INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 242

Alienated Extensions and Common Knowledge Worlds

by

Robert S. Simon

April 1995

University of Bielefeld

33501 Bielefeld, Germany

Alienated Extensions and Common Knowledge Worlds

Abstract

This article concerns the interactive modal propositional calculus, using the multi-agent epistemic logic $S5$. For a finite alphabet and a finite number of agents of at least two there are uncountably many minimal common knowledge worlds in which only the tautologies are held in common knowledge.

1 Introduction

This paper investigates further the approach to the interactive modal propositional calculus explored by Robert Aumann in his paper "Notes on Interactive Epistemology" [Au2]. Working with a finite alphabet of primitive propositions, a finite number of agents, formulae constructed finitely by "negation", the "and" and "or" conjuctions, and symbols for the knowledge of each of the agents, and the multi-agent epistemic logic $S5$, I was confronted, like many others, with two aspects of the relationship between semantics and syntactics. One has well known completeness results, which present an equivalence between syntactic and semantic knowledge. On the other hand, this equivalence takes place only when considering for any given state an agent's knowledge or lack of knowledge of all the formulae of the language, a very rich kind of "knowledge" not expressible within the language. Several authors have moved toward more complex logical structures in an effort to overcome this problem; however, this problem has lead me instead to a study of what is implied from intermediate levels of "knowledge" beyond what can be expressed within the language but short of a knowledge determination for every formula in the language.

In particular, what can one conclude about the semantic formulation of common knowledge from the set of formulae held in common knowledge?

Concerning the above question, I have a pessimistic result. There is an uncountable set of "minimal common knowledge worlds" of the canonical state space that can never be ruled out by any finite set of formulae and the fact that the only formulae in common knowledge are the tautologies.

In order to prove this result, I have created a special model for the multi-agent epistemic logic $S5$ when there is a finite alphabet and a finite number of agents. This model is constructed in hierachichal stages, and the end result is represented by a Cantor set of a Euclidean space. At all intermediate finite stages of this construction there is asymetry with respect to the agents. I did this for two reasons. First, I thought that an understanding of how an agent refines its knowledge concerning lower stages in a way consistent with those lower stages and the knowledge of other agents is done best by concentrating on this single agent. Second, to understand the intricacies of common knowledge I wanted to understand as explicitly as possible what it means for the knowledge of one agent to remain constant while the knowledge of other agents is variable. (For other constructions of universal models see

Fagin, Halpern, and Vardi [Fa-Ha-Va] or Heifetz and Samet [He-Sa].)

The main result of Section 5, the existence of at least one common knowledge world in which only the tautologies are common knowledge, has already been proven using a previous existing construction [Fa-Ha-Va]. The main result of this paper, mentioned above and proved in Section 6, seems to be original, but probably can be proven using the existing universal models. Whether the construction of the model presented in this paper has unique heuristic value for this or other results cannot be answered at this time.

Throughout this article, the multi-agent epistemic logic $S5$ will be assumed, also refered to as $S5_n$ when n is the number of agents [Hu-Cr, Ha-Mo].

2 Background

Construct the set $\mathcal{L}_I(X)$ of legitimate formulae using the alphabet set X of primitive propositions with a set of agents indexed by I in the following way:
1) If $x \in X$ then $x \in \mathcal{L}_I(X)$,
2) If $g \in \mathcal{L}_I(X)$ then $(\neg g) \in \mathcal{L}_I(X)$,
3) If $g, h \in \mathcal{L}_I(X)$ then $(g \wedge h) \in \mathcal{L}_I(X)$.
4) If $g \in \mathcal{L}_I(X)$ then $(k_j g) \in \mathcal{L}_I(X)$ for every $j \in I$.
5) Only formulae constructed through application of the four above rules are members of $\mathcal{L}_I(X)$.
We can consider $g \vee h$ to be $\neg(\neg g \wedge \neg h)$ and $g \Rightarrow h$ to be $\neg g \vee h$. Where no ambiguity exists, we can drop the paretheses.

We will work with the canonical indexing of a finite set of n agents, namely $I_n := \{1, 2, \cdots, n\}$.

For a discussion of the $S5$ logic system, see Hughes and Cresswell, An Introduction to Modal Logic [Hu-Cr]; and for the multi-agent variation $S5_n$, see Halpern and Moses [Ha-Mo]. Briefly, the $S5_n$ logic system is defined by two rules of inference, modus ponens and necessitation, and five types of axioms. Modus ponens means that if f is a theorem and $f \Rightarrow g$ is a theorem, then g is also a theorem. Necessitation means that if f is a theorem then $k_j f$ is also a theorem for all $1 \leq j \leq n$. The axioms are the following, for every $f, g \in \mathcal{L}_{I_n}(X)$ and $1 \leq j \leq n$:
1) all formulae resulting from tautologies of the propositional calculus through substitution,

2) $(k_j f \wedge k_j(f \Rightarrow g)) \Rightarrow k_j g$,
3) $k_j f \Rightarrow f$,
4) $k_j f \Rightarrow k_j(k_j f)$,
5) $\neg k_j f \Rightarrow k_j(\neg k_j f)$.

A list of formulae in $\mathcal{L}_{I_n}(X)$ is called "complete" if for every formula $f \in \mathcal{L}_{I_n}(X)$ either f or $\neg f$ is in this list. A list of formulae is called "consistent" if no finite subset of this list leads to a logical contradiction, (using the $S5_n$ logic system.) Define a formula $f \in \mathcal{L}_{I_n}(X)$ to be "possible" if $\neg f$ is not a tautology of $S5_n$.

