INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 273

An Extension of the Raiffa-Kalai-Smorodinsky Solution to Bargaining Problems with Claims

by

Anke Gerber

June 1997

University of Bielefeld

33501 Bielefeld, Germany

An Extension of the Raiffa–Kalai–Smorodinsky Solution to Bargaining Problems with Claims

Anke Gerber

June 1997

Abstract

We attempt to unify the theory of bargaining problems with and without claims. The claims of the players define a natural status quo for the underlying bargaining problem which can therefore be solved using a solution for bargaining problems (without claims). Following this procedure we define an extension of the Raiffa–Kalai–Smorodinsky solution to bargaining problems with claims. It is shown that this solution is uniquely characterized by a set of axioms including a monotonicity axiom.

1 Introduction

Recently the theory of bargaining problems originating in Nash's seminal paper [9] has been enriched by the introduction of a claims point which is not an element of the feasible set (see Chun and Thomson [3], Bossert [2]). Various interpretations of the claims point have been provided by the literature, the most prominent one referring to bankruptcy problems as studied in O'Neill [10], Aumann and Maschler [1], Curiel, Maschler and Tijs [4] and Dagan and Volij [5]. If a firm goes bankrupt the estate typically is not large enough to meet the creditors' claims. Thus, the estate has to be divided in a way that takes into account the creditors' claims without being able to satisfy them all. Since it is assumed that the creditors' utility is a linear function of money a bankruptcy problem belongs to the class of problems with transferable utility. By using game theoretical methods the authors cited above try to rationalize and generalize the ancient rules for dealing with such problems that can be found in the Babylonian Talmud.

Another problem that can be modelled as a bargaining problem with claims is a labor–management conflict about the division of the profits of a firm. In this case the disagreement point is given by the "worst case scenario" of a strike or a lockout and the claims of the two parties are given by their expectations about how the conflict should be solved. These expectations might have been formed on the ground of the parties' payoffs in similar past situations, where changes in the feasible set make them incompatible in the current situation. Or the expectations might be based on the resolution of the conflict in related industries, where differences in the feasible set cause the payoffs to be incompatible in the present conflict. Note that we do not address the question of credibility of the claims point. Rather, the claims point is taken as exogenously given as are the disagreement point and the feasible set.

Bargaining problems with claims can also be used to model coalition formation and payoff distribution in a general NTU game (see Gerber [6]). In each coalition the players' payoffs should depend on what they can achieve outside this coalition. If these "outside opportunities" are not feasible for the given coalition we are in a situation that can be modelled by a bargaining problem with claims.

There is a close connection between bargaining problems with claims and location games or, more general, games with a *bliss point*. A location conflict arises if e.g. a planning agency has to decide about the location of an attractive (or undesirable) object and the utility functions of the agents do not have a common satiation point. Thus, the satiation points of the agents define a bliss point which is not feasible and therefore is similar in spirit to a claims point. However, the additional structural element of a disagreement point is missing in a location conflict and this is the main difference to a situation we model by a bargaining problem with claims. Nevertheless, the general problem of finding a "fall-back" position within the feasible set is the same. For a treatment of this problem in the context of games with a bliss point we refer to Rosenmüller [14] who proposes an extension of the Nash solution to this class of games.

The literature so far has treated the theory of bargaining problems with and without claims as essentially different providing separate solutions to both problems. This does not seem to be satisfactory since the only difference is the existence of a claims point in one of the problems. Thus, two interesting questions can be raised, both of which we try to answer in this paper. 1. Is there a canonical way in which we can derive a bargaining problem without claims from a bargaining problem with claims, so that we can find a solution to the latter by applying some well known solution concept for the former and 2. is there a canonical way in which one can embed the class of bargaining problems without claims into the class of bargaining problems with claims, so that one can look for an extension of bargaining solutions to the class of bargaining problems with claims?

The answer to the first question is provided by deriving from the claims point an adjusted threatpoint for the underlying bargaining problem taking into account the original threatpoint of the bargaining problem with claims. This new threatpoint gives each player the maximum utility the other players unanimously concede to him and it thus can be interpreted as a minimally equitable agreement. Having derived a bargaining problem without claims from the given problem with claims a natural next step is to apply some well known solution concept to the former. In this paper we concentrate on the Raiffa–Kalai–Smorodinsky (RKS) solution and show that on the class of bargaining problems with claims the extended RKS solution is uniquely characterized by a set of axioms including a monotonicity axiom.

The answer to the second question provides a justification for the term "extended" that we use for the solution we have derived above. We show that the RKS solution for a bargaining problem without claims is identical to the extended RKS solution for the equivalent bargaining problem with claims where equivalence refers to the embedding procedure we define.

It turns out that the extended RKS solution assigns payoffs to the players that in general are not bounded by the claims point. Since this is an undesirable property in some situations that we model by a bargaining problem with claims — though by far not in all — we also propose a modification of the extended RKS solution which satisfies boundedness by claims. This solution is given by the RKS solution applied to the bargaining problem that arises if we restrict the feasible set to those utility allocations that are bounded by the claims point and adjust the threatpoint as before. Under a slight modification of two axioms we also obtain a characterization of the claims–bounded extended RKS solution.

The idea to derive a bargaining problem without claims from a bargaining problem with claims and then to apply a well known solution concept to the former is not completely new. In fact Dagan and Volij [5] proceeded in this way in the special case of a bankruptcy problem. They show that the RKS solution applied to the deduced bargaining problem, where the feasible set restricted to the payoffs that are bounded by the claims and the threatpoint is defined as in our paper, induces the *adjusted proportional rule*. The adjusted proportional rule, which is an extension of the *contested garment principle* that can be found in the Talmud, has been introduced and axiomatically characterized by Curiel, Maschler and Tijs [4]. Thus, the claims–bounded extended RKS solution is an extension of the contested garment principle to general (not necessarily bankruptcy) problems with claims.

The paper is organized as follows. Section 2 provides the basic definitions and points out the connection between bargaining problems with and without claims. In section 3 we present the characterization of the extended RKS solution. Section 4 deals with the boundedness of a solution by the claims point. The claims–bounded extended RKS solution is defined and characterized. We discuss the relation between this solution and division rules for bankruptcy games. Finally, section 5 closes the paper with some concluding remarks.

