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A Formal Approach to Nash's Program

by

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Abstract

We provide a formal framework which allows for a precise formulation of the Nash program for n–person decision problems (i.e., n–person cooperative games). Two classes of decision problems are investigated: Finite decision problems (i.e., decision problems that may be resolved by means of finite extensive games), and decision problems that lead to multistage games with observed actions. In both cases some mild (technical) assumptions imply the rejection of Nash's program. We suggest to replace Nash's program by the following weaker assumption.

The weak program of Nash: Let N be a set of n rational decision makers, $n \geq 2$, and let δ be a decision problem for N. Then there is a nonempty set Γ of n–person noncooperative extensive games that may resolve δ. The choice of a game from Γ, in order to solve δ, is left to nature.

§ 1 Introduction

As far as we know there is no formal definition of the general decision problem for n rational agents, $n \geq 2$. However, game theory provides, at least implicitly, the following operational meaning to the foregoing problem. Let $N = \{1, \dots, n\}$, $n \geq 2$, be a set of rational decision–makers, and let δ be a decision problem for N. Then, according to game theory (see, e.g., Myerson [1991, p. 1]) the agents can choose an n–person game G such that some solution of G provides a solution to δ. Furthermore, by Nash's program for cooperative games (see Harsanyi and Selten [1988, Section 1.11]), we may assume that G is an n–person noncooperative extensive game. These statements are precisely formulated in Section 2.

We propose a precise formulation of the Nash program for certain classes of decision problems. In Section 3 we consider finite decision problems, that is, decision problems that can be resolved by a finite set of finite (noncooperative) extensive games. Our version of Nash's program for this class of games is incompatible with the following assumption. The decision–makers use a game form that satisfies unanimity in order to select the noncooperative game to be played. This result is generalized in Section 4, under suitable restrictions, to multi–stage games with observable actions. Section 5 is devoted to discussion and applications of the formal results. If our model is accepted, then the main conclusion is that in situations where not only the strategies are to be chosen by the decision–makers, but also the (game) environment itself, the players cannot be explicitly rational (i.e., they have to rely on the strategic behavior which is inherent in their genes).

The failure of Nash's program implies that the general multiperson decision problem must be reformulated (or include also cooperative games). In Section 6 we suggest the following new formulation. Let N and δ be defined as in the first paragraph of this section. Then, according to our assumption, there exists a nonempty set Γ of n–person noncooperative extensive games such that each game $G \in \Gamma$ has at least one reasonable Nash equilibrium that solves δ. δ is solved by the following two–stage procedure. In stage 1 nature chooses $G \in \Gamma$. Then, in stage 2, the members of N have to find some reasonable Nash equilibrium of G.

§ 2 A Framework for Rational Interaction between Rational Decision–Makers

Let N = {1, ... , n} be a set of rational agents. If n = 1 then the structure of the general decision situation that agent 1 may face is well known: He has to select one alternative from a recognized set of decision alternatives. His choice is solely determined by his preference relation on the set of alternatives. The rationality of agent 1 implies that he will choose a maximal alternative, with respect to his preference relation, if such an alternative is available.

If $n \geq 2$ then a description of a framework for rational interaction between the n agents seems to be highly complicated. However, such a description is provided by game theory. Indeed Myerson [1991, p. 1] writes: "Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare. As such, game theory offers insights of fundamental importance for scholars in all branches of the social sciences, as well as for practical decision–makers. The situations that game theorists study are not merely recreational activities, as the term "game" might unfortunately suggest. "Conflict analysis" or "interactive decision theory" might be more descriptively accurate names for the subject, but the name "game Theory" seems to be here to stay." Following Myerson we may introduce the following definition.

Definition 2.1. *A full framework for rational interaction between n rational decsion–makers is an n–person game.*

There are two kinds of n–person games: cooperative and noncooperative (see Harsanyi and Selten [1988, Section 1.2]). Roughly, a game is cooperative if the option of making binding agreements is available to the players in at least one situation. The game is noncooperative if the players are unable to make enforceable agreements which are not specified by the game itself. Now, according to Nash [1951], the analysis of any cooperative game G can be based on the solution of a suitable noncooperative game G*. (The choice of G* may not be unique.). This principle is known as Nash's program for cooperative games (see Harsanyi and Selten[1988, Section 1.11] and Myerson [1991, Section 8.1]).

We now recall the well known fact that every noncooperative game in strategic form can be represented in extensive form. For a definition of finite games in extensive form the reader is referred to Myerson [1991, Section 2.1]. Infinite games in extensive form are analyzed in Aumann [1964]. Although our discussion in Section 4 deals with infinite games, a knowledge of finite games in extensive form is sufficient for full understanding of our result. In view of the foregoing discussion we may formulate Corollary 2.2.

Corollary 2.2. A full framework for rational interaction between n rational decision–makers is an n–person noncooperative game in extensive form.

Our operational interpretation of Corollary 2.2 is as follows. If $N = \{1, .., n\}$, $n \geq 2$, is a set of agents that face some decision problem δ, then there exists a nonempty set $\Gamma(\delta)$ of noncooperative extensive games with the following three properties:

(2.1) Each game in $\Gamma(\delta)$ has at least one "reasonable" Nash equilibrium (NE) that resolves δ.

(2.2) If $G \in \Gamma(\delta)$ is given (i.e., it is common knowledge among the players), then the members of N can choose any n–tuple of strategies of G.

(2.3) The interaction between the members of N that is necessary for the solution of δ is restricted to a play of an n–tuple of strategies of some game in $\Gamma(\delta)$.

The agents are **rational**, in the foregoing framework, if they always choose an NE of some game in $\Gamma(\delta)$.

§ 3 Finite Decision Problems

Let $N = \{1, ... , n\}$, $n \geq 2$, be a set of rational decision–makers, and let δ be a decision problem for N. Further, let $\gamma(\delta)$ be the family of all sets $\Gamma(\delta)$ (of noncooperative extensive games), that satisfy (2.1)–(2.3).