A formula $f \in \mathcal{L}_I(X)$ is common knowledge in a list of formulae $\mathcal{A} \subseteq \mathcal{L}_I(X)$ if $f \in \mathcal{A}$ and for every $m \geq 1$ and every function $a : \{1, \cdots, m\} \to I$ the formula $k_{a(m)} \cdots k_{a(1)} f$ is in $\mathcal{A}$ [Le].

Consider any set S with partitions $\{\mathcal{Q}^i \mid i \in I\}$ of S, sometimes called an Aumann structure [Au1]. For each $i \in I$ define a mapping $K_i : \mathcal{P}(S) \to \mathcal{P}(S)$, from the set of subsets of S to itself, by

$$K_i(A) := \{a \in A \mid a \in B \in \mathcal{Q}^i \Rightarrow B \subseteq A\} \quad \text{[Au1]}.$$

(Notice that $K_i(A) = \emptyset$ is possible when $A \neq \emptyset$.) One can interpret $\mathcal{Q}^i$ as the collection of sets representing the finest instrument providing discrete measurements available to the ith agent, that is $A \in \mathcal{Q}^i$ is a set such that for every $a \in A$ and $b \in S$ the ith agent can discriminate between a and b if only if $b \notin A$ [Au1].

With a set S and partitions $\{\mathcal{Q}^i \mid i \in I\}$ of S one can define a semantic concept of common knowledge. Consider the meet partition $\vee_{i \in I} \mathcal{Q}^i$, which is the finest partition coarser than $\mathcal{Q}^i$ for all $i \in I$. The set $A \subseteq S$ is common knowledge at s if and only if $s \in B \in \vee_{i \in I} \mathcal{Q}^i$ implies that $B \subseteq A$ [Au1]. Equivalently, one can define A to be common knowledge at $s \in S$ if and only if for all $1 \leq m < \infty$ and functions $a : \{1, \cdots, m\} \to I$ it follows that $s \in K_{a(m)}(\cdots (K_{a(1)}(A)) \cdots)$ [Au1]. A "common knowledge world" is a set $B \subseteq S$ such that $K_i(B) = B$ for all $i \in I$ [Au2]. The set of minimal common knowledge worlds is the same as the members of the partition $\vee_{i \in I} \mathcal{Q}^i$.

If in addition to a set S and partitions $\{\mathcal{Q}^i \mid i \in I\}$ we have a an alphabet X and a mapping $\psi : X \to \mathcal{P}(S)$, the quintuple $\mu = (S; I; \{\mathcal{Q}^i \mid i \in I\}; X; \psi)$ is called a "model." A model is one easy way to generate complete and consistent lists of formulae. We can define a mapping $\phi^\mu : \mathcal{L}_I(X) \to \mathcal{P}(S)$ inductively on the structure of the formulae in the following way:

Case 1 $f = x \in X$: $\phi^\mu(x) := \psi(x)$.
Case 2 $f = \neg g$: $\phi^\mu(f) := S - \phi^\mu(g)$,
Case 3 $f = g \wedge h$: $\phi^\mu(f) := \phi^\mu(g) \cap \phi^\mu(h)$,
Case 4 $f = k_i(g)$: $\phi^\mu(f) := K_i(\phi^\mu(g))$.
For any point $s \in S$ one can consider the list of formulae defined by $V^\mu(s) := \{f \in \mathcal{L}_I(X) \mid s \in \phi^\mu(f)\}$. Such a list of formulae is complete and consistent due to Case 2 and the fact that the implication of f and $\neg f$ from the use of the multi-agent $S5$ logic system would imply the containment of the point s in both $\phi^\mu(f)$ and $\phi^\mu(\neg f)$. (See Hughes and Cresswell [Hu-Cr].)

For a given alphabet X and the canonical index I_n of n agents, Aumann labeled the set of all consistent and complete lists of formulae in $\mathcal{L}_{I_n}(X)$ with the symbol $\Omega = \Omega(X, I_n)$ [Au2]. Can one consider the canonical state space Ω as a model itself? Yes. For $1 \leq i \leq n$ consider the partition $\mathcal{Q}^i$ of Ω generated by inverse images of the function $\beta^i : \Omega \to \mathcal{P}(\mathcal{L}_{I_n}(X))$ defined by $\beta^i(\omega) := \{f \in \mathcal{L}_{I_n}(X) \mid k_i(f) \in \omega\}$. Consider the mapping $\psi : X \to \mathcal{P}(\Omega)$ defined by $\psi(x) := \{\omega \in \Omega \mid x \in \omega\}$. Now we have a model $\Omega = (\Omega; I_n; \mathcal{Q}^1 \cdots, \mathcal{Q}^n; X; \psi)$. Following closely the proofs of previously existing completeness theorems, Aumann showed that $\phi^\Omega(f) = \{\omega \mid f \in \omega\}$ for every $f \in \mathcal{L}_{I_n}(X)$ and $V^\Omega(\omega) = \omega$ for every $\omega \in \Omega$ [Au2]. (See also Halpern and Moses [Ha-Mo] and Hughes and Cresswell [Hu-Cr].) For the purposes of this paper, we will call this result the "Completeness Theorem."