2 Bargaining Problems with and without Claims

In the following $\mathbf{R}^n$, $n \in \mathbb{N}$, will denote the n-dimensional euclidean space and the set $N = \{1, \ldots, n\}$, $n \geq 2$, will denote the *player set*. The notation for vector inequalities is $\geq$, $>$, $\gg$. For $x \in \mathbf{R}^n$, $a \in \mathbf{R}$, and $i \in N$ the vector $(a, x_{-i}) \in \mathbf{R}^n$ is defined to be the vector $(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)$. The *comprehensive convex hull* of the vectors $a^1, \ldots, a^k \in \mathbf{R}^n$ is given by

$$\text{CoCon}\{a^1, \ldots, a^k\} = \left\{ x \in \mathbf{R}^n \,\middle|\, x \leq \sum_{i=1}^{k} \lambda_i a^i,\ \lambda_i \geq 0,\ i = 1, \ldots, k,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

For $S \subseteq \mathbf{R}^n$ let

$$\text{WPO}(S) = \{x \in S \mid y \in \mathbf{R}^n,\ y \gg x \Rightarrow y \notin S\}$$

be the set of *weakly Pareto optimal* points in S and let

$$\text{PO}(S) = \{x \in S |\ y \in \mathbf{R}^n, y > x \Rightarrow y \notin S\}$$

be the set of *Pareto optimal* points in S.

Definition 2.1 *An* **n–person bargaining problem** *is a tuple* (S, d), *where*

1. $S \subseteq \mathbf{R}^n$ *is convex, closed and comprehensive.*[1]

2. $d \in S$ *and* $\exists\, x \in S$ *with* $x \gg d$.

3. $\{x \in S |\ x \geq d\}$ *is bounded.*

Given a bargaining problem (S, d) the set S is the set of feasible utility allocations for N and d is the threatpoint or status quo which marks the outcome of the game if the players cannot agree on a feasible point. The conditions in Definition 2.1 are standard. Let $\mathbf{\Sigma}^n$ be the class of all n–person bargaining problems. For a bargaining problem $(S, d) \in \mathbf{\Sigma}^n$ let

$$\text{IR}(S, d) = \{x \in S |\ x \geq d\}$$

be the set of *individually rational* points in S. The class $\tilde{\mathbf{\Sigma}}^n \subseteq \mathbf{\Sigma}^n$ is given by

$$\tilde{\mathbf{\Sigma}}^n = \{(S, d) \in \mathbf{\Sigma}^n |\ \text{WPO}(S) \cap \text{IR}(S, d) \subseteq \text{PO}(S)\}.$$

For $(S, d) \in \mathbf{\Sigma}^n$ the *utopia point* $u(S, d)$ is defined by

$$u_i(S, d) = \max\{x_i |\ x \in \text{IR}(S, d)\},\ i = 1, \ldots, n.$$

A *solution* on a class of bargaining problems $\mathbf{D}^n \subseteq \mathbf{\Sigma}^n$ is a mapping $f : \mathbf{D}^n \to \mathbf{R}^n$ such that $f(S, d) \in S$ for all $(S, d) \in \mathbf{D}^n$. For a given bargaining problem (S, d) the *Raiffa–Kalai–Smorodinsky solution* selects the weakly Pareto optimal point on the line connecting the disagreement point with the utopia point $u(S, d)$. For the case of $n = 2$ the Raiffa–Kalai–Smorodinsky solution was proposed by Raiffa [12] and was later axiomatically characterized by Kalai and Smorodinsky [8].

[1] A set $S \subseteq \mathbf{R}^n$ is *comprehensive* if $x \in S$ and $y \leq x$ implies that $y \in S$.

Definition 2.2 $k : \Sigma^n \to \mathbf{R}^n$ *is the* **Raiffa–Kalaï–Smorodinsky (RKS) solution** *if for* $(S,d) \in \Sigma^n$

$$k(S,d) = (1-\bar{\lambda})d + \bar{\lambda}u(S,d),$$

where $\bar{\lambda} = \max\{\lambda \in \mathbf{R} |\ (1-\lambda)d + \lambda u(S,d) \in S\}$.

As outlined in the introduction there are situations in which the players' claims or expectations should be considered. To this end we define a bargaining problem with claims as follows.

Definition 2.3 *An* **n–person bargaining problem with claims** *is a triple* (S,d,c), *where*

1. $(S,d) \in \Sigma^n$.
2. $c \in \mathbf{R}^n \setminus S,\ c \gg d$.

Let Σ^n_c be the class of all n-person bargaining problems with claims. A *solution* on a class of bargaining problems with claims $\mathbf{D}^n_c \subseteq \Sigma^n_c$ is a mapping $F : \mathbf{D}^n_c \to \mathbf{R}^n$ such that $F(S,d,c) \in S$ for all $(S,d,c) \in \mathbf{D}^n_c$. The class $\tilde{\Sigma}^n_c \subseteq \Sigma^n_c$ is given by

$$\tilde{\Sigma}^n_c = \{(S,d,c) \in \Sigma^n_c |\ (S,d) \in \tilde{\Sigma}^n\}.$$

Now we can define the extended RKS solution (see Figure 1).

Definition 2.4 *The* **extended RKS solution** $K : \tilde{\Sigma}^n_c \to \mathbf{R}^n$ *is given by*

$$K(S,d,c) = k\left(S, t(S,d,c)\right),\ (S,d,c) \in \tilde{\Sigma}^n_c,$$

where $t_i(S,d,c) = \max\left\{d_i, \max\{x_i |\ (x_i, c_{-i}) \in S\}\right\},\ i = 1,\ldots,n.$[2]

Observe that the extended RKS solution is well defined: For $(S,d,c) \in \tilde{\Sigma}^n_c$ and $t = t(S,d,c)$ either $t = d$ or there exists $i \in N$ such that $t_i > d_i$. Then $(t_i, c_{-i}) \in S$ and since $t_j \leq c_j$ for all $j = 1,\ldots,n$ we get $(t_i, c_{-i}) \geq t$. Therefore,

[2] By definition $\max(\emptyset) = -\infty$.

