions in $N \backslash [i]$ ($\iota(S)$ is the last player of S)

$$\bar{c}(S) := \bar{\gamma}(\iota(S)) = \begin{cases} \min(\gamma[i] - x, \gamma(\iota(S))) & \text{if } \iota(S) \leq [i] \\ \gamma(\iota(S)) - x & \text{if } \iota(S) \geq [i] \end{cases}$$

The D–M reduced game gives to a coalition $S \subset N \backslash [i]$ the coalition value

$$\bar{c}(S) := \min(c(S), c(S \cup [i]) - x) = \begin{cases} \min(\gamma(\iota(S)), \gamma[i] - x) & \text{if } \iota(S) \leq [i] \\ \gamma(\iota(S)) - x & \text{if } \iota(S) \geq [i] \end{cases}$$

These games are the same (also for $S = N \setminus [i]$).

For the modified nucleolus we proceed as follows. Suppose that ${}^{\mu}Nu(N, c) = ((\gamma[j] - \lambda)_+)_{[j] \in N}$ and ${}^{\mu}Nu(N \setminus [i], \bar{c}) = ((\bar{\gamma}[j] - \bar{\lambda})_+)_{j \in N \setminus [i]}$. It suffices to prove that $(\gamma[j] - \lambda)_+ = (\bar{\gamma}[j] - \bar{\lambda})_+$ for $[j] \neq [i]$.

We assume that $[i] \neq [n]$ or $[i] = [n]$ but $\gamma[n] = \gamma[n-1]$.

By definition λ and $\bar{\lambda}$ are solutions of the equations

$$\sum_{j=1}^{n} (\gamma[j] - t)_+ = \gamma[n] \quad \text{and} \quad \sum_{j=1, j \neq i}^{n} (\bar{\gamma}[j] - \bar{t})_+ = \bar{\gamma}[n].$$

The first equation can be written as $\sum_{j=1, j \neq i}^{n} (\gamma[j] - t)_+ = \gamma[n] - (\gamma[i] - t)_+$.

If we take $x := (\gamma[i] - \lambda)_+$, the LHS of the second equation contains terms $(\gamma[j] - x - \bar{t})_+$ and if we take $t = x + \bar{t}$ both LHS's are equal. So we must prove that $\gamma[n] - (\gamma[i] - \lambda)_+ = (\gamma[n] - x)_+$. This is true because of $\gamma[n] \geq \gamma[i] \geq x = (\gamma[i] - \lambda)_+$.

◁

Remark: The only case we did not define μ-reduced games is if $[i] = [n]$ and $\gamma[n] > \gamma[n-1]$. It is not difficult to check that the μ-reduced airport problem game should be defined by $\bar{\gamma}[j] = (\gamma[j] - \bar{x})_+$ with $\bar{x} := x - (\gamma[n] - \gamma[n-1])$ and $x \geq \gamma[n] - \gamma[n-1]$. So, the payment of the deleted player is first used to pay his marginal $(\gamma[n] - \gamma[n-1])$ and the remaining part $\bar{x}$ is used to decrease the costs of the other players. In ν-reduced games it is not necessary to make this exception because then we always start with the marginal. The inequality $x \geq \gamma[n] - \gamma[n-1]$ is always true in core allocations.

In this section we give axiomatizations for the Shapley value, the nucleolus and the modified nucleolus **on the class of airport games.** The axioms that these solution concepts will have in common are: Pareto optimality (PO), equal treatment (ETP), and covariance (COV) (an axiom to be defined). For airport games with at most two players these properties characterize a unique solution rule. For airport problems with one player Pareto optimality fixes the solution $\sigma_{[1]} = \gamma[1]$. For *symmetric* 2-player airport problems (i.e. $\gamma[1] = \gamma[2]$) equal treatment and Pareto optimality fixes the solution $\sigma_{[1]} = \sigma_{[2]} = \frac{1}{2}\gamma[1] = \frac{1}{2}\gamma[2]$. We are left with the airport problems with $\gamma[1] < \gamma[2]$.

For a solution rule σ on a subset $\mathcal{B}$ of airport problems we define *covariance* (COV) by:

(i) If $(N,\gamma) \in \mathcal{B}$, $\alpha > 0$, and $(N,\alpha\gamma) \in \mathcal{B}$, then $\sigma(N,\alpha\gamma) = \alpha\sigma(N,\gamma)$.