The Completeness Theorem has fascinating consequences for the minimal common knowledge worlds of $\Omega = (\Omega; I_n; \mathcal{Q}^1 \cdots \mathcal{Q}^n; X; \psi)$. Assume that $\mu = (S, I_n, \mathcal{P}^1, \cdots, \mathcal{P}^n; X; \psi)$ is a model and f is common knowledge in $V^\mu(s)$ for some $s \in S$. If $s \in B \in \mathcal{P}^1 \vee \cdots \vee \mathcal{P}^n$ and $s' \in B$ then it is an easy induction proof to show that f is also common knowledge in $V^\mu(s')$. On the other hand, assume that for some $B \in \mathcal{P}^1 \vee \cdots \vee \mathcal{P}^n$ that $B \subseteq \phi^\mu(f)$. If there were some $s \in B$ such that f were not common knowledge in $V^\mu(s)$, then $s \notin \phi^\mu(k_{a(j)} \cdots k_{a(1)} f)$ for some j and $a : \{1, \cdots, j\} \to \{1, \cdots, n\}$ would mean that $A \not\subseteq \phi^\mu(k_{\alpha(j-1)} \cdots k_{m(1)} f)$ for the $A \in \mathcal{P}^{\alpha(j)}$ containing s. Also by induction we would get an $s' \in B$ with $s' \notin \phi^\mu(f)$, a contradiction. Therefore we have a nice elementary result:

Lemma 0: If $s \in B \in \mathcal{P}^1 \vee \cdots \vee \mathcal{P}^n$ then f is common knowledge in the list $V^\mu(s)$ if and only if $B \subseteq \phi^\mu(f)$ (Lemma 4.1, [Ha-Mo]).

For the state space Ω, using the Completeness Theorem, Lemma 0 would imply:

For any $B \in \mathcal{Q}^1 \vee \cdots \vee \mathcal{Q}^n$ $\{f \mid f$ is common knowledge in ω for some $\omega \in B\} = \{f : f$ is common knowledge in ω for all $\omega \in B\} = \{f : f \in \omega$ for all $\omega \in B\}$.

With the help of the above application of Lemma 0 to $(\Omega; I_n; \mathcal{Q}^1, \cdots \mathcal{Q}^n; X; \psi)$, one could restate the main result of this paper:

Proposition 3: There is an uncountable subset $\mathcal{T} \subseteq \mathcal{Q}^1 \vee \cdots \vee \mathcal{Q}^n$ such that for any $B \in \mathcal{T}$ and any possible formula $f \in \mathcal{L}_{I_n}(X)$ there is a point $\omega \in B$ such that $f \in \omega$.

If one gives the state space $\Omega = \Omega(X, I_n)$ the natural topology whose base of open sets is $\{\phi^{\Omega}(f) \mid f \in \mathcal{L}_{I_n}(X)\}$, then Proposition 3 claims that every member of $\mathcal{T}$ is dense in the entire state space Ω.

Section 3 contains a construction of a special model for the multi-agent epistemic logic $S5$ with a finite alphabet and a finite number of agents. Section 4 establishes some relationships between this construction and models for this logic in general, especially concerning the state space Ω. Section 5 shows a canonical way to extend a finite stage of this special model called the "alienated extension," and proves the existence of at least one dense minimal common knowledge world in Ω. Section 6 contains a proof of the main result, through introducing an uncountable set of variations on the alienated extension.

3 A Construction of the Model Ω_∞

To investigate the state space $\Omega = \Omega(X, I_n)$ we construct another model which turns out to be semantically equivalent to Ω, but which is more tractible for the present purposes. Since the construction is complicated, we give first a small sketch of it.

We create a decreasing sequence of closed sets $\Omega_0 \supseteq \Omega_1 \supseteq \cdots$ in a Euclidean space and partitions $\mathcal{F}_i$ of Ω_i, $i = 1, 2, \cdots$. (If the alphabet X were infinite, the model would have to be embedded into a different space with the appropriate topological properties.) The first partition $\mathcal{F}_1$ describes basic possible non-epistemic information of the first agent. The second partition $\mathcal{F}_2$ describes basic possible non-epistemic information of the second agent as well as information concerning the first agent's information partition $\mathcal{F}_1$. This process does not stop at the nth agent; at the n+1 stage, the first agent's information partition $\mathcal{F}_1$ is refined by taking into account the information partitions of the other agents defined in the first round. This refinement must not contradict the first agent's information on the first stage, which adds some technical conditions to stages greater than n which are not needed for stages up to and including n. This process does not stop at any finite stage, and we are interested in the limiting structure. For topological reasons, in each step we choose a space Ω_i smaller than Ω_{i-1}.

In order to define the process and make arguments concerning finite stages we introduce auxillery partitions $\mathcal{G}_i$ of Ω_i. $\mathcal{G}_i$ represents the finest partition relevant to the ith stage.

If $\mathcal{Q}$ is a partition of a set S and T is a subset of S, one can always consider the partition of T induced by $\mathcal{Q}$, namely $\mathcal{Q}|_T := \{A \cap T \mid A \in \mathcal{Q}, A \cap T \neq \emptyset\}$. After describing the construction, we will be concerned with the set $\Omega_\infty := \cap_{i=0}^{\infty} \Omega_i$ and the partitions of Ω_∞ induced by $\mathcal{G}_i$ and $\mathcal{F}_i$ for finite $i < \infty$. The liberty is taken to write that $\mathcal{F}_j$ and $\mathcal{G}_k$ are partitions of Ω_i for all i greater than or equal to j and k, including $i = \infty$, instead of writing $\mathcal{F}_j|_{\Omega_i}$ and $\mathcal{G}_k|_{\Omega_i}$.

The sequence of partitions $\mathcal{G}_i$ will be strictly refining for increasing i; and for each $1 \leq j \leq n$ the sequence of partitions $\mathcal{F}_{j+kn}$ will be strictly refining for increasing k. The intersection $\Omega_\infty := \cap_{i=1}^{\infty} \Omega_i$ will be a Cantor set, along with a mapping $\psi_\infty : X \to \mathcal{P}(\Omega_\infty)$ and knowledge partitions of Ω_∞ for each of the n agents, corresponding to the limits of the partitions $\mathcal{F}_i$ modulo n. (The limit partition of $\mathcal{G}_i$ is not interesting, since it will be the discrete partition of Ω_∞.)