Definition 3.1 δ **is finite** if

(3.1) every $\Gamma \in \gamma(\delta)$ is finite;

(3.2) if $\Gamma \in \gamma(\delta)$ and $G \in \Gamma$, then G is a finite (noncooperative) extensive game.

This section is entirely devoted to the study of finite decision problems.

Now let δ be a finite decision problem, let $\Gamma(\delta) \in \gamma(\delta)$, and assume that $\Gamma(\delta)$ is not a singleton. Although all the games in $\Gamma(\delta)$ are models of δ, the members of N may have conflicting preferences over $\Gamma(\delta)$ (see, e.g., Harsanyi and Selten [1988, p. 22]). Thus, in order to solve δ, the problem of choosing $G \in \Gamma(\delta)$ must be resolved first. So, actually the decision–makers face the following two–stage decision problem δ^*:

Stage 1: Choose $G \in \Gamma(\delta)$.
Stage 2: If $G \in \Gamma(\delta)$ is chosen, then select a Nash equilibrium of G.

At this point two assumptions are possible:

Assumption I: the choice of $G \in \Gamma(\delta)$ (Stage 1) is done by N.
Assumption II: Stage 1 is resolved by some agent who is not a member of N.

3.1 **The solution of Stage 1 under Assumption I**

We impose only one condition, unanimity, on the choice procedure at Stage 1. More precisely, we introduce the following definition.

Definition 3.2. A **choice procedure** for Stage 1 is a (generalized) strategic game $M = \langle \Gamma(\delta), .., \Gamma(\delta); f \rangle$, where $f: \Gamma(\delta)^N \to \Gamma(\delta)$ satisfies
(3.3) $f(G, \ldots, G) = G$ for all $G \in \Gamma(\delta)$.
(Notice that the outcomes of M are (extensive) games.)

f is called the **choice function** (of M). In M the players choose their strategies simultaneously, and the outcome is determined by the choice function. The players may also use mixed strategies. The outcome in this case is the resulting probability distribution over $\Gamma(\delta)$. The following example illustrates the foregoing discussion.

Example 3.3 Let $\Gamma(\delta) = \{G_1, G_2\}$ and let $M = \langle \Gamma(\delta), \Gamma(\delta); f \rangle$ be given by the following matrix:

	G_1	G_2
G_1	G_1	G_1
G_2	G_1	G_2

Table 1

Then M is a choice procedure. If player 1 chooses the (mixed) strategy $(\frac{1}{2}, \frac{1}{2})$ and player 2 chooses $(\frac{1}{3}, \frac{2}{3})$, then the outcome is the probability distribution $(\frac{2}{3}, \frac{1}{3})$ on $\Pi(\delta)$.

3.2 The impossibility of Nash's program under Assumption I.

Let, again, δ be a decision problem such that $\Pi(\delta)$ is not a singleton. If M is a choice procedure for Stage 1, then the decision problem δ^* is equivalent to the following n–person noncooperative extensive game G^*: (i). First play M in order to choose $G \in \Pi(\delta)$. (ii) If $G \in \Pi(\delta)$ is chosen (the choice is common knowledge) then play G. As we have already remarked, every strategic game has an extensive representation. Therefore, we may treat M as extensive game. Thus, G^* is, indeed, an extensive game.

The following example illustrates the foregoing discussion.

Example 3.4. Let $\Pi(\delta)$ and M be as in Example 3.3. Let further G_1 and G_2 be given by:

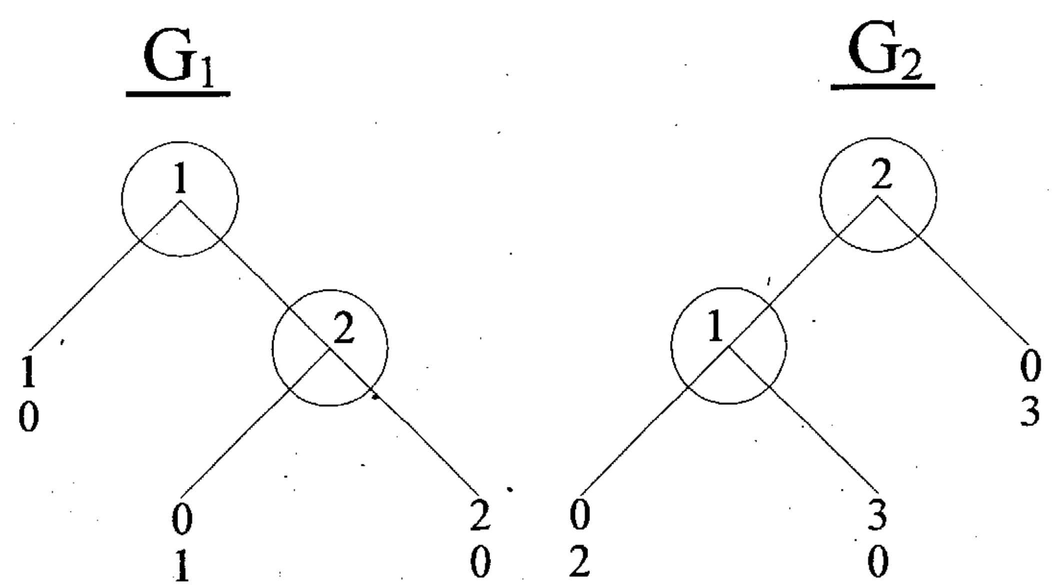

Fig. 1

Then G^* is given by:

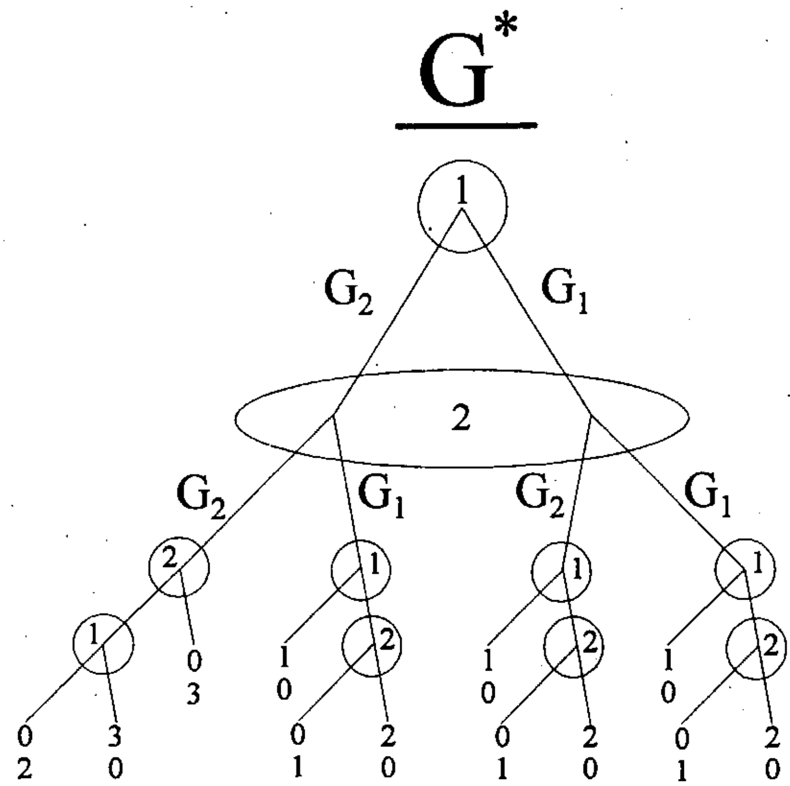

Fig. 2

The extensive form of M is not unique[1]. Therefore the representation of G^* is not unique. However, the set of Nash Equilibria of G^* is not affected by this nonuniqueness.

We now observe that, by (3.3), every $G \in \Gamma(\delta)$ is a proper subgame of G^*. Therefore, if $G \in \Gamma(\delta)$ and b is an n–tuple of (behavior) strategies for G, then b is **not** an n–tuple of (behavior) strategies for G^*. Thus, we arrive at the following conclusion.

Theorem 3.5. Let δ be a decision problem such that $\Gamma(\delta)$ is not a singleton. Under Assumption I and (3.3) Nash's program is false for δ.

Proof: By (2.3) the interaction between the players is restricted to a play of some n–tuple of (behavior) strategies of some game $G \in \Gamma(\delta)$. Thus the players are unable to choose an n–tuple of (behavior) strategies in G^*. However, by Assumption I and (2.2)

[1] However, see Peleg, Rosenmüller, and Sudhölter (1995).

the players can choose any n–tuple of (behavior) strategies of G^*. Thus, Corollary 2.2 and Assumption I are incompatible. Because Nash's program implies Corollary 2.2, it is also incompatible with Assumption I. Q.E.D.

3.3 Discussion

We conclude from Theorem 3.5 that if we want to maintain Nash's program, then we have to reject Assumption I. Now, Assumption II is very broad and does not tell what exactly happens when it is adopted. In Section 5 we assume that Stage 1 is resolved by nature, and reformulate Nash's program accordingly.

The assumption that $\Gamma(\delta)$ is not a singleton (see Theorem 3.5), should be scrutinized. Actually, we have assumed (implicitly) more than that, namely, that the decision–makers have conflicting preferences over $\Gamma(\delta)$. A "minimal conflict" may be defined as follows. There exists, $i, j \in N$, $i \neq j$. and $G_k \in \Gamma(\delta)$, $k = i, j$, such that the following conditions are satisfied. Let $NE(G_k)$ be the set of Nash equilibria of G_k, $k = i, j$. Then: (i) $NE(G_k) \neq \emptyset$, $k = i, j$; and (ii) the maximum payoff to i(j) in $NE(G_i)$ ($NE(G_j)$) is greater than the maximum in $NE(G_j)$ ($NE(G_i)$).

§ 4 Infinite Decision Problems

4.1 Multi–stage games with observed actions

Multi–stage games with observed actions (MSG's) model many economic and biological conflicts (see Fudenberg and Tirole [1991, Ch.4]). Therefore, we have chosen to represent infinite decision problems by MSG's. In this section we shall define a family of MSG's.

Let Q be the Hilbert cube. We recall that Q is the infinite Cartesian product

$$Q = [0, 1] \times [0, \tfrac{1}{2}] \times \dots \times [0, \tfrac{1}{n}] \times \dots$$

We also recall that every compact metric space is homeomorphic to a closed subset of Q (see Schurle [1979, p. 88]). Because we allow only sets of actions (of the players) that are compact metric spaces, Q is the basic building block of our model.

Definition 4.1. A multi–stage game with observed actions (MSG) is a list $G = < \underset{\sim}{\varphi}^1, \dots, \underset{\sim}{\varphi}^n; u^1, \dots, u^n >$ with the following properties.

(4.1) φ_0^i is a non–empty closed subset of Q, $i \in N$. Thus, the set $A_1 = \varphi_0^1 \times \dots \times \varphi_0^n$ is a nonempty closed subset of Q^N. Assume now that $\varphi^i_{t_0-1}$, $i \in N$, where $t_0 \geq 1$, and $A_{t_0} \subset Q^{Nt_0}$ $(= (Q^N)^{t_0})$ have already been defined, and A_{t_0} is a nonempty closed subset of Q^{Nt_0}. Then $\varphi^i_{t_0}: A_{t_0} \to Q$ is an upper hemicontinuous correspondence for $i \in N$, and

(4.2) $A_{t_0+1} = \{(x, y^1, \dots, y^n) \mid x \in A_{t_0}$ and $y^i \in \varphi^i_{t_0}(x)$ for all $i \in N\}$ (clearly, A_{t_0+1} is a non-empty closed subset of $Q^{N(t_0+1)}$).

(4.3) $u^i: Q^{NI} \to R$ is continuous for all $i \in N$. (Here $I = \{1, 2, 3, \dots\}$ is the set of natural numbers.)