(ii) If $(N,\gamma) \in \mathcal{B}$, $\delta \geq \gamma[n-1] - \gamma[n]$, where $\gamma[0] = 0$ in case $n = 1$, and $(N,\bar{\gamma}) \in \mathcal{B}$, where $\bar{\gamma} = \gamma[i] + \delta e_{[n]}$, then $\sigma(N,\bar{\gamma}) = \sigma(N,\gamma) + \delta e_{[n]}$. Here $e_{[i]}$ denotes the i-th unit vector $(0,\ldots,0,1,0,\ldots,0)$.

For more general cost games covariance says how a solution changes if an additive game is added. Moreover, COV guarantees that the solution rule is positively homogeneous. To stay inside the class of airport games only additive games $(0,0,\ldots,0,a)$ can be added with $a \geq \gamma[n-1] - \gamma[n]$, as in airport games only the last player $[n]$ can have a non vanishing marginal.

If we add the covariance property, there is a unique solution for 2-player airport problems: $\sigma_{[1]} = \frac{1}{2}\gamma[1]$ and $\sigma_{[2]} = \gamma[2] - \frac{1}{2}\gamma[1]$.

Remark: A solution rule on a subset $\mathcal{B}$ of $\mathcal{A}$ satisfying COV and WCONS is automatically Pareto optimal. Indeed, for one-person airport problems COV guarantees PO. Note that the global assumption of SIVA is crucial for this property. Let (N,γ) be any airport problem in $\mathcal{B}$ with at least two players. Clearly the solution applied to this airport problem is Pareto optimal if the solution restricted to $N \setminus [1]$ is Pareto optimal for the corresponding reduced airport problem. Therefore an inductive argument and weak consistency finish the proof.

In the following theorem we give axioms for the nucleolus and the modified nucleolus on the class of (nonnegative) airport games. The following proposition is an extension of Proposition 2 and a necessary tool for the proof of Theorem 12. It describes monotonicity properties of the nucleolus and the modified nucleolus.

Proposition 11. *Let (N,γ) be a nonnegative airport problem.*

(a) If t is a number with $0 \leq t \leq \gamma[n]$ and γ_t is the cost function defined by $\gamma_t[i] := \gamma[i] \wedge (\gamma[n] - t)$, then $Nu(N,\gamma_t) \leq Nu(N,\gamma)$.

(b) If t is a number with $0 \leq t \leq \gamma[n]$ and γ_t is a cost function defined by $\gamma_t[i] := (\gamma[i] - t)_+$, then ${}^{\mu}Nu(N,\gamma_t) \leq {}^{\mu}Nu(N,\gamma)$.

(c) If t is a number with $0 \leq t \leq \gamma[1]$ and γ_t is a cost function defined by

$\gamma_t[i] := \gamma[i] - t$, *then* $Nu(N,\gamma_t) \le Nu(N,\gamma)$.

Proof: (a) For airport problems with one player the statement follows trivially from (PO). We assume the claim holds for airport problems with less than k players and suppose that (N,γ) has $k \ge 2$ players. Let $Nu(N,\gamma) = x$ and $Nu(N,\gamma_t) = x_t$. Using the generating function for the nucleolus, we find

$$x[1] = \min_{i<n} \frac{\gamma[i]}{i+1} \ge \min_{i<n} \frac{\gamma[i] \wedge (\gamma[n]-t)}{i+1} = x_t[1].$$

There is equality if and only if $\frac{\gamma[n]-t}{n} \ge x[1]$. If $x[1] = x_t[1]$, weak consistency of the nucleolus (WCONS) gives $x_{\bar{N}} = Nu(\bar{N},\bar{\gamma})$ with $\bar{\gamma}[i] = \gamma[i] - x[1]$ and $(x_t)_{\bar{N}} = Nu(\bar{N},\bar{\gamma}_t)$ with

$$\bar{\gamma}_t[i] = \gamma_t[i] - x[1] = (\gamma[i] - x[i]) \wedge (\gamma[n] - x[i]) - t) = \bar{\gamma}[i] \wedge (\bar{\gamma}[n] - t).$$

The reduced airport problems satisfy the conditions of Proposition 11 (a) too and the induction hypothesis gives the desired result for coordinates of players $[i]$ with $i \ge 2$.