X will always be a fixed non-empty finite set and n will always be a fixed natural number greater than or equal to 2. Let $\mathbf{R}^t$ be a Euclidean space with $t \geq 1$. (One may wish to choose t large enough to allow for interesting geometric relationships.) Let

$$\Omega_0 := \bigcup_{i=1}^{2^{|X|}} R_i$$

where each $R_i \subseteq \mathbf{R}^t$ is a t-dimensonal compact set and $R_i \cap R_j = \emptyset$ for every

$i \neq j$.

We define a function $\psi_0 : X \to \mathcal{P}(\Omega_0)$. Consider any bijection $\alpha : \{0,1\}^X \to \{R_1, \cdots, R_{2^{|X|}}\}$, and let $\pi_i : \{0,1\}^X \to \{0,1\}$ be the projection onto the ith coordinate. Define

$$\psi_0(x_i) := \alpha(\pi_i^{-1}(1)).$$

Define $\mathcal{G}_0$ to be the natural partition of Ω_0 with $2^{|X|}$ members, ($\mathcal{G}_0 = \{R_1, \cdots, R_{2^{|X|}}\}$,) and define $\mathcal{F}_0$ to be the trivial partition of Ω_0, namely $\{\Omega_0\}$.

For any partition $\mathcal{Q}$ let $\overline{\mathcal{Q}}$ be the non-empty members of the field generated by $\mathcal{Q}$, namely the sets created by the finite unions of members of $\mathcal{Q}$.

For the sake of formality, for all $i < 0$ let $\Omega_i = \Omega_0$ and $\mathcal{F}_i = \mathcal{G}_i = \mathcal{F}_0 = \{\Omega_0\}$. We will define Ω_i, $\mathcal{G}_i$, and $\mathcal{F}_i$ inductively with the assumption that Ω_j, $\mathcal{F}_j$ and $\mathcal{G}_j$ have been defined for all $j < i$. We will see after the fact that $\mathcal{G}_i = \mathcal{F}_i \wedge \mathcal{G}_{i-1}$ for all $i \geq 1$.

Because $\mathcal{F}_i$ will be a refinement of $\mathcal{F}_{i-n}$, let us consider an arbitrary $A \in \mathcal{F}_{i-n}$. Consider the double nested partition of $A \cap \Omega_{i-1}$ created by the non empty intersections of $A \cap \Omega_{i-1}$ with the members of the partitions $\mathcal{G}_{i-n-1}$ and $\mathcal{G}_{i-1}$, (assuming that $\mathcal{G}_{i-1}$ is finer than $\mathcal{G}_{i-n-1}$.) Let these partitions of $A \cap \Omega_{i-1}$ be denoted by $\mathcal{G}^A_{i-n-1}$ and $\mathcal{G}^A_{i-1}$, respectively. Let $B^A_i, B^A_2, \cdots, B^A_{m(A)}$ be the members of $\mathcal{G}^A_{i-n-1}$. (If $i \leq n$ then there is only one $A \in \mathcal{F}_{i-n}$, namely $\Omega_{i-n} = \Omega_0$, $m(\Omega_{i-n}) = 1$, $\mathcal{G}^{\Omega_{i-n}}_{i-n-1} = \{\Omega_{i-1}\}$, and $B_1^{\Omega_{i-n}} = \Omega_{i-1}$.) For $1 \leq j \leq m(A)$ let $l^A(j)$ be the number of members of $\mathcal{G}^A_{i-1}$ contained in B^A_j. (If $i \leq n$, then $l^{\Omega_{i-n}}(1)$ is the number of members of $\mathcal{G}_{i-1}$.) For every $C \in \mathcal{G}^A_{i-1}$, assuming that $C \subseteq B^A_j$, choose

$$\rho(C) := 2^{(l^A(j)-1)} \prod_{k=1,\cdots,m(A),\ k \neq j} (2^{(l^A(k))} - 1)$$

mutually disjoint t-dimensional compact sets contained in C, called $D^C_1, D^C_2, \cdots, D^C_{\rho(C)}$, such that diameter $(D^C_m) \leq 1/i$ for all $1 \leq m \leq \rho(C)$. (If $i \leq n$ then $\rho(C) = 2^{|\mathcal{G}_{i-1}|-1}$ for every $C \in \mathcal{G}^{\Omega_0}_{i-1}$.) After this proceedure is performed for every $A \in \mathcal{F}_{i-n}$ and $C \in \mathcal{G}^A_{i-1}$ let

$$\Omega_i := \bigcup_{A \in \mathcal{F}_{i-n}} \bigcup_{C \in \mathcal{G}^A_{i-1}} \bigcup_{j=1}^{\rho(C)} D^C_j \quad \subset \Omega_{i-1}.$$

Let $\mathcal{G}_i := \{D_j^C \mid A \in \mathcal{F}_{i-n}, C \in \mathcal{G}_{i-1}^A, 1 \leq j \leq \rho(C)\}$. (Therefore $\mathcal{G}_i$ is a refinement of $\mathcal{G}_{i-1}$ as partitions of Ω_i.)