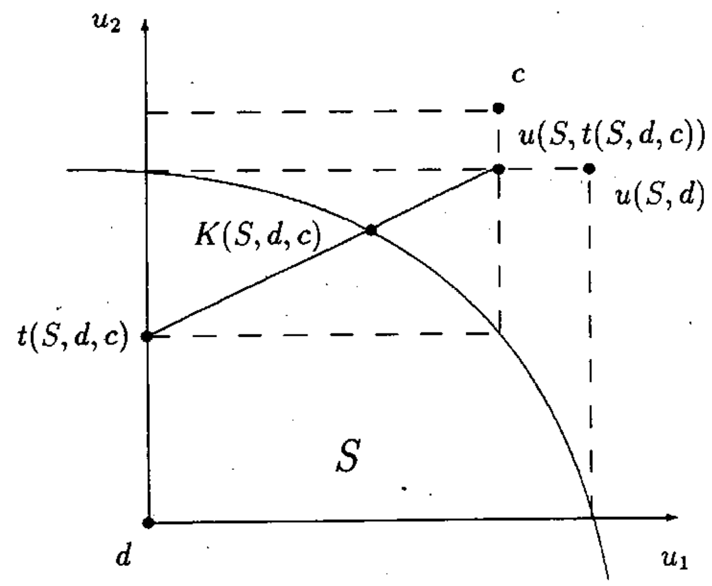

Figure 1: The extended Raiffa–Kalai–Smorodinsky solution.

by comprehensiveness of S it is true that $t \in S$. Further, there exists $x \in S$ such that $x \gg t$. Suppose not, then $t \in \mathrm{PO}(S)$ and therefore $t_i > d_i$ for some $i \in N$. By definition of t_i this implies $(t_i, c_{-i}) \in \mathrm{PO}(S)$ and we conclude that $t_j = c_j$ for all $j \neq i$. Again by definition of t_j and since $c \notin S$ this is only possible if $c_j = d_j$ for all $j \neq i$ which contradicts $c \gg d$. Also, since $t(S, d, c) \geq d$ the set $\{x \in S | \ x \geq t(S, d, c)\}$ is bounded. This implies $(S, t(S, d, c)) \in \tilde{\Sigma}^n$.

The interpretation of the new adjusted threatpoint or better status quo $t(S, d, c)$ is straightforward. No player can expect someone else to settle with less than the amount that is necessary to satisfy the claims of the other players and no rational player will accept any payoff below the disagreement utility level. Thus, all players will agree that the final outcome should not leave any player i with less than $t_i(S, d, c)$. In the context of bankruptcy problems Curiel, Maschler and Tijs [4] call $t_i(S, d, c)$ the "minimal right of claimant i", i.e. "the amount that is not claimed by any of the others". As Herrero [7] puts it: $t(S, d, c)$ "represents a 'natural concession' from coalition $N \setminus \{i\}$ to agent i." Following Herrero we will call $t(S, d, c)$ the *minimally equitable agreement.* [3]

[3] Herrero [7] presents a unifying approach to bargaining problems with a reference point and bargaining problems with a claims point. The threatpoint $t(S, d, c)$ in her paper plays the role of a natural reference point thus mapping any bargaining problem with claims to a bargaining

The definition of the extended RKS solution consists of two steps. First the original bargaining problem with claims is transformed into a problem without claims by adjusting the threatpoint in a way that takes into account the claims point. In the second step the RKS solution is applied.

Note that the utopia point $u(S, t(S, d, c))$ in general neither coincides with $u(S, d)$ (cf. Figure 1) nor has the property that $u(S, t(S, d, c)) \leq c$ (see the following example). The latter is only true if $n = 2$.

Example 2.5 Let $n = 3$ and let $S = \text{CoCon}\{(1,0,0),(0,2,0),(0,0,3)\}$, $d = (0,0,0)$, $c = (0.5, 1, 1.5)$. Then $t(S, d, c) = d = (0,0,0)$ and $u(S, t(S, d, c)) = (1,2,3) \gg c$.

Up to now we have argued that there is a natural way in which one can deduce a bargaining problem without claims from a problem with claims $(S, d, c) \in \tilde{\Sigma}^n_c$. We now pose the reverse question: Is there a natural way in which one can embed $\tilde{\Sigma}^n$ into $\tilde{\Sigma}^n_c$?

Remark 2.6

1. $\tilde{\Sigma}^n$ can be embedded into $\tilde{\Sigma}^n_c$ in the following way. Let $(S, d) \in \tilde{\Sigma}^n$. Then $u(S, d) \notin S$, $u(S, d) \gg d$, and therefore $(S, d, u(S, d)) \in \tilde{\Sigma}^n_c$. Since $t(S, d, u(S, d)) = d$ the game $(S, d, u(S, d))$ is an embedding of (S, d) into $\tilde{\Sigma}^n_c$.

2. Given the embedding defined above K is indeed an extension of the RKS solution $k : \tilde{\Sigma}^n \to \mathbf{R}^n$ to bargaining problems with claims: It is straightforward to see that for $(S, d) \in \tilde{\Sigma}^n$

$$K(S, d, u(S, d)) = k(S, d).$$

We believe that the embedding of $\tilde{\Sigma}^n$ into Σ^n_c we have defined in Remark 2.6 is natural since the utopia point is usually interpreted as a claims point for the underlying bargaining game. It represents the maximal expectations the players can have about their payoffs if they believe that the others are rational, meaning that the others will not accept anything less than their disagreement level. However, if we only impose the condition that $t(S, d, c) = d$ for regarding

---

problem with reference point.

$(S,d,c) \in \tilde{\Sigma}^n_c$ as an embedding of $(S,d) \in \tilde{\Sigma}^n$ into $\tilde{\Sigma}^n_c$ there are several other possibilities for choosing the claims point c as we will see in the following.