An MSG $G = < \underset{\sim}{\varphi}^1, \dots, \underset{\sim}{\varphi}^n; u^1, \dots, u^n >$ is played as follows. At time $t = 1$ player i chooses (simultaneously with the other players), an action $a_1^i \in \varphi_0^i$, $i = 1, \dots, n$. We now proceed by induction. Assume that the profiles of actions $a_1 = (a_1^1, \dots, a_1^n)$, … $a_{t-1} = (a^1_{t-1}, \dots, a^n_{t-1})$, $t \geq 2$, have already been chosen. Then at time t player i chooses, simultaneously with all other players, an action $a_t^i \in \varphi^i_{t-1}(a_1, \dots, a_{t-1})$. Let $\underset{\sim}{a} = < a_1, a_2, \dots >$ be the sequence of action profiles that was generated in the foregoing way. Then the payoff to player $i \in N$ is $u^i(\underset{\sim}{a})$.

We shall now show that all discounted infinite repetitions of finite strategic games are included in our model. We shall only consider 2 × 3 two–person games. The proof of the foregoing claim in the general case is similar.

Example 4.2. Let g be a 2 × 3 two–person game:

	S_1^2	S_2^2	S_3^2
S_1^1	a_{11}, b_{11}	a_{12}, b_{12}	a_{13}, b_{13}
S_2^1	a_{21}, b_{21}	a_{22}, b_{22}	a_{23}, b_{23}

and let $0 < \delta_1, \delta_2, < 1$ be the discount factors of 1 and 2 respectively. Denote

$$\hat{e}_j = < \overbrace{0, \ldots, 0,}^{j-1} \tfrac{1}{j}, 0, \ldots >, j = 1, 2, \ldots$$

The vectors $< \hat{e}_1, 0 >$, $< \hat{e}_2, 0 >$, $< 0, \hat{e}_1 >$, $< 0, \hat{e}_2 >$, and $< 0, \hat{e}_3 >$ are linearly independent in $Q \times Q$. Let Σ^i be the set of mixed strategies of player $i = 1, 2$. We do not distinguish between Σ^i and its canonical embedding in $Q^{\{i\}}$. Define $\varphi^i_t = \Sigma^i$, $i = 1, 2$, and $t = 0, 1, \ldots$ Also, the map $h: \Sigma^1 \times \Sigma^2 \to Q \times Q$ given by

$$h(p, q, r) = < p\hat{e}_1 + (1-p)\hat{e}_2, q\hat{e}_1 + r\hat{e}_2 + (1-q-r)\hat{e}_3 >$$

$0 \le p \le 1$, $q, r \ge 0$, $q+r \le 1$, is an embedding of $\Sigma^1 \times \Sigma^2$ in $Q \times Q$. Define v^1 on $h(\Sigma^1 \times \Sigma^2)$ by

$$v^1(h(p, q, r)) = pqa_{11} + pra_{12} + \ldots + (1-p)(1-q-r)a_{23}.$$

Then v^1 can be extended to a (bounded) continuous function on $Q \times Q$. Thus, if we define u^1 by

$$u^1(\underset{\sim}{a}) = u^1((a^1_1, a^2_1), (a^1_2, a^2_2), \ldots, (a^1_t, a^2_t), \ldots) = \sum_{t=1}^{\infty} \delta^t_1 v^1(a^1_t, a^2_t),$$

then u^1 is continuous on Q^{2I}. u^2 is defined similarly. As the reader may easily verify, the MSG $< \underset{\sim}{\varphi}^1, \underset{\sim}{\varphi}^2; u^1, u^2 >$ is equivalent to the infinite repetition of g.

Remark 4.3. An MSG $G = < \underset{\sim}{\varphi}^1, \underset{\sim}{\varphi}^2, \ldots, \underset{\sim}{\varphi}^n; u^1, \ldots, u^n >$ is **finite** if there exist $T \ge 0$ and functions $v^i: Q^{NT} \to R$ such that $u^i(a_1, \ldots, a_T, \ldots) = v^i(a_1, \ldots, a_T)$ for all $\underset{\sim}{a} = < a_1, \ldots, a_T, \ldots > \in Q^{NI}$ and all $i \in N$. If G is finite and T has the foregoing property, then we shall assume without loss of generality that $\varphi^i_t \equiv \{0\}$ for all $i \in N$ and $t \ge T$.

We denote by H the family of all MSG's.

4.2 A metric on H

Let $G = < \underset{\sim}{\varphi}^1, , \ldots, \underset{\sim}{\varphi}^n; u^1, \ldots, u^n >$ be an MSG. We denote $\psi^i_0 = \varphi^i_0$ for all $i \in N$ and

(4.4) $\psi^i_t = \{(x, y) \mid x \in A_t \text{ and } y \in \varphi^i_t(x)\}$ for all $i \in N$ and $t \ge 1$ (see (4.2)).

Then $A_1 = \psi^1_0 \times \ldots \times \psi^n_0$ and

(4.5) $A_{t+1} = \{(x, y^1, \ldots, y^n) \mid (x, y^i) \in \psi^i_t \text{ for all } i \in N\}$, $t = 1, 2, \ldots$

Furthermore, ψ^i_t, $i \in N$, $t \ge 0$, is a nonempty closed subset of $Q^{tN} \times Q^{\{i\}}$, and $G = < \underset{\sim}{\psi}^1, \ldots, \underset{\sim}{\psi}^n; u^1, \ldots, u^n >$ is an equivalent representation of G. (Here $\underset{\sim}{\psi}^i = < \psi^i_0, \ldots, \psi^i_t, \ldots >$).

Consider now the reverse approach. Let ψ_0^i, $i \in N$, be a nonempty closed subset of $Q^{\{i\}}$, and let $A_1 = \psi_0^1 \times \ldots \times \psi_0^n$. Assume now that ψ_{t-1}^i, $i \in N$, and A_t have already been defined, and A_t is a nonempty closed subset of Q^{tN}. Then we may choose for each $i \in N$ a nonempty closed subset ψ_t^i of $Q^{tN} \times Q^{\{i\}}$ such that

(4.6) the projection of ψ_t^i on Q^{tN} is A_t.