For any $A \in \mathcal{F}_{i-n}$ and $C \in \mathcal{G}_{i-1}^A$ there are exactly $\rho(C)$ different subsets of $A \cap \Omega_{i-1}$ in $\overline{\mathcal{G}_{i-1}^A}$ containing the set C with a non-empty intersection with every $B_j^A, 1 \leq j \leq m(A)$. (If $i \leq n$ then the non-empty intersection condition can be dropped, as there would be only one member in $\underline{\mathcal{G}_{i-n-1}^{\Omega_{i-n}}}$, namely Ω_{i-1}.) The number of all such subsets of $A \cap \Omega_{i-1}$, members of $\overline{\mathcal{G}_{i-1}^A}$ with non-empty intersections with every member of $\mathcal{G}_{i-n-1}^A$, is

$$q(A) := \prod_{j=1}^{m(A)} (2^{(l^A(j))} - 1),$$

and let $\mathcal{S}(A) := \{S_1, S_2, \cdots, S_{q(A)}\}$ be the collection of all such subsets. (If $i \leq n$ then $q(\Omega_{i-n}) = 2^{|\mathcal{G}_{i-1}|} - 1$.) One can create a collection of mutually disjoint sets $E_1, E_2, \cdots, E_{q(A)} \in \overline{\mathcal{G}_i}$ such that

1) $\cup_{C \in \mathcal{G}_{i-1}^A,\, E_j \cap C \neq \emptyset} C = S_j$ for every $1 \leq j \leq q(A)$,

2) for every $1 \leq j \leq q(A)$ and $C \in \mathcal{G}_{i-1}^A$, the intersection of E_j with C is either empty or a single member of $\mathcal{G}_i$.

3) $\cup_{j=1}^{q(A)} E_j = A \cap \Omega_i$.

The refinement $\mathcal{F}_i$ of $\mathcal{F}_{i-n}$ within $A \cap \Omega_i$ is defined to be the partition $\{E_1, \cdots, E_{q(A)}\}$ of $A \cap \Omega_i$. From the above conditions, for all $i \geq 1$ it follows that $\mathcal{F}_i \wedge \mathcal{G}_{i-1} = \mathcal{G}_i$ as partitions of Ω_i, and for all $i \geq 1$ and $A \in \mathcal{F}_{i-n}$, $\mathcal{G}_{i-1}^A = \{C \in \mathcal{G}_{i-1} \mid C \subseteq A\}$.

From now on we shall assume that $\mathcal{F}_i$ and $\mathcal{G}_i$ are partitions of Ω_∞.

Notice that, by the Nested Interval Principle, for any $0 \leq i < \infty$ and for any set $S \in \mathcal{G}_i$ there are points of Ω_∞ contained in S. By the condition on the diameter of a member of $\mathcal{G}_i$ any decreasing sequence of sets $C_i \in \mathcal{G}_i$ will intersection in a point.

For $1 \leq j \leq n$ we define the partition $\mathcal{F}_i^j$ of Ω_∞ to be $\mathcal{F}_l$ where l is the greatest number less than or equal to i such that $l = j (\text{mod } n)$, and we define

$$\mathcal{F}_\infty^j := lim_{i \to \infty} \quad \mathcal{F}_i^j,$$

whereby the limiting partition of a refining sequence of partitions is defined by the relation $a \sim b$ if and only if a and b share the same partition member for every partition of the sequence. Let $\psi_\infty : X \to \mathcal{P}(\Omega_\infty)$ be defined by $\psi_\infty(x) := \Omega_\infty \cap \psi_0(x)$. For every i, including $i = \infty$, we consider the model

$\lambda^i = (\Omega_\infty; X; \mathcal{F}_i^1, \cdots, \mathcal{F}_i^n; \psi_\infty)$, and let $\phi_i := \phi^{\lambda^i} : \mathcal{L}_{I_n}(X) \to \mathcal{P}(\Omega_\infty)$ be the corresponding mapping defined in Section 2. Notice that an image of ϕ_i is always either a member of $\overline{\mathcal{G}_i}$ or the empty set.

4 Basic Results

Lemma 1 If $g, h \in \mathcal{L}_{I_n}(X)$ are such that $\phi_i(g) = \phi_\infty(g)$ for all $i \geq m_1 \geq 0$ and $\phi_i(h) = \phi_\infty(h)$ for all $i \geq m_2 \geq 0$ then
1)$\phi_i(\neg g) = \phi_\infty(\neg g)$ for all $i \geq m_1$,
2) $\phi_i(g \wedge h) = \phi_\infty(g \wedge h)$ for all $i \geq max(m_1, m_2)$, and
3) $\phi_i(k_j g) = \phi_\infty(k_j g)$ for all $i \geq l$ such that l is the least number satisfying $l = j(\mod n)$ and $l \geq max(1, m_1)$.

Proof: The first two cases follow from the fact that $\phi_i(\neg g) = \Omega_\infty - \phi_i(g)$ and $\phi_i(g \wedge h) = \phi_i(g) \cap \phi_i(h)$.

For the third case, assume that $a \in A \in \mathcal{F}_l$.

Assume that $a \in \phi_l(k_j g)$, which means that A is contained in $\phi_l(g)$. By assumption, $A \subseteq \phi_i(g)$ for all $i \geq l$. But since $\mathcal{F}_i^j$ are all refinements of $\mathcal{F}_l$ for $i \geq l$, including the case $i = \infty$, we conclude for all $i \geq l$ and $a \in A' \in \mathcal{F}_i^j$ that $A' \subseteq \phi_i(g)$, and therefore $a \in \phi_i(k_j g)$.

Assume that $a \notin \phi_l(k_j g)$. This implies that there is a $D \in \mathcal{G}_l$ such that $D \subseteq A$ and $D \not\subseteq \phi_l(g)$. Since $\phi_l(g)$ is a member of $\overline{\mathcal{G}_l} \cup \{\emptyset\}$, we conclude, also by assumption, that $D \cap \phi_i(g) = \emptyset$ for all $i \geq l$, including $i = \infty$.