Let $(S,d) \in \tilde{\Sigma}^n$. For $i = 1, \ldots, n$ define

$$C_i = \{x \in \mathbf{R}^n | \; (d_i, x_{-i}) \in \mathrm{PO}(S) \cap \mathrm{IR}(S,d)\}$$

and $c(S,d) = \bigcap_{i=1}^n C_i$. It is easy to see that given the conditions on (S,d) the set $c(S,d)$ is nonempty and defines a unique point in $\mathbf{R}^n$.[4] Also, $c(S,d) \notin S$, $c(S,d) \gg d$ and therefore $(S,d,c(S,d)) \in \tilde{\Sigma}^n_c$. We have the following lemma.

Lemma 2.7 *1. Let $(S,d,c) \in \tilde{\Sigma}^n_c$ be such that $c \geq c(S,d)$. Then*

$$t(S,d,c) = d.$$

2. Let $(S,d,c) \in \tilde{\Sigma}^n_c$ be such that $c < c(S,d)$. Then

$$t(S,d,c) > d.$$

Proof:

1. The claim directly follows from the fact that $(d_i, c_{-i}(S,d)) \in \mathrm{PO}(S)$.
2. Let $(S,d,c) \in \tilde{\Sigma}^n_c$ be given with $c < c(S,d)$. Let $j \in N$ be such that $c_j < c_j(S,d)$ and let $i \neq j$. Then $(d_i, c_{-i}) < (d_i, c_{-i}(S,d)) \in \mathrm{PO}(S)$ and therefore $t_i(S,d,c) > d_i$.

Q.E.D.

The lemma shows that in principle any $c \geq c(S,d)$ but no $c < c(S,d)$ could define an embedding of $(S,d) \in \tilde{\Sigma}^n$ into $\tilde{\Sigma}^n_c$. Nevertheless, the embedding defined by choosing $c = u(S,d)$ seems to be the most natural one.

[4] If $n = 2$ then $c(S,d) = u(S,d)$. However, in general it is only true that $c(S,d) \leq u(S,d)$ as Example 2.5 shows. In this example $c(S,d) = c$ but $u(S,d) = (1,2,3)$.

3 Characterization of the Extended RKS Solution

In this section we will show that the extended RKS solution is uniquely characterized by a set of axioms. Since this characterization is a consequence of the axiomatization of the RKS solution for bargaining games we first recall the latter. In order to do so we need some additional definitions (cf. Rosenmüller [13], Def. 2.4 and Remark 2.8).

Let $\pi : N \to N$ be a permutation. Then π also induces a mapping $\pi : \mathbf{R}^n \to \mathbf{R}^n$, which we denote by the same symbol, via

$$\pi(x)_i = x_{\pi^{-1}(i)},\ i = 1, \ldots, n.$$

Again using the same symbol, π defines mappings $\pi : \mathbf{\Sigma}^n \to \mathbf{\Sigma}^n$, $\pi : \mathbf{\Sigma}_c^n \to \mathbf{\Sigma}_c^n$, respectively, via

$$\pi(S,d) = (\pi(S), \pi(d)),\ (S,d) \in \mathbf{\Sigma}^n,$$

$$\pi(S,d,c) = (\pi(S), \pi(d), \pi(c)),\ (S,d,c) \in \mathbf{\Sigma}_c^n,$$

respectively.

A mapping $L : \mathbf{R}^n \to \mathbf{R}^n$ is a *positive affine transformation* if there exist $a, b \in \mathbf{R}^n$, $a \gg 0$, such that for all $x \in \mathbf{R}^n$ and for all $i \in N$, $L_i(x) = a_i x_i + b_i$. Using the same symbol, any positive affine transformation L also induces mappings $L : \mathbf{\Sigma}^n \to \mathbf{\Sigma}^n$, $L : \mathbf{\Sigma}_c^n \to \mathbf{\Sigma}_c^n$, respectively, via

$$L(S,d) = (L(S), L(d)),\ (S,d) \in \mathbf{\Sigma}^n,$$

$$L(S,d,c) = (L(S), L(d), L(c)),\ (S,d,c) \in \mathbf{\Sigma}_c^n,$$

respectively.

Now let $f : \mathbf{D}^n \to \mathbf{R}^n$ be a solution on a class of bargaining games $\mathbf{D}^n \subseteq \mathbf{\Sigma}^n$. The following axioms are used to characterize the RKS solution.

(PO*) **Pareto Optimality**: For all $(S,d) \in \mathbf{D}^n : f(S,d) \in \mathrm{PO}(S)$.

(SY*) **Symmetry**: For all $(S,d) \in \mathbf{D}^n$ if for all permutations $\pi : N \to N$ it is true that $S = \pi(S)$ and $d = \pi(d)$ then $f_i(S,d) = f_j(S,d)$ for all $i, j \in N$.

(INV*) Invariance under Positive Affine Transformations: For all $(S,d) \in \mathbf{D}^n$ and for all positive affine transformations $L : \mathbf{R}^n \to \mathbf{R}^n$ if $L(S,d) \in \mathbf{D}^n$ then $f(L(S,d)) = L(f(S,d))$.

(IM*) Individual Monotonicity: For all $(S,d),(S',d) \in \mathbf{D}^n$, if $S \subseteq S'$ and if for some $i \in N$ and all $j \neq i$, $u_j(S,d) = u_j(S',d)$, then $f_i(S,d) \leq f_i(S',d)$.

The following theorem is well known (see for example [15]).

Theorem 3.1 *The RKS solution k is the unique solution on* $\tilde{\Sigma}^n$ *that satisfies* (PO*), (SY*), (INV*) *and* (IM*).

Now let $F : \mathbf{D}^n_c \to \mathbf{R}^n$ be a solution on a class of bargaining problems with claims $\mathbf{D}^n_c \subseteq \Sigma^n_c$. We consider the following axioms.

(PO) Pareto Optimality: For all $(S,d,c) \in \mathbf{D}^n_c : F(S,d,c) \in \mathrm{PO}(S)$.

(SY) Symmetry: For all $(S,d,c) \in \mathbf{D}^n_c$ if for all permutations $\pi : N \to N$ it is true that $S = \pi(S)$, $d = \pi(d)$ and $c = \pi(c)$ then $F_i(S,d,c) = F_j(S,d,c)$ for all $i,j \in N$.