Furthermore, we may define A_{t+1} by (4.5). If we now define for $i \in N$ $\varphi_0^i = \psi_0^i$ and $\varphi_t^i : A_t \to Q$, $t \geq 1$, by

$$y \in \varphi_t^i(x) \text{ iff } (x, y) \in \psi_t^i,$$

then $G = < \underset{\sim}{\varphi}^1, , \ldots, \underset{\sim}{\varphi}^n; u^1, \ldots, u^n >$ is an MSG (for every choice of continuous functions $u^1, \ldots, u^n$). Thus, if $\underset{\sim}{\psi}^1, \ldots, \underset{\sim}{\psi}^n$ have the foregoing properties, then $G = < \underset{\sim}{\psi}^1, \ldots, \underset{\sim}{\psi}^n; u^1, \ldots, u^n >$ is a game. We shall now define a natural metric on H.

For a natural number q let F^q be the set of all nonempty and closed subsets of Q^q, and let D^q be the Hausdorff metric on F^q. Then (F^q, D^q) is compact metric space (see Hildenbrand [1974, p. 17]). Also, let $U = C(Q^{NI}, R)$ be the set of all (real–valued) continuous functions on Q^{NI} with the maximum norm. Then U is a (separable) metric space. Clearly,

$$H \subset \mathop{X}_{t=0}^{\infty} \mathop{X}_{i=1}^{n} F^{nt+1} \times U^N.$$

Thus, H is a (separable) metric space. Moreover, one can prove that there exists a (nonempty) closed subset F of the (compact) space $\mathop{X}_{t=0}^{\infty} \mathop{X}_{i=1}^{n} F^{nt+1}$ such that $H = F \times U^N$ (see the Appendix).

4.3 Choice procedures for families of games in H

Let Γ be a compact subset of H with two members at least. We shall now define and analyze choice procedures for Γ. Our analysis will enable us to examine the Nash program for games in H.

Because Γ is compact there is an embedding $h: \Gamma \to Q$. We denote $\hat{\Gamma} = h(\Gamma)$. The following Definition is the analog of Definition 3.2.

Definition 4.4. A choice procedure for Γ (and h) is a (generalized) strategic game $M = < \hat{\Gamma}, \ldots, \hat{\Gamma}; f >$, where $f: \hat{\Gamma}^N \to \Gamma$ (the choice fucntion) is a continuous function that satisfies

(4.7) $f(x, ..., x) = h^{-1}(x)$ for all $x \in \hat{\Gamma}$..

Remark 4.5. The "dictatorial" choice functions,namely, $f^i(x^1, ..., x^n) = h^{-1}(x^i)$ for all $(x^1, ..., x^n) \in \hat{\Gamma}^N$ and for some $i \in N$, are continuous and satisfy (4.7). Hence there exist choice functions.

Remark 4.6. We observe again that the outcomes of a choice procedure are extensive games (and not payoff vectors).

Let $M = < \hat{\Gamma}, ..., \hat{\Gamma}; f >$ be a choice procedure for Γ. We now define the following two—stage game $G^* = G^* (\Gamma, f, h)$:

Stage 1: Play M in order to choose $G \in \Gamma$.
Stage 2: If $G \in \Gamma$ is chosen (the choice is common knowledge), then play G.

The game G^* will play an important role in the next subsection, where we shall investigate the validity of Nash's program for infinite decision problems. Here we only prove the following result.

Theorem 4.7. $G^*(\Gamma, f, h) \in H$.

Proof: Define $\psi^i_{*0} = \hat{\Gamma}$ for all $i \in N$. Then ψ^i_{*0} is a closed subset of Q (recall that Γ is compact and h is an embedding). For $t \geq 1$ and $i \in N$ define ψ^i_{*t} in the following way.

$$< a_1, ..., a_t, b > \in \psi^i_{*t} \Leftrightarrow < a_2, ..., a_t, b > \in \psi^i_{f(a_1),t-1}$$

for all $< a_1, ..., a_t, b > \in Q^{tN} \times Q^{\{i\}}$. (If $G \in \Gamma$ then we write $G = < \underset{\sim}{\varphi}^1_G, , ..., \underset{\sim}{\varphi}^n_G; u^1_G, ..., u^n_G > = < \underset{\sim}{\psi}^1_G, ..., \underset{\sim}{\psi}^n_G; u^1_G, ..., u^n_G >$.) We have to prove that ψ^i_{*t} is closed. If $a_1(k) \to a_1, ..., a_t(k) \to a_t$, $b(k) \to b$, and $< a_2(k), ..., a_t(k), b(k) > \in \psi_{f(a_1(k)),t-1}$, then $f(a_1(k)) \to$ $\to f(a_1)$ because f is continuous. By the definition of the metric in H, $\psi_{f(a_1(k)),t-1} \to \psi_{f(a_1),t-1}$ (in the Hausdorff metric). Because $< a_2(k), ..., a_t(k), b(k) > \in \psi_{f(a_1(k)),t-1}$, $k = 1,2, ...$, it follows that $< a_2, ..., a_t, b > \in \psi_{f(a_1),t-1}$, that is, $< a_1, a_2, ..., a_t, b > \in \psi^i_{*t}$. If we now define

$b \in \varphi^i_{*t}\,(a_1, ..., a_t) \Leftrightarrow \langle a_1, ..., a_t, b \rangle \in \psi^i_{*t},$

then φ^i_{*t} is upper hemicontinuous. Clearly, $G^* = \langle \underset{\sim}{\varphi}^1_*, ..., \underset{\sim}{\varphi}^n_*; u^n_*, ..., u^n_* \rangle$ where u^1_*, ..., u^n_* are given below. Thus, it only remains to prove that the payoff functions of G^* are continuous.