Next we claim that for every $0 \leq m < \infty$ there is a decreasing sequence of sets $D_m \in \mathcal{G}_{l+mn}$, $D_0 = D$, such that D_m is contained in the member of $\mathcal{F}_{l+mn}$ containing a. This was shown already for $m = 0$. Assuming that $a \in A_{m-1} \in \mathcal{F}_{l+(m-1)n}$ and $D_{m-1} \in \mathcal{G}_{l+(m-1)n}$ with $D_{m-1} \subseteq A_{m-1}$ for $m \geq 1$, consider the $A_m \in \mathcal{F}_{l+mn}$ such that $a \in A_m$. Let C_{m-1} be the element of $\mathcal{G}_{l+(m-1)n-1}$ such that $D_{m-1} \subseteq C_{m-1}$. By the construction of $\mathcal{F}_{l+mn}$ it follows that $A_m \cap C_{m-1} \neq \emptyset$. Since A_m and C_{m-1} are both members of $\overline{\mathcal{G}_{l+mn}}$ one can choose a $D_m \in \mathcal{G}_{l+mn}$ with $D_m \subseteq A_m \cap C_{m-1}$. Since $D_{m-1} = A_{m-1} \cap C_{m-1}$ from the construction of $\mathcal{G}_{l+(m-1)n}$ and $l + (m-1)n \geq l \geq 1$, it follows that $D_m \subseteq D_{m-1}$.

The intersection point of the sequence D_m will also share with a the same member of $\mathcal{F}_\infty^j$, which completes the claim. q.e.d.

We define the "depth" of a formula inductively on the structure of the formulae. If $x \in X$, then depth $(x) := 0$. If $f = \neg g$ then depth $(f) :=$ depth (g); if $f = g \wedge h$ then depth $(f) :=$ max (depth (g) , depth (h)); and if $f = k_j(g)$ then depth $(f) :=$ depth $(g) + 1$.

Corollary 1: For all $f \in \mathcal{L}_{I_n}(X)$ and all $i \geq n$ depth (f), $\phi_i(f) = \phi_\infty(f)$.

Remark: See Lemma 2.5 of [Fa-Ha-Va].

Proof: The statement is true for a formula $x \in \mathcal{L}_{I_n}(X)$ such that $x \in X$. The rest follows from induction on the structure of the formulae and Lemma 1.

Lemma 2: Let $\mu = (\Omega'; I_n; \mathcal{Q}^1, \cdots, \mathcal{Q}^n; X; \psi')$ be any model with the alphabet X and n agents. There is a mapping $\gamma^\mu : \Omega' \to \Omega_\infty$ such that for every $f \in \mathcal{L}_{I_n}(X)$ and $\omega' \in \Omega'$, $\omega' \in \phi^\mu(f)$ if and only if $\gamma^\mu(\omega') \in \phi_\infty(f)$.

Proof: Define the map $\gamma_0^\mu : \Omega' \to \mathcal{G}_0$ by

$$\gamma_0^\mu(\omega') := \Omega_\infty \cap \alpha(1_{\psi'(x_1)}(\omega'), \cdots, 1_{\psi'(x_{|X|})}(\omega')).$$

For the sake of formality, let us assume that $\gamma_i^\mu = \Omega_\infty$ for all $i < 0$. Given that $\gamma_k^\mu : \Omega' \to \mathcal{G}_k$ is already defined for all $k < i$, define $\gamma_i^\mu : \Omega' \to \mathcal{G}_i$ for $i \geq 1$ in the following way. Let $1 \leq j \leq n$ satisfy $j = i (\text{mod } n)$. For any $\omega' \in A' \in \mathcal{Q}^j$, let $\gamma_i^\mu(\omega') := \gamma_{i-1}^\mu(\omega') \cap A$ where $A \in \mathcal{F}_i$ satisfies $\{C \in \mathcal{G}_{i-1} \mid C \cap A \neq \emptyset\} = \{C \in \mathcal{G}_{i-1} \mid {\gamma_{i-1}^\mu}^{-1}(C) \cap A' \neq \emptyset\}$. By the construction of $\mathcal{F}_i$ there can be at most one $A \in \mathcal{F}_i$ corresponding to this subset, and by induction the function is well defined for all $i \leq n$. Let us suppose for the sake of contradiction that γ_i^μ is not well defined for some $i > n$ but well defined for all $m < i$. That means that there exists some $\omega' \in A' \in \mathcal{Q}^j$ with $j = i (\text{ mod } n)$ and some $B \in \mathcal{G}_{i-n-1}$ with $B \cap A^* \neq \emptyset$ and ${\gamma_{i-1}^\mu}^{-1}(B) \cap A' = \emptyset$, where A^* is the member of $\mathcal{F}_{i-n}$ such that $\gamma_{i-n}^\mu(\omega') = A^* \cap \gamma_{i-n-1}^\mu(\omega')$. It follows by the definition of γ_{i-n}^μ that ${\gamma_{i-n-1}^\mu}^{-1}(B) \cap A' \neq \emptyset$. Since there exists an $\omega' \in A'$ such that $\gamma_{i-1}^\mu(\omega') \subseteq \gamma_{i-n-1}^\mu(\omega') = B$, we have a contradiction.

Keeping with the above sets, notice that for all $i \geq 1$ the $A \in \mathcal{F}_i$ used to define $\gamma_i^\mu(\omega')$ is the same for all ω' in A', which means that $\gamma_i^\mu(A') \subseteq A$. Let us examine an arbitrary $D \in \mathcal{G}_i$ with $D \subseteq A$. There exists exactly one $C \in \mathcal{G}_{i-1}$ with $D \subseteq C$, and it follows from $i \geq 1$ that $C \cap A = D$. By the definition of γ_i^μ, the set ${\gamma_{i-1}^\mu}^{-1}(C) \cap A'$ is not empty and its image under γ_i^μ is D. Therefore $\gamma_i^\mu(A') = A$ (as long as $i \geq 1$.)