(INV) Invariance under Positive Affine Transformations: For all $(S,d,c) \in \mathbf{D}^n_c$ and for all positive affine transformations $L : \mathbf{R}^n \to \mathbf{R}^n$ if $L(S,d,c) \in \mathbf{D}^n_c$ then $F(L(S,d,c)) = L(F(S,d,c))$.

(IND) Independence of Non–Equitable Alternatives: For all (S,d,c), $(S',d',c') \in \mathbf{D}^n_c$ if

$$\{x \in S \mid x \geq t(S,d,c)\} = \{x \in S' \mid x \geq t(S',d',c')\}$$

then $F(S,d,c) = F(S',d',c')$.

(IM) Individual Monotonicity: For all $(S,d,c),(S',d,c) \in \mathbf{D}^n_c$ if $S \subseteq S'$ and if for some $i \in N$ and all $j \neq i$,

$$\{x_j \mid x \in S, x \geq t(S,d,c)\} = \{x_j \mid x \in S', x \geq t(S',d,c)\},$$

then $F_i(S,d,c) \leq F_i(S',d,c)$.

The axioms (PO), (SY) and (INV) are standard in bargaining theory and need no further explanation. The axiom (IND) is an adaptation to our context of bargaining problems with claims of the axiom *independence of non–individually rational alternatives* that was introduced by Peters [11]. The justification for this axiom is straightforward. In a bargaining problem with claims no player can expect someone else to settle with less than the minimally equitable agreement. Therefore, non–equitable alternatives should not play any role in the determination of the solution outcome. (IM) describes the behavior of a solution in a situation in which the feasible set expands. If the expansion is such that the equitable alternatives for all players but i remain the same then it should be player i who gains from the expansion (cf. figure 2).

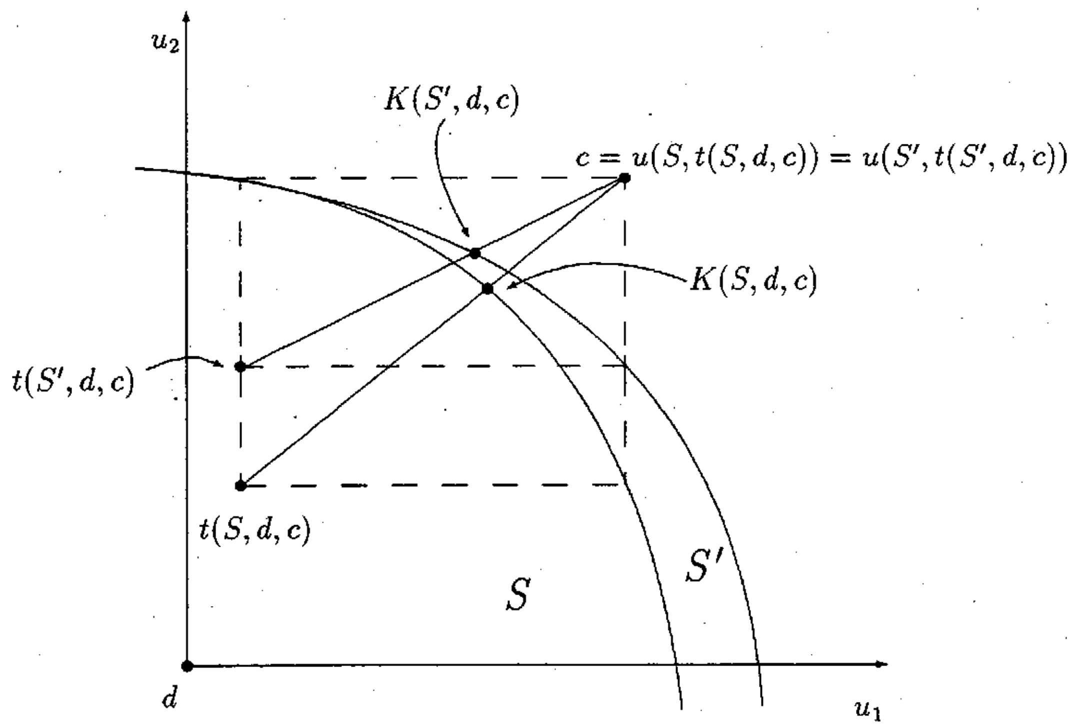

Figure 2: Individual monotonicity of the extended RKS solution.

We get the following result.

Theorem 3.2 *The extended RKS solution K is the unique solution on $\tilde{\Sigma}_c^n$ which satisfies (PO), (SY), (INV), (IND) and (IM).*

Proof:

1. K satisfies the axioms. Since (PO), (SY) and (INV) directly follow from the respective properties of k we only have to show that K satisfies (IND) and (IM). To this end let $(S,d,c),(S',d',c') \in \tilde{\Sigma}_c^n$ be such that

$$\{x \in S \mid x \geq t(S,d,c)\} = \{x \in S' \mid x \geq t(S',d',c')\}. \tag{1}$$

Then it is immediate that $t(S,d,c) = t(S',d',c')$ and $u(S,t(S,d,c)) = u(S',t(S',d',c'))$. From (1) and the definition of K we conclude that K satisfies (IND).

To show (IM) let $(S,d,c),(S',d,c) \in \tilde{\Sigma}_c^n$ be given with $S \subseteq S'$ and

$$\{x_j \mid x \in S, x \geq t(S,d,c)\} = \{x_j \mid x \in S', x \geq t(S',d,c)\} \tag{2}$$

for some $i \in N$ and all $j \neq i$. As before this directly implies that

$$t_j(S,d,c) = t_j(S',d,c) \text{ for all } j \neq i. \tag{3}$$

Since $S \subseteq S'$ we have

$$t_i(S,d,c) \leq t_i(S',d,c). \tag{4}$$

(3) together with $S \subseteq S'$ implies

$$u_i(S,t(S,d,c)) \leq u_i(S',t(S',d,c)) \tag{5}$$

and by (2)

$$u_j(S,t(S,d,c)) = u_j(S',t(S',d,c)) \text{ for all } j \neq i. \tag{6}$$

For $\lambda \geq 0$ define $g(\cdot;\lambda) : \tilde{\Sigma}_c^n \to \mathbf{R}^n$ by

$$g((S,d,c);\lambda) = (1-\lambda)t(S,d,c) + \lambda u(S,t(S,d,c)).$$

Because of (3), (4), (5), and (6) for all $\lambda \geq 0$ we have

$$g_i((S,d,c);\lambda) \leq g_i((S',d,c);\lambda), \tag{7}$$

$$g_j((S,d,c);\lambda) = g_j((S',d,c);\lambda) \text{ for all } j \neq i. \tag{8}$$