A sequence $\underset{\sim}{a} \in Q^{NI}$ is **attainable** for G^* if $a_1 \in \hat{\Gamma}$, and $\langle a_2, ..., a_t \rangle \in A_{f(a_1),t-1}$ for $t \geq 2$. Let A be the set of all attainable sequences for G^*. We shall prove that A is a closed subset of Q^{NI}. Let $\underset{\sim}{a}(k) = \langle a_1(k), a_2(k), ..., \rangle \to \underset{\sim}{a} = \langle a_1, a_2, ... \rangle$, $\underset{\sim}{a}(k) \in A$, $k = 1, 2, ...$ Then, $a_1 = \lim_k a_1(k)$ is in $\hat{\Gamma}$. Thus, it is sufficient to prove that (the sets) $A_{f(a_1(k)),t}$ converge (in the Hausdorff metric) to $A_{f(a_1),t}$ for $t \geq 1$. We prove this by the following argument. Choose $t \geq 1$ and $i \in N$. Then $\psi^i_{f(a_1(k)),t}$ converge, in the Hausdorff metric, to $\psi^i_{f(a_1),t}$ (because $f(a_1(k))$ converge to $f(a_1)$). Now, $A_{f(a_1(k)),t}$ is the domain of definition of $\varphi^i_{f(a_1(k)),t}$, that is, it is the projection of $\psi^i_{f(a_1(k)),t}$ on Q^{Nt}. Hence $A_{f(a_1(k)),t}$ converge, in the Hausdorff metric, to $A_{f(a_1),t}$.

Let $i \in N$. The payoff function u^i_* is defined on A by

$$u^i_*(\underset{\sim}{a}) = u^i_{f(a_1)}(a_2, a_3, ...).$$

First we show that u^i_* is continuous on A. Let $\underset{\sim}{a}(k) = \langle a_1(k), .. a_t(k), .. \rangle \to \underset{\sim}{a} = \langle a_1, ... \rangle$ on A. Because f(.) is continuous, $f(a_1(k)) \to f(a_1)$. Hence, by the definition of the metric on H, $u_{f(a_1(k))}$ converge uniformly to $u_{f(a_1)}$. Thus, by Munkres [1975, p. 132]

$$u^i_{f(a_1)}(a_2, a_3, ...) = \lim_{k \to \infty} u^i_{f(a_1(k))}(a_2(k), a_3(k), ...). \text{ Hence, } u^i_*(\underset{\sim}{a}) = \lim_{k \to \infty} u^i_*(\underset{\sim}{a}(k)).$$

Thus, u^i_* is continuous on A. We now may extend u^i_* to a continuous function on Q^{NI}.
Q.E.D.

§ 4.4 **The Nash program with MSG's**

Let $N = \{1, ..., n\}$, $n \geq 2$, be a set of rational decision makers, and let δ be a decision problem for N. Nash's program (with games in H) is equivalent to the following claim.

Claim 4.8. There exists a nonempty compact subset $\Pi(\delta) \subset H$ with the following three properties:

(4.8) Each $G \in \Gamma(\delta)$ has at least one reasonable Nash equilibrium that resolves δ.

(4.9) If $G \in \Gamma(\delta)$ is given (i.e., it is common knowledge among the players), then the members of N can choose any n–tuple of strategies of G.

(4.10) The interaction between the members of N that is necessary for the solution of δ is restricted to a play of an n–tuple of strategies of some game in $\Gamma(\delta)$.

Remark 4.9. The assumption that $\Gamma(\delta)$ is compact is equivalent to the following two conditions: (i)$\Gamma(\delta)$ is closed; and (ii) the projection of $\Gamma(\delta)$ on U^n (U is the space of utility functions) is compact (see Subsection 4.2 and the Appendix). (i) is clearly acceptable. (ii) is reasonable because all the games in $\Gamma(\delta)$ have the same payoff space, namely, the payoff space of δ. In particular, uniform boundedness and equicontinuity seem to be acceptable for the family of utility functions which is considered in our case.

We shall make the following assumption.

(4.11) The members of N have conflicting preferences over $\Gamma(\delta)$ (in particular, $\Gamma(\delta)$ is not a singleton).

A discussion of (4.11) is contained in Subsection 3.3. If the players themselves choose a game in $\Gamma(\delta)$, then we shall assume that they use some choice procedure (see Definition 4.4). At this point two assumptions are possible.

Assumption I: The players use some choice procedure M to choose a game G in $\Gamma(\delta)$.

Assumption II: A game $G \in \Gamma(\delta)$ is chosen by an outside agent.

We shall prove that Assumption I may contradict claim 4.8. First we need the following definitions. Let $G_k = \langle \underset{\sim}{\psi}_k^1, \ldots, \underset{\sim}{\psi}_k^n; u_k^1, \ldots, u_k^n \rangle$, k = 1, 2, be two games in H.

Definition 4.10. G_2 **is isomorphic to a subgame of** G_1 if there exist $T \geq 0$ and $a_T \in Q^{NT}$ such that

(4.12) $\psi_{2,t}^i = \{x_t \mid \langle a_T, x_t \rangle \in \psi_{1,T+t}^i\}$, $i \in N$ and $t = 0, 1, 2, \ldots$.

(4.13) $u_2^i(\underset{\sim}{a}) = u_1^i(a_T, \underset{\sim}{a})$, $i \in N$ and $\underset{\sim}{a} \in Q^{NI}$.

We now add the following assumption.
(4.14) If $G_1 \in \Gamma(\delta)$ then there exists $G_2 \in \Gamma(\delta)$ such that G_2 is not isomorphic to any subgame of G_1.

Theorem 4.11. Assume that (4.7) and (4.14) hold. Then claim 4.8 (Nash's program for δ) and Assumption I are incompatible.