Next we claim that if depth$(f) = d$ then $\omega' \in \phi^\mu(f)$ if and only if

$\gamma^{\mu}_{dn}(\omega') \subseteq \phi_{dn}(f)$. We proceed by induction on the structure of the formulae. The claim is true for a formula $x \in X$. Due to Corollary 1, the only case presenting a problem is $f = k_j(g)$ for some $1 \leq j \leq n$ and for depth $(g) = d - 1 \geq 0$. Let l be the largest number less than or equal to nd with $l = j(\bmod n)$. Let $\omega' \in A' \in \mathcal{Q}^j$ and let $\gamma^{\mu}_{nd}(\omega') \subseteq A \in \mathcal{F}_l = \mathcal{F}^j_{nd}$. It follows that $A \subseteq \phi_{nd}(g)$ if and only if $A \subseteq \phi_l(g)$ if and only if $\gamma^{\mu}_l(A') \subseteq \phi_l(g)$ if and only if $\gamma^{\mu}_{(d-1)n}(A') \subseteq \phi_{(d-1)n}(g)$ if and only if $A' \subseteq \phi^{\mu}(g)$; the first implication follows by Corollary 1, the second implication by the above mentioned fact that $\gamma^{\mu}_l(A') = A$ from $l \geq 1$, the third implication by Corollary 1 and the fact that for all $\omega' \in A'$ $\gamma^{\mu}_l(\omega') \subseteq \gamma^{\mu}_{(d-1)n}(\omega') \in \mathcal{G}_{(d-1)n}$ and $\phi_{(d-1)n}(g) \in \overline{\mathcal{G}_{(d-1)n}} \cup \{\emptyset\}$, and the fourth implication by the induction hypothesis.

Next define $\gamma^{\mu} : \Omega' \to \Omega_{\infty}$ by $\gamma^{\mu}(\omega') := \cap_{i=0}^{\infty}\gamma^{\mu}_i(\omega')$. The rest follows from Corollary 1. q.e.d.

Lemma 3: For every $0 \leq i < \infty$ and every $C \in \mathcal{G}_i$ there exists a formula f_C such that $\phi_m(f_C) = C$ for all $m \geq i$, including $m = \infty$.

Remark: See also Lemma 2.9 of [Fa-Va].

Proof: We proceed by induction on i. For $C \in \mathcal{G}_0$ and $C = R \cap \Omega_{\infty}$ with $R \in \{R_1, \cdots, R_{2^{|X|}}\}$, let

$$f_C := \bigwedge_{\pi_i \alpha^{-1}(R)=1} x_i \bigwedge_{\pi_i \alpha^{-1}(R)=0} \neg x_i.$$

For all $C \in \mathcal{G}_0$, $\phi_0(f_C) = C$ implies by Lemma 1 that $\phi_i(f_C) = C$ for all $i \geq 0$. We assume that f_B is defined for every $B \in \mathcal{G}_k$ with $0 \leq k < i$. Assume that $C \in \mathcal{G}_i$, $C \subseteq A \in \mathcal{F}_i$, $j = i \pmod n$ and $C \subseteq D \in \mathcal{G}_{i-1}$. Define

$$f_C := f_D \wedge (\bigwedge_{B \in \mathcal{G}_{i-1},\ B \cap A \neq \emptyset} \neg k_j(\neg f_B) \bigwedge_{B \in \mathcal{G}_{i-1},\ B \cap A = \emptyset} k_j(\neg f_B)).$$

From the fact that $\mathcal{G}_i = \mathcal{F}_i \wedge \mathcal{G}_{i-1}$ for all $i \geq 1$ the sets $\{\phi_i(f_C) \mid C \in \mathcal{G}_i\}$ are pairwise disjoint. By Lemma 1 applied to the above definition of f_C, it suffices to show that $C \subseteq \phi_i(f_C)$ for all $C \in \mathcal{G}_i$. As above, let A be the member of $\mathcal{F}_i$ containing C and $j = i(\bmod n)$. By the induction hypothesis $A \cap B = \emptyset$ and $B \in \mathcal{G}_{i-1}$ imply that $A \cap \phi_i(f_B) = \emptyset$, which in turn implies that $A \subseteq \phi_i(\neg f_B)$ and $C \subseteq \phi_i(k_j(\neg f_B))$. On the other hand, $A \cap B \neq \emptyset$, $B \in \mathcal{G}_{i-1}$ and the induction hypothesis imply that $A \not\subseteq \phi_i(\neg f_B)$ and $C \not\subseteq \phi_i(k_j(\neg f_B))$.

Since $C \in \mathcal{G}_i$ and $\phi_i(k_j(\neg f_B)) \in \overline{\mathcal{G}_i} \cup \{\emptyset\}$ we have $C \cap \phi_i(k_j(\neg f_B)) = \emptyset$ and $C \subseteq \phi_i(\neg k_j(\neg f_B))$. Likewise the induction hypothesis guarantees that $C \subseteq D = \phi_i(f_D)$. The rest follows from the definition of ϕ_i. q.e.d.

Proposition 1: If $\Omega = (\Omega; I_n; \mathcal{Q}^1, \cdots, \mathcal{Q}^n; X, \psi')$ is the canonical state space on the alphabet X with n agents, the mapping $\gamma^\Omega : \Omega \to \Omega_\infty$ is a bijection.

Proof: Two distinct members of Ω differ on the containment of some $f \in \mathcal{L}_{I_n}(X)$. By Lemma 2 and the Completeness Theorem one is mapped by γ^Ω into $\phi_\infty(f)$ and the other into $\phi_\infty(\neg f) = \Omega_\infty - \phi_\infty(f)$. Therefore γ^Ω is injective.