Let $\bar{\lambda}$ be such that $K(S,d,c) = g((S,d,c);\bar{\lambda})$. There are two possibilities. Either $K_i(S,d,c) \le t_i(S',d,c) \le K_i(S',d,c)$ or $K_i(S,d,c) = g_i((S,d,c);\bar{\lambda}) > t_i(S',d,c)$. In the latter case

$$g_i((S',d,c);0) = t_i(S',d,c) < g_i((S,d,c);\bar{\lambda}) \le g_i((S',d,c);\bar{\lambda}).$$

Then there exists $0 < \lambda' \le \bar{\lambda}$ such that $g_i((S',d,c);\lambda') = g_i((S,d,c);\bar{\lambda})$. For all $j \ne i$ we have $g_j((S',d,c);\lambda') \le g_j((S',d,c);\bar{\lambda}) = g_j((S,d,c);\bar{\lambda})$. Therefore, $g((S',d,c);\lambda') \le g((S,d,c);\bar{\lambda}) \in S \subseteq S'$ which implies $K_i(S',d,c) \ge g_i((S',d,c);\lambda') = g_i((S,d,c);\bar{\lambda}) = K_i(S,d,c)$. Thus K satisfies (IM).

2. It remains to be shown that K is unique. Let $F : \tilde{\Sigma}_c^n \to \mathbf{R}^n$ satisfy the axioms and let $(S,d,c) \in \tilde{\Sigma}_c^n$. Because of (IND) and since $t(S,t(S,d,c),c) = t(S,d,c)$ w.l.o.g. we can assume that $t(S,d,c) = d$. Also, by (INV) w.l.o.g. let $t(S,d,c) = \mathbf{0}$ and $u(S,d) = u(S,t(S,d,c)) = e$ (cf. figure 3, where for ease of presentation we only display the set of equitable agreements).[5] Observe that we use the fact that there exists $x \in S,\ x \gg t(S,d,c) = d$. Let $x^* = K(S,d,c)$. Obviously, $x^* = \alpha e$ for some $\alpha < 1$. Define $(S',d',c') \in \tilde{\Sigma}_c^n$ as follows. Let $d' = d$, $c' = c$ and let $S' = \text{CoCon}\{e^1,\dots,e^n,x^*\}$.

   Let $c'' = u(S,d) = e$. Then $(S',d',c'') \in \tilde{\Sigma}_c^n$ is symmetric and by (SY) and (PO) we conclude that $F(S',d',c'') = K(S',d',c'') = x^*$. It is easy to see that $t(S',d',c'') = d' = \mathbf{0}$. Since $S' \subseteq S$, $d' = d$, $c' = c$ we have $d' \le t(S',d',c') \le t(S,d,c) = d = d' = \mathbf{0}$, i.e. $t(S',d',c') = t(S',d',c'')$ and therefore

$$\{x \in S' |\ x \ge t(S',d',c')\} = \{x \in S' |\ x \ge t(S',d',c'')\}.$$

   By (IND) this implies $F(S',d',c') = F(S',d',c'')$. By construction, for all $i \in N$, it is true that

$$\{x_i |\ x \in S, x \ge t(S,d,c)\} = \{x_i |\ x \in S', x \ge t(S',d',c')\}.$$

   Also $S' \subseteq S$ and therefore (IM) implies $F(S,d,c) \ge F(S',d',c'') = x^*$. By (PO) of K we have $F(S,d,c) = x^*$ and the theorem is proved.

Q.E.D.

[5] By $\mathbf{0}$ we denote the zero vector in $\mathbf{R}^n$ and by e we denote the vector for which $e_i = 1$ for all $i = 1,\dots,n$.

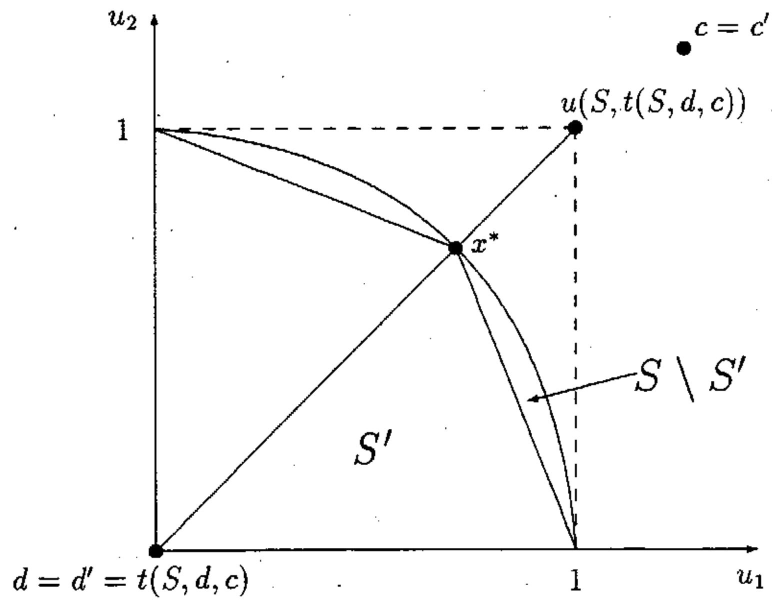

Figure 3: Construction of (S', d', c').

Remark 3.3 It is straightforward to show that the axioms are independent.