If $x[1] > x_t[1]$, we have $t > \gamma[n] - n\,x[1]$. If we increase t from $t = 0$ to $t = \gamma[n] - n\,Nu(N,\gamma)_{[1]}$, the nucleolus of (N,γ_t) decreases weakly (by the first part of the proof) and for larger values of t we get equal split i.e. $x_t[i] = x_t[j] = \frac{\gamma[n]-t}{n}$ for all i,j. This finishes the proof of part (a).

(b) We have a number λ such that $\sum_{[i]\in N} (\gamma[i] - \lambda)_+ = \gamma[n]$ and we look for a number λ_t with$\sum_{[i]\in N} ((\gamma[i]-t)_+ - \lambda_t)_+ = \gamma[n]-t$. If we subtract the two equalities we find $\sum_{[i]\in N} [(\gamma[i] - \lambda)_+ - (\gamma[i] - (t+\lambda_t))_+] = t$. Then

$(\gamma[i] - \lambda)_+ > \gamma[i] - (t+\lambda_t))_+$ for some player $[i] \in N$ and therefore,$\lambda < t + \lambda_t$.

Then $(\gamma[j] - \lambda)_+ \ge (\gamma[j] - (t+\lambda_t))_+$ for all players $[j] \in N$.

This finishes the proof of (b).

(c) The proof follows from Proposition 2, if we show that

$$g(N,\gamma) - t \le g(N,\gamma_t) \le g(N,\gamma).$$

where g is the generating function of the nucleolus. This amounts to

$$\min_{i<n} \frac{\gamma[i]}{i+1} - t \le \min_{i<n} \frac{\gamma[i]-t}{i+1} \le \min_{i<n} \frac{\gamma[i]}{i+1}.$$

This is clearly true. ◁

Theorem 12. *The nucleolus is the unique solution rule on the class of nonnegative airport problems satisfying the equal treatment property (ETP), covariance (COV), and ν-consistency (ν-CONS). The modified nucleolus is the unique solu-*

tion rule on the class of nonnegative airport problems satisfying ETP, COV, and μ-CONS.

Proof: The proof of both statements has the same structure. Let us abbreviate the notation for a moment by $\sigma := \sigma(N,\gamma)$ and $\nu := Nu(N,\gamma)$ (or $= {}^{\mu}Nu(N,\gamma)$). Let (N,γ) be an airport problem with a minimal number of players with $\sigma \neq \nu$. If $\sigma[i] = \nu[i]$, the ν-(or μ-)reduced game is also a game with $\sigma \neq \nu$. So we may assume that all coordinates of σ and ν are different.

If $\sigma[1] < \nu[1]$, we prove, by using Proposition 11, that $\sigma_{N\setminus[1]} > \nu_{N\setminus[1]}$. Then in particular, $\sigma[n] > \nu[n]$. From this fact we prove, again by using Proposition 11, that $\sigma_{N\setminus[n]} < \nu_{N\setminus[n]}$. Then $\sigma[n-1] > \nu[n-1]$ and $\sigma[n-1] < \nu[n-1]$.

We start with the nucleolus. Suppose that σ is a solution rule for airport problems satisfying COV, ETP and ν–CONS, hence WCONS and thus PO by the last remark. If $\sigma \neq Nu$, there is a number k and an airport problem (N,γ) with k players such that $\sigma(N,\gamma) \neq Nu(N,\gamma)$ and for all airport problems with less players the rules σ and Nu give the same outcome. We know that $k \geq 3$. Let x be $\sigma(N,\gamma)$ and let y be $Nu(N,\gamma)$. We may assume that $x[i] \neq y[i]$ for every player $[i]$. Suppose that $x[1] > y[1]$. If we consider the reduced airport problems with respect to $x[1]$ and $y[1]$ (notation: $(\bar{N},\bar{\gamma}_x)$ and $(\bar{N},\bar{\gamma}_y)$), we get $\bar{\gamma}_x[i] = \gamma[i] - x[1] = \gamma[i] - y[1] - t$ with $t = x[1] - y[1]$. We may apply Proposition 11 (c) and find $Nu(\bar{\gamma}_x) \leq Nu(\bar{\gamma}_y)$. By weak consistency (WCONS) for Nu and the induction hypothesis we find $x_{\bar{N}} \leq \sigma(\bar{\gamma}_y)$. Weak consistency for σ gives $x_{\bar{N}} \leq y_{\bar{N}}$. Therefore, $x[n] \leq y[n]$. As we assumed no equality, we have $x[n] < y[n]$. If we look at the two ν–reductions of the problem to $N\setminus[n]$ with respect to $x[n]$ and $y[n]$ (notation $(\tilde{N},\tilde{\gamma}_x)$ and $(\tilde{N},\tilde{\gamma}_x)$), we get the cost functions $\tilde{\gamma}_x[i] := \gamma[i] \wedge (\gamma[n] - x[n])$ and $\tilde{\gamma}_y[i] := \gamma[i] \wedge (\gamma[n] - y[n]) = \gamma[i] \wedge (\gamma[n] - x[n] - \tilde{t})$ with $\tilde{t} := y[n] - x[n]$. Application of Proposition 11, (a) gives $Nu(\tilde{\gamma}_x) \geq Nu(\tilde{\gamma}_y)$. By the induction hypothesis and ν-consistency, we obtain $x_{\tilde{N}} = Nu(N,\gamma)_{\tilde{N}} \geq \sigma(\tilde{\gamma}_y) = y_{\tilde{N}}$. In fact, $x_{\tilde{N}} > y_{\tilde{N}}$. For player $[n-1]$ we find $x[n-1] > y[n-1]$ and $x[n-1] < y[n-1]$. Contradiction!