Proof: Assume, on the contrary, that claim 4.8 and Assumption I are both true. Let $h:\Gamma(\delta) \rightarrow Q$ be an embedding, let $h(\Gamma(\delta)) = \hat{\Gamma}$, and let $M = < \hat{\Gamma}, \ldots, \hat{\Gamma}; f >$ be the choice procedure that is used by N (according to Assumption I). By Assumption I and (4.9) the players will solve δ by choosing a suitable strategy in $G^*(\Gamma(\delta), f, h)$ (see Subsection 4.3 for the definition of G^*). However, by (4.14), $G^*(\Gamma(\delta), f, h)$ is not isomorphic to any subgame of a game in $\Gamma(\delta)$. Indeed, if $G_1 \in \Gamma(\delta)$ and $G^*(\Gamma(\delta), f, h)$ is isomorphic to a subgame of G_1, then every $G \in \Gamma(\delta)$ is isomorphic to a subgame of G_1 by (4.7). Thus, (4.14) is contradicted. Now, if $G^*(\Gamma(\delta), f, h)$ is not isomorphic to any subgame of a game in $\Gamma(\delta)$, then, by (4.10), the players cannot use $G^*(\Gamma(\delta), f, h)$ in order to solve δ. Q.E.D.

We remark that behavior strategies may be allowed in the proof of Theorem 4.11.

Remark 4.12. Theorem 4.11 may be false without Assumption (4.14). For example, let $N = \{1, 2\}$, let G be a 2 × 2 two–person game, and let $\Gamma(\delta)$ consist of G and the infinite game G_1:

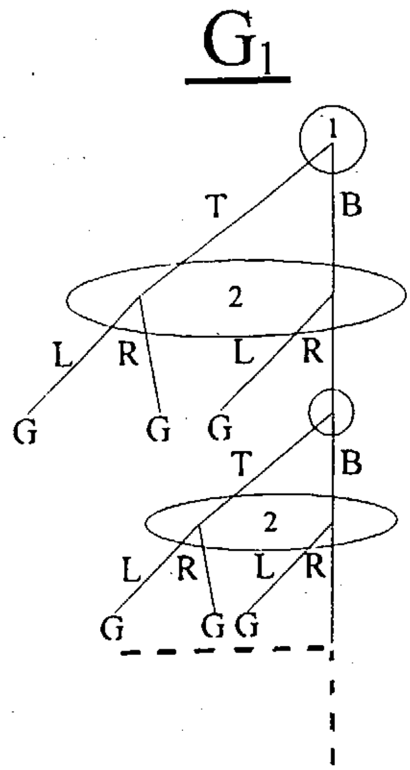

Fig. 3

If M is given by the following (generalized) strategic game

	G	G_1
G	G	G
G_1	G	G_1

Table 2

then $G^*(\Gamma(\delta), M)$ is insomorphic to G_1.

§ 5 Discussion and Applications

In Sections 5 and 6 we discuss only finite decision problems. We now recall the standard problem in noncooperative game theory: For a given n–person noncooperative game G in extensive form the players must find some reasonable Nash equilibrium of G for use in the game. Thus, the framework for rational interaction is given, and the players' only task is to find and follow the equilibrium strategies which are recommended by game theory. Therefore, it seems that our Theorem 3.5 has no bearing on game theory. Nevertheless, we shall now argue that the theorem has some implications on the foundations of game theory and on its applications to economics and psychology.

Example 5.1. Consider two individuals R (Robinson Crusoe) and F (Friday) who are alone on an island. They face the decision problem of sharing in the island. Their interaction, since their first meeting, must be described by a play of some two–person noncooperative game G in extensive form (see (2.3)). By Theorem 3.5, the choice of G cannot be made By R and F. As R and F are the only individuals on the island, the question who chooses G naturally arises. Unfortunately, as far as we can see, there is no simple answer to this question. We suggest the following way out. The game G is chosen by nature from a nonempty set Γ of two–person noncooperative games in extensive form. Further, nature is assumed to have a probability distribution π on Γ which is unknown to R and F. More concretely, G was selected spontaneously when R and F first met, according to the parameters Γ and π, by the strategic behavior which is inherent in the genes of R and F. We shall precisely define our solution in Section 5.

Remark 5.2. In our view Example 5.1 might have some applications to psychology. First we notice that the interaction between R and F, according to our solution, was determined by themselves in a (partially) unconscious manner. Furthermore, by Theorem 3.5, it is impossible to find a fully rational explanation (i.e., an explanation based on a play of a Nash equilibrium in a game chosen by R and F), for the interaction between R and F. This example might be generalized to describe the interaction, subject to the standard of behavior of a society, between any two (or more) individuals in the society, provided that these individuals have "strong personalities" so that their actions cannot be manipulated by the rest of the society.

There are also economic situations where the extensive game to be played must be chosen by the agents. We shall now mention two cases.

Example 5.3. Consider a meeting of a committee with no final agenda. Then the choice of the agenda and the voting procedures amounts to choosing the extensive game that will be played when the committee will make its decisions. Again, we may use (2.3) and Theorem 3.5 to argue that the interaction between the members of the committee cannot be fully rationalized in an explicit manner. Indeed, by (2.3) there must be a noncooperative extensive game G such that the interaction between the members of the committee, including the choice of the agenda, is described by a play of an n–tuple of strategies in G (n denotes the member of committee members). Thus, by the theorem, G cannot be chosen by the members of the committee.

Example 5.4. Consider the process of merger of two firms. It seems to us that there are no a priori rules for bargaining on the conditions for the merger. Thus, the extensive game that models the bargaining before the merger is chosen by the two firms. Again, there is no complete and explicit rationalization for the foregoing process.

§ 6 Incomplete Games in Extensive Form

An incomplete game in extensive form has the same mathematical structure as an extensive game except that some of the rules of the game may not be common knowledge.

For example, a Bayesian game with the property that the priors of the players are private information is an incomplete game (see Battigalli [1995]). In the sequel we shall restrict ourselves to finite games in order to simplify the presentation. Motivated by Example 5.1 and Remark 5.2 we introduce the following definition.