4 The Claims–Bounded extended RKS Solution

This section deals with an additional axiom that is often imposed upon a solution for bargaining problems with claims, namely boundedness by the claims point. There are situations modelled by a bargaining problem with claims that require a solution to fulfill this axiom. For example in a bankruptcy problem no player should end up with a payoff larger than the amount he has claimed. On the other hand there are also situations in which imposing a boundedness axiom seems to be inappropriate. If, for example, the claims point reflects the players' "opportunities" outside a given coalition in an NTU game we would consider a solution unfair which always bounds the players' payoffs by what they can achieve outside the coalition.

Let $F : \mathbf{D}_c^n \to \mathbf{R}^n$ be a solution on a class of bargaining problems with claims $\mathbf{D}_c^n \subseteq \mathbf{\Sigma}_c^n$.

(BC) Boundedness by Claims: For all $(S,d,c) \in \mathbf{D}_c^n$:

$$F(S,d,c) \leq c.$$

Since $u(S,t(S,d,c)) \leq c$ for all $(S,d,c) \in \tilde{\Sigma}_c^2$ the following lemma is straightforward.

Lemma 4.1 *The extended RKS solution K satisfies (BC) on $\tilde{\Sigma}_c^2$.*

In general K does not satisfy (BC) as can be seen from the following example.

Example 4.2 Let $n = 3$, $S = \text{CoCon}\{(1,0,0),(0,2,0),(0,0,3)\}$, $d = (0,0,0)$, $c = (0.25,2,3)$. Then

$$\begin{aligned} t(S,d,c) &= (0,0,0), \\ u(S,t(S,d,c)) &= (1,2,3), \\ K(S,d,c) &= (1/3,2/3,1). \end{aligned}$$

In the following we define a modification of the extended RKS solution which satisfies (BC). For $(S,d,c) \in \Sigma_c^n$ let $S_c = \{x \in S | \ x \leq c\}$.

Definition 4.3 *The* **claims–bounded extended RKS solution** $K^{\mathbf{b}} : \tilde{\Sigma}_c^n \to \mathbf{R}^n$ *is given by*

$$K^{\mathbf{b}}(S,d,c) = k(S_c, t(S,d,c)), \ (S,d,c) \in \tilde{\Sigma}_c^n.$$

$K^{\mathbf{b}}$ is well defined: It remains to be shown that there exists $x \in S_c$ with $x \gg t(S,d,c)$. Since there exists $y \in S$, $y \gg t(S,d,c)$ and since $t_i(S,d,c) < c_i$ for all $i \in N$, the claim follows by comprehensiveness of S. Thus, $(S_c, t(S,d,c)) \in \Sigma^n$.

Example 4.4 Let $(S,d,c) \in \tilde{\Sigma}_c^3$ be given as in Example 4.2. Then

$$K^{\mathbf{b}}(S,d,c) = \frac{1}{9}(1,8,12).$$

In general $(S_c, t(S,d,c) \notin \tilde{\Sigma}^n$ and therefore the following lemma is not obvious.

Lemma 4.5 *$K^{\mathbf{b}}$ fulfills (PO).*

Proof: Let $(S,d,c) \in \tilde{\Sigma}_c^n$ and suppose $K^{\mathbf{b}}(S,d,c) = k(S_c, t(S,d,c)) \notin \mathrm{PO}(S)$. Let $t = t(S,d,c)$. We first show that $u(S_c,t) \notin S$. Suppose $u(S_c,t) \in S$ and suppose there exists $i \in N$ such that $u_i(S_c,t) < c_i$. Then $u_i(S_c,t) = u_i(S,t)$ and since $(u_i(S,t), t_{-i}) \in \mathrm{PO}(S)$ this implies that $u_j(S_c,t) = t_j$ for all $j \neq i$. This contradicts the fact that there exists $x \in S_c$, $x \gg t$. Therefore, $u(S_c,t) = c$ which is a contradiction to $c \notin S$. Thus, $u(S_c,t) \notin S$.

Since $u(S_c,t) \gg t$ the fact that $u(S_c,t) \notin S$ implies that $c \gg k(S_c,t)$. Further, because of $k(S_c,t) \notin \mathrm{PO}(S)$ there exists $y \in S$, $y \gg k(S_c,t)$. For $\epsilon > 0$ small enough we therefore have $k(S_c,t) + \epsilon e \in S_c$ which contradicts $k(S_c,t) \in \mathrm{WPO}(S)$.

Q.E.D.

In order to get a characterization of $K^{\mathbf{b}}$ we replace (IND) and (IM) by the following axioms:

(IND2) Independence of Non–Equitable and Unclaimed Alternatives: For all (S,d,c), $(S',d',c') \in \mathbf{D}_c^n$ if

$$\{x \in S_c |\ x \geq t(S,d,c)\} = \{x \in S'_{c'} |\ x \geq t(S',d',c')\}$$

then

$$F(S,d,c) = F(S',d',c').$$

(IM2) Individual Monotonicity on the Relevant Bargaining Region: For all $(S,d,c), (S',d,c) \in \mathbf{D}_c^n$ if $S \subseteq S'$ and if for some $i \in N$ and all $j \neq i$,

$$\{x_j |\ x \in S_c, x \geq t(S,d,c)\} = \{x_j |\ x \in S'_c, x \geq t(S',d,c)\},$$

then

$$F_i(S,d,c) \leq F_i(S',d,c).$$

Theorem 4.6 *$K^{\mathbf{b}}$ is the unique solution on $\tilde{\Sigma}_c^n$ that satisfies (PO), (SY), (INV), (IND2) and (IM2).*

The proof is analogous to the proof of Theorem 3.2 and is therefore skipped.