For the modified nucleolus the proof is similar. Using the μ-consistency and Proposition 2, we find, as before, $\sigma(N,\gamma)_{[1]} > {}^{\mu}Nu(N,\gamma)_{[1]}$ implies $\sigma(N,\gamma)_{[n-1]} \leq {}^{\mu}Nu(N,\gamma)_{[n-1]}$ as well as $\sigma(N,\gamma)_{[n]} \leq {}^{\mu}Nu(N,\gamma)_{[n]}$ and the latter inequality implies $\sigma(N,\gamma)_{[n-1]} \geq {}^{\mu}Nu(N,\gamma)_{[n-1]}$. This means $\sigma(N,\gamma)_{[n-1]} = {}^{\mu}Nu(N,\gamma)_{[n-1]}$.

The induction hypothesis and μ–consistency gives $\sigma(N,\gamma) = {}^{\mu}Nu(N,\gamma)$. ◁

Theorem 13. *The symmetric Shapley value is the unique solution rule on the class of nonnegative airport problems satisfying Pareto optimality, the equal treatment property, and strong monotonicity.*

Proof: For an airport problem (N,γ) we introduce $h(N,\gamma) := \#\{[i] \mid \gamma[i] < \gamma[n]\}$. The proof is by induction to $h(N,\gamma)$. If $h(N,\gamma) = 0$, we have a completely symmetric situation and $\sigma_{[i]} = \frac{\gamma[n]}{n}$ for all players $[i] \in N$ by PO and ETP. The same is true for the Shapley value. If the theorem holds for $h(N,\gamma) < k$ and (N,γ) is an airport problem with $h(N,\gamma) = k$, we have $\gamma[1] \le \gamma[2] \le \cdots \le \gamma[k] < \gamma[k+1] = \cdots = \gamma[n]$. If we define $\bar{\gamma}$ by $\bar{\gamma}[i] := \gamma[i] \wedge \gamma[k]$, we have by the induction hypothesis $\sigma(N,\bar{\gamma}) = \phi(\bar{\gamma})$. Notice that the marginals of the players $[i]$ with $i \le k$ do not change if we go from γ to $\bar{\gamma}$ and therefore, $\sigma(N,\gamma)_{[i]} = \sigma(N,\bar{\gamma})_{[i]} = \phi(N,\bar{\gamma})_{[i]} = \phi(N,\gamma)_{[i]}$ for $i \le k$ by strong monotonicity of σ and ϕ. The equal treatment property gives $\sigma(N,\gamma)_{[j]} = \sigma(N,\gamma)_{[n]}$ and $\phi(N,\gamma)_{[j]} = \phi(N,\gamma)_{[n]}$ for $j > k$ and Pareto optimality give $\sigma(N,\gamma)_{[n]} = \phi(N,\gamma)_{[n]}$. ◁

To finish this paper we will summarize the results in the following table showing the interdependencies and independencies of properties discussed in this paper. We use the abbreviations used in this paper and, moreover, (FR) for fair ranking, (CM) for coalitional monotonicity, (SM) for strong monotonicity and (PM) for population monotonicity. We assume that $|\Omega| \ge 3$. The solution rule ρ is the–not REAS–rule introduced in Section 1. Notice the interdependencies for solution rules on $\mathcal{A}_+$

FR⇒ETP, SM⇒CM, ν–CONS⇒WCONS and μ–CONS⇒WCONS.