Definition 6.1. A *simple incomplete* n–person game in extensive form is a finite sequence $< G_1, p_1;, ...; G_k, p_k >$ such that:

(i) G_j, $j = 1, ..., k$ is an n–person game in extensive form;

(ii) $p_j > 0$, $j = 1, ..., k$, and $\sum_{j=1}^{k} p_j = 1$.

$< G_1, p_1;, ...; G_k, p_k >$ is played in the following way. First, nature chooses an integer $1 \leq j \leq k$ and publicly signals the players that the game G_j will be played. The rules of G_j then become common knowledge. However, the probabilities $p_1, ..., p_k$ are unknown to the players if $k \geq 2$. In the second phase of the incomplete game the (ordinary) game G_j is played. As in Remark 5.2, we adopt the "psychological interpretation" of incomplete games: $p_1, ..., p_k$ are determined "subconsciously" by the strategic behavior which is inherent in the genes of the players.

Using Definition 6.1 we suggest to replace Corollary 2.2 by the following postulate.

(*) *Sufficiency of simple incomplete games:* A full framework for rational interaction between n rational decision–makers is a simple incomplete n–person game $< G_1, p_1;, ...; G_k, p_k >$, where $G_1, ..., G_k$ are n–person noncooperative games in extensive form.

The operational interpretation of (*) is as follows. Let $N = \{1, ..., n\}$, $n \geq 2$, be a group of n rational decision–makers, and let δ be a decision problem for N. Then there is a (nonempty) set Γ of n–person noncooperative games in extensive form which may resolve δ. δ is resolved by the following three–stage procedure.

Stage 1: Nature chooses an incomplete simple game $< G_1, p_1;, ...; G_k, p_k >$ where $G_j \in \Gamma$ for $j = 1, ..., k$.

Stage 2: Nature publicly chooses an index j_0, $1 \leq j_0 \leq k$.

Stage 3: The game G_{j_0} is played by N in order to find a solution for δ.

Remark 6.2. Stage 2 may be replaced by:

Stage 2*: The agents choose spontaneously, by the strategic behavior inherent in their genes, the game G_{j_0} that will be played in Stage 3.

The new formulation allows for an explicit presentation of the improvement of the solution of δ by the evolutionary process. Hence, it has an advantage over the first formulation.

Remark 6.2. In our interpretation of (*) we use an assumption which might be called the "weak program of Nash". We now separately formulate this assumption.

The weak program of Nash: Let N be a set of n rational decision–makers, $n \geq 2$, and let δ be a decision problem for N. Then there is a (nonempty) set Γ of n–person noncooperative games in extensive form which may resolve δ. The choice of a game from Γ, in order to solve δ, must be left to nature.

Appendix

Throughout the appendix we use the notations of Section 4.

Theorem A.1. There exists a closed subset F of $\mathop{\times}\limits_{t=0}^{\infty} \mathop{\times}\limits_{i=1}^{n} F^{nt+1}$ such that $H = F \times U^N$.

Proof: By the definition of H there is a subset F of $\mathop{\times}\limits_{t=0}^{\infty} \mathop{\times}\limits_{i=1}^{n} F^{nt+1}$ such that $H = F \times U^N$. Indeed F is the set of all n–tuples of sequences $\underset{\sim}{\psi} = < \underset{\sim}{\psi}^1, \ldots, \underset{\sim}{\psi}^n >$ with the following properties:

(A.1) ψ_0^i is a nonempty closed subset of Q, $i \in N$.

(A.2) For $t \geq 1$ ψ_t^i is a nonempty closed subset of $Q^{tN} \times Q^{\{i\}}$ that satisfies (4.6) (where A_t is given by (4.5)).

Let $\underset{\sim}{\psi}(k) \in F$, h = 1, 2, ... If $\underset{\sim}{\psi}(k) \to \underset{\sim}{\psi}$ in $\mathop{\times}\limits_{t=0}^{\infty} \mathop{\times}\limits_{i=1}^{n} F^{nt+1}$ then $\psi_t^i(k) \to \psi_t^i$ in the Hausdorff metric for t = 0, 1, 2, ... Therefore, $A_t(k) \to A_t$ where A_t is the projection of ψ_t^i, $i \in N$, on Q^{tN}, t = 1, 2, ..., because of (4.6). Thus, $\underset{\sim}{\psi}$ satisfies (4.6). (A.1) is satisfied by definition. Thus, it remains to prove that $\underset{\sim}{\psi}$ satisfies (4.5) for $t \geq 2$, that is

$A_t = \{< x, y^1, \ldots, y^n > | x \in A_{t-1} \text{ and } (x, y^i) \in \psi_{t-1}^i \text{ for all } i \in N\}$ for t = 2, 3, ... Now

$A_t(k) = \{< x, y^1, \ldots, y^n > | x \in A_{t-1}(k) \text{ and } (x, y^i) \in \psi_{t-1}^i(k), i \in N\}$.

As $A_t = \lim\limits_{k \to \infty} A_t(k)$,

$A_t \subset \{< x, y^1, \ldots, y^n > | x \in A_{t-1} \text{ and } (x, y^i) \in \psi_{t-1}^i, i \in N\}$

Let $< x, y^1, \ldots, y^n > \notin A_t$ such that $x \in A_{t-1}$. Then $d(A_t(k), < x, y^1, \ldots, y^n >) \geq \delta$ where $\delta > 0$, for $k \geq K(\delta)$ and there is a sequence $x(k) \in A_{t-1}(k)$, $x(k) \to x$. Therefore, for some $i \in N$, there exists $\epsilon > 0$ such that $d((x(k), y_i), \psi_{t-1}^i(k)) \geq \epsilon$ for $k \geq K(\epsilon)$. Hence, $(x, y_i) \notin \psi_{t-1}^i$. This proves that

$A_t \supset \{< x, y^1, \ldots, y^n > | x \in A_{t-1} \text{ and } (x, y_i) \in \psi_{t-1}^i, i \in N\}$

Q.E.D.

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