Remark 4.7 Let $(S,d,c) \in \tilde{\Sigma}^n_c$. Then $K(S,d,c) \leq c$ does not imply that $K^{\mathbf{b}}(S,d,c) = K(S,d,c)$. To see this consider the following example which is a slight modification of Example 4.2. Let $n = 3$, $S = \text{CoCon}\{(1,0,0),(0,2,0),(0,0,3)\}$, $d = (0,0,0)$, $c = (0.5,2,3)$. Then

$$\begin{aligned} K(S,d,c) &= \frac{1}{3}(1,2,3) \ll c, \text{ but} \\ K^{\mathbf{b}}(S,d,c) &= \frac{1}{5}(1,4,6). \end{aligned}$$

In the following we briefly discuss the connection between $K^{\mathbf{b}}$ and division rules for bankruptcy problems. A *bankruptcy problem* is a pair (E,c), where $c \in \mathbf{R}^n_{++}$ and $0 < E < C := \sum_{i=1}^n c_i$.[6] Since in a bankruptcy problem no player should end up with more than what he claimed, zero creditors can be excluded from the set of players. A bankruptcy problem (E,c) can be represented by a bargaining problem with claims $(S(E,c),\mathbf{0},c) \in \tilde{\Sigma}^n_c$, where

$$S(E,c) = \left\{ x \in \mathbf{R}^n | \sum_{i=1}^n x_i \leq E \right\}.$$

A *division rule* is a function g that assigns to each bankruptcy problem (E,c) a vector $g(E,c) \in \mathbf{R}^n$ such that $\sum_{i=1}^n g_i(E,c) = E$. Division rules for bankruptcy problems can already be found in the Babylonian Talmud, like for example the *contested garment (CG) principle* g^{CG} for two–creditor problems which is defined as follows:

$$g^{\text{CG}}(E,(c_1,c_2)) = \left(\frac{E + c_1^E - c_2^E}{2}, \frac{E + c_2^E - c_1^E}{2} \right),$$

where $c_i^E = \min\{c_i, E\}$. Curiel, Maschler and Tijs [4] have extended the CG principle to the n–creditor case. The *adjusted proportional (AP) rule* g^{AP} they define is given by

$$g^{\text{AP}}(E,c) = v + \left(c^E - v\right) \left(\sum_{j=1}^n \left(c_j^E - v_j\right) \right)^{-1} \left(E - \sum_{j=1}^n v_j \right),$$

[6] We exclude the cases $E = 0$ and $C = E$ for technical reasons. In fact these cases are not interesting since if $E = 0$ there is nothing to distribute and if $C = E$ all claims can be met. So in neither case a conflict arises.

where for $i = 1, \ldots, n$, $v_i = v_i^{(E,c)} = \max\left\{E - \sum_{j \neq i} c_j, 0\right\}$. Observe that $v = t(S(E,c), \mathbf{0}, c)$. The rationale for the AP rule is the following. First each creditor receives v_i which is that part of the estate that is not claimed by the other players. Then the remaining part of the estate is divided in proportion to the relevant claims c^E that are still outstanding (this also explains the name of the rule). Dagan and Volij [5] show that the RKS solution applied to $\left(\left(S(E,c)\right)_c, v\right)$ induces the AP rule, in other words we have the theorem:

Theorem 4.8 *Let (E,c) be a bankruptcy game. Then*

$$K^{\mathrm{b}}(S(E,c), \mathbf{0}, c) = g^{\mathrm{AP}}(E,c).$$

The theorem shows that the claims–bounded extended RKS solution is an extension of the AP rule — and therefore of the CG principle — to general bargaining problems with claims.

5 Conclusion

We have shown that there are two natural ways to derive a bargaining problem from a given bargaining problem with claims. Which one is more appropriate depends on whether the modelled situation requires that only those utility payoffs are feasible which are bounded by the claims point. Thus, any solution on the class of problems without claims also defines a solution to a bargaining problem with claims and therefore the theory of bargaining problems with and without claims can be unified. Also, we can embed the class of bargaining problems into the class of bargaining problems with claims so that any solution defined on the latter class in the way described above will be an extension of the original bargaining solution. In this paper we have concentrated on the RKS solution and have derived the extended and the claims–bounded extended RKS solution. However, the same analysis can be carried out using any bargaining solution. One interesting feature of the claims–bounded extended RKS solution is that it extends the AP rule and therefore the ancient CG principle for bankruptcy games to the class of general bargaining problems with claims.

It would be interesting to get a characterization for extensions of other bargaining solutions to the class of bargaining problems with claims. This will be the topic of future research.

References

[1] AUMANN, R. J. and M. MASCHLER (1985) "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud", *Journal of Economic Theory* **36**, 195–213.

[2] BOSSERT, W. (1993) "An Alternative Solution to Bargaining Problems with Claims", *Mathematical Social Sciences* **25**, 205–220.

[3] CHUN, Y. and W. THOMSON (1992) "Bargaining Problems with Claims, *Mathematical Social Sciences* **24**, 19–33.

[4] CURIEL, I. J., MASCHLER, M. and S. H. TIJS (1987) "Bankruptcy Games", *Zeitschrift für Operations Research* **31**, A143–A159.

[5] DAGAN, N. and O. VOLIJ (1993) "The Bankruptcy Problem: A Cooperative Bargaining Approach", *Mathematical Social Sciences* **26**, 287–297.

[6] GERBER, A. (1996) "Coalition Formation in General NTU Games", *IMW Working Paper* **No. 264**, Bielefeld University.

[7] HERRERO, C. (1994) "Bargaining with Reference Points —Bargaining with Claims: Egalitarian Solutions Reexamined", *IVIE WP-AD* **94-15**, Universidad de Alicante.

[8] KALAI, E. and M. SMORODINSKY (1975) "Other Solutions to Nash's Bargaining Problem", *Econometrica* **43**, 513–518.

[9] NASH, J. F. (1950) "The Bargaining Problem", *Econometrica* **18**, 155-162.

[10] O'NEILL, B. (1982) "A Problem of Rights Arbitration from the Talmud", *Mathematical Social Sciences* **2**, 345–371.

[11] PETERS, H. J. M. (1986) "Simultaneity of Issues and Additivity in Bargaining", *Econometrica* **54**, 153–169.

[12] RAIFFA, H. (1953) "Arbitration Schemes for Generalized Two–Person Games", *Annals of Mathematics Studies* **28**, 361–387.

[13] ROSENMÜLLER, J. (1981) *The Theory of Games and Markets*, North–Holland, Amsterdam–New York–Oxford.

[14] ROSENMÜLLER, J. (1984) "Selection of Values for Non–Sidepayments Games", *Methods of OR* **48**, 340–398.

[15] THOMSON, W. (1980) "Two Characterizations of the Raiffa Solution", *Economics Letters* **6**, 225–231.