	PO	ETP	COV	SM	ν-CONS	μ-CONS	FR	CM	PM	REAS	WCONS
ϕ	+	+	+	+	−	−	+	+	+	+	+
Nu	+	+	+	−	+	−	+	+	+	+	+
${}^{\mu}Nu$	+	+	+	−	−	+	+	+	+	+	+
τ	+	+	+	−	−	−	+	(+)	(−)	+	−
ϕ_w	+	−	+	+	(−)	(−)	−	+	+	+	+
ρ	+	+	+	−	−	−	+	(+)	(−)	−	+

For the Shapley value ϕ we proved the positive results. This solution rule cannot be ν– or μ–consistent by Theorem 12 and $\phi \ne Nu, {}^{\mu}Nu$ (see the table at the end of Section 3). For the nucleolus and the modified nucleolus we proved the positive results and we gave an example showing that these solution rules are not strongly monotonic. The other negative results follow from Theorem 12 and the fact that $Nu \neq {}^{\mu}Nu$. For the τ-value τ we proved that it does not satisfy WCONS and therefore, the other consistency properties either. It cannot be strongly monotonic by Theorem 13. The positive results for the weighted Shapley value ϕ_w (we assume that the weight are not all equal) are proved in the paper, and also $\neg$FR and $\neg$ETP. The solution rule ρ was designed to violate REAS. It cannot satisfy SM by Theorem 13 nor ν-CONS or μ-CONS by Theorem 12, if it satisfies COV. It satisfies COV, as the 'discount' λ is not dependent on $\gamma[n]$ (see the formula in Section 1). The brackets in the table are the properties that are not discussed in the paper. Probably the weighted Shapley values are neither ν- nor μ-consistent and τ and ρ are coalitionally monotonic but not population monotonic.

The Theorems 12 and 13 axiomatize the Shapley value and both nucleoli solutions each by three properties, respectively. To show this we need solution rules that satisfy all characterizing properties except one. Again, $|\Omega| \geq 3$ is assumed.

	PO	ETP	COV	SM	ν-CONS	μ-CONS	WCONS
ϕ	+	+	+	+	−	−	+
Nu	+	+	+	−	+	−	+
${}^{\mu}Nu$	+	+	+	−	−	+	+
σ_1	+	+	−	−	−	+	+
σ_2	−	+	+	+	−	−	−
σ_3	+	+	−	−	+	−	+
σ_4	+	−	−	+	+	+	+
σ_5	+	−	+	−	+	−	+
σ_6	+	−	+	−	−	+	+

The solution rule σ_1 is defined by $\sigma_1(N,\gamma)_{[i]} = 0$ if $\gamma[i] < \gamma[n]$ and $\sigma_1(N,\gamma)_{[i]} = \dfrac{\gamma[n]}{p}$ if $\gamma[i] = \gamma[n]$ and $p := |\{i \mid \gamma[i] = \gamma[n]\}|$.

The solution rule σ_2 is defined by $\sigma_2(N,\gamma)_{[i]} = 0$ for $[i] \neq [n]$ and $\sigma_2(N,\gamma)_{[n]} = \gamma[n] - \gamma[n-1]$ where $\gamma[0] = 0$.

The solution rule σ_3 is defined via its generating function $g(N,\gamma) = \min_{k\in N}(\gamma[k]/k)$. By Theorem 1 this solution rule applied to every nonnegative airport problem yields an element of the core of the corresponding cost game. An inductive argument on the number of players shows that this element is lexicographically maximal within those core elements satisfying the fair ranking property.

The solution rule σ_4 is defined by $\sigma_4(N,\gamma)_{[i]} = 0$ for $[i] \neq [n]$ and $\sigma_4(N,\gamma)_{[n]} = \gamma[n]$. The solution rule σ_5 depends on an exceptional potential player $\star$ in the universal player set Ω and is defined by distinguishing the following cases. If $\star \notin N$, then $\sigma_5(N,\gamma) = Nu(N,\gamma)$. If $\star \in N$ and $(N,\gamma) \in \mathcal{A}_+$, define $\sigma_5(N,\gamma)_i = Nu(N\setminus\{\star\},\bar{\gamma})_i$ for $i \neq \star$ and define $\sigma_5(N,\gamma)_\star$ to be the unique real number such that σ_5 satisfies PO. Here $\bar{\gamma}$ denotes the restriction of γ to $N \setminus \{\star\}$. Note that in the second case the exceptional player only pays something if he is the last player. Then he pays $\gamma[n] - \gamma[n-1]$. Moreover, it should be remarked that the $\nu-$ or $\mu-$reduced airport problem with respect to the exceptional player coincides with the restriction of the airport problem to all other players.

The solution rule σ_6 is defined analogously to σ_5 by interchanging the roles of Nu and ${}^{\mu}Nu$.

It is straightforward to verify the properties of the solution rules on $\mathcal{A}_+$ summarized in the table. Therefore Nu, σ_2, σ_4 show the logical independence of PO, ETP, and SM, whereas ϕ, σ_3, σ_5 or ϕ, σ_1, σ_6 show that ETP, COV, and ν-CONS or ETP, COV, and μ-CONS, respectively, are logically independent.

Remarks: In this paper single valuedness has been used as a global assumption for solution rules. This assumption can be avoided with the help of some modifications. A *set valued* solution rule on a set $\mathcal{B}$ of games or airport problems assigns a set of payment vectors (instead of a singleton) to each member of $\mathcal{B}$. All consistency properties mentioned so far can be redefined for set valued solution rules. Indeed, a set valued solution σ satisfies *consistency*, if the reduced problems with respect to each member of the solution belong to the domain of σ and the resticted vectors belong to the solutions of the reduced problems or games. For the definition of a stronger version (not used in this context) of consistency for set valued solution rules see Yanovskaya (1994). Pareto optimality, reasonableness, the equal treatment property, fair ranking, and covariance, can be generalized to set valued solution rules by demanding the foregoing properties for every element of the solution of a given problem.

For set valued solutions Theorems 1 and 12 remain valid, if single valuedness is added as an assumption. In Theorem 12 SIVA can be replaced by nonemptiness and PO. In this version of the result only Pareto optimality for one-person problems is really needed and both characterizations are, again, axiomatizations. This can be seen by considering the empty solution which shows that the nonemtiness assumption is logically independent. The solution which assigns the set of real numbers to every one-person airport problem and the nucleolus or modified nucleolus respectively to every other airport problem shows that PO is needed.

It should be noted that the characterization of the nucleolus rule is in fact an axiomatization of the kernel, defined by Davis and Maschler (1965) (cf. Maschler, Peleg, and Shapley (1979)). The kernel is a singleton, hence coincides with the nucleolus, in this case since the corresponding cost games under consideration are concave (see Maschler, Peleg, and Shapley (1972)). This is the reason why the infinity assumption on the universe Ω of players, which is a necessary condition in Sobolev's (1975) axiomatization of the prenucleolus (cf. Sudhölter (1993)), can be avoided. An alternative proof of this part of Theorem 12 can be given by showing that the solution has to be contained in the kernel (cf. Peleg (1986)). It should be noted that the same properties, only ν-CONS has to be replaced by consistency w.r.t. D-M reduced games) yield an axiomatization of the nucleolus for the set of concave cost games with player set contained in Ω.

The assertions of Theorems 3, 8, 13 concerning the (positively weighted) Shapley value remain valid on the set $\mathcal{A}$ of all airport problems, since nonnegativity does not apply in the corresponding proofs.

There is a generalization of the concept of positively weighted Shapley values. For an index subset I of the natural numbers and an ordered partition $\mathcal{S} = (S_j)_{j \in I}$ of Ω the $(w, \mathcal{S})$*-weighted Shapley value* $\phi_{w,\mathcal{S}}$ is the linear solution rule defined by $\phi_{w,\mathcal{S}}(N, u_S^*)_i := \dfrac{w(i)}{w(\tilde{S})}$ if $i \in \tilde{S}$ and $\phi_{w,\mathcal{S}}(N, u_S^*)_i := 0$ if $i \notin \tilde{S}$, where $\tilde{S} = S \cap S_{max\{j \in I | S_j \cap S \neq \emptyset\}}$.

Analogously to the proof of Theorem 3 it can be seen that this solution is weakly consistent and generated by $g(N, \gamma) := \dfrac{w[1]}{w(\tilde{S})} \gamma[1]$ if $[1] \in \tilde{S}$ and $g(N, \gamma) := 0$ otherwise. Moreover, all other results of the preceding sections according to **positively** weighted Shapley values remain valid for weighted Shapley values.

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