Since $(\Omega_\infty; I_n; \mathcal{F}^1_\infty, \cdots, \mathcal{F}^n_\infty; X; \psi_\infty)$ is a model, every point of Ω_∞ has a consistent and complete list of formulae determined by $V_\infty := V^{\lambda^\infty} : \Omega_\infty \to \Omega$ (as defined in Section 2 with V_∞ corresponding to ϕ_∞.) Observe that $f \in \omega \in \Omega$ if and only if $\omega \in \phi^\Omega(f)$ if and only if $\gamma^\Omega(\omega) \in \phi_\infty(f)$ if and only if $f \in V_\infty(\gamma^\Omega(\omega))$, the first implication by the Completeness Theorem, the second implication by Lemma 2, and the third implication by definition. Therefore $V_\infty \gamma^\Omega$ is the identity mapping of Ω to itself. Lemma 3 demonstrates the injectivity of V_∞. With the previous paragraph this implies the surjectivity of γ^Ω. q.e.d.

Notice that if we give the state space Ω the natural topology as defined at the end of the Section 2 then the mapping $\gamma^\Omega : \Omega \to \Omega_\infty$ is also a homeomorphism between compact sets, due to Proposition 1, Corollary 1, Lemma 3, and the condition on the diameter of the members of $\mathcal{G}_i$. Because of this and Lemma 2, for most purposes we can consider the above constructed model $\lambda_\infty = (\Omega_\infty; I_n; \mathcal{F}^1_\infty, \cdots \mathcal{F}^n_\infty; X; \psi_\infty)$ to be that of the state space $\Omega = \Omega(X, I_n)$.

5 Alienated Extension

For any set $C \in \mathcal{G}_i$ with $i \geq 0$ we define a point $p(C) \in C$ called the "alienated extension of C". (See also the "no-information extension" of [Ha-Fa-Va] and [He-Sa].) Consider the $A \in \mathcal{F}_{i-n+1}$ such that $C \subseteq A$, and consider the unique $A' \in \mathcal{F}_{i+1}$ such that $A' \cap C' \neq \emptyset$ whenever $C' \in \mathcal{G}_i$ and $C' \subseteq A$. Define $p_i(C) := C$ and $p_{i+1}(C) := A' \cap C \in \mathcal{G}_{i+1}$; for $k > i$ define p_k inductively by $p_k(C) := p_k(p_{k-1}(C))$; and define $p(C) := \cap_{k=i}^\infty p_k(C)$.

For any $i \geq 0$ and any $C \in \mathcal{G}_i$ we call the point $p(C) \in \Omega_\infty$ a point of "alienation." We call the $2^{|X|}$ points $\{p(C) \mid C \in \mathcal{G}_0\}$ the points of maximal alienation.

Lemma 4: Let $C' \subseteq C$ with $C' \in \mathcal{G}_i$, $C \in \mathcal{G}_{i-1}$ and $i \geq 1$. If $p(C) \in A \in \mathcal{F}^j_\infty$ with $j \neq i (\text{mod } n)$ then $p(C') \in A$.

Proof: Let l be the largest number strictly less than i and equivalent to j (mod n), and let A_l be the member of $\mathcal{F}_l$ such that $A \subseteq A_l$. $C \in \mathcal{G}_{i-1}$ means that $C \subseteq A_l$, which also means that $p_{l+n-1}(C) \subseteq A_l$ and $p_{l+n-1}(C') \subseteq A_l$.

It suffices to show that both $p_{l+(k+1)n-1}(C)$ and $p_{l+(k+1)n-1}(C')$ are contained in A_{l+kn} for all $k \geq 0$, where A_{l+kn} is the member of $\mathcal{F}_{l+kn}$ containing A. Proceed by induction on k; the claim for $k = 0$ was shown above. Assume that $p_{l+kn-1}(C) \subseteq A_{l+(k-1)n}$ and $p_{l+kn-1}(C') \subseteq A_{l+(k-1)n}$ for $k \geq 1$. Then the $A' \in \mathcal{F}_{l+kn}$ defining $p_{l+kn}(C')$ is equal to A_{l+kn} because it is the unique $A' \in \mathcal{F}_{l+kn}$ defining $p_{l+kn}(D)$ for all $D \in \mathcal{G}_{l+kn-1}$ with $D \subseteq A_{l+(k-1)n}$. It follows that both $p_{l+(k+1)n-1}(C)$ and $p_{l+(k+1)n-1}(C')$ are contained in A_{l+kn}. q.e.d.

For any set S with n partitions $\{\mathcal{Q}^i \mid i \in I\}$ of S define s to be i-adjacent to s' if s and s' belong to the same member of $\mathcal{Q}^i$. For $s, s' \in S$, define adjacency-distance $(s, s') := min\{m \mid$ there exists a sequence $s = s_0, s_1, \cdots, s_m = s'$ and a mapping $a : \{1, \cdots, m\} \rightarrow I$ such that for all $1 \leq j \leq m$ s_j is $a(j)$ adjacent to s_{j-1}.$\}$. Let adjaceny-distance $(s, s') := \infty$ if no such sequence exists; and let adjacency distance $(s, s) := 0$. The equivalence relation defined by $s \sim s'$ if and only if adjaceny-distance $(s, s') < \infty$ defines the partition $\vee_{i \in I} \mathcal{Q}_i$.

Proposition 2: There is a minimal common knowledge world of $(\Omega_\infty; I_n; \mathcal{F}^1_\infty, \cdots, \mathcal{F}^n_\infty; X; \psi_\infty)$ that contains all points of alienation, (and therefore it is dense in Ω_∞.)

Remark: See Corollary 4.13 of [Fa-Ha-Va].

Proof: Consider an arbitrary $i \geq 0$ and $C \in \mathcal{G}_i$. By Lemma 4, adjacency-distance $(p(C_i), p(C_0)) \leq i$ where C_0 is the member of $\mathcal{G}_0$ containing C. We need only show that adjacency-distance $(p(C_0), p(C'_0)) < \infty$ for any two distinct members C_0 and C'_0 of $\mathcal{G}_0$. But j-adjacency of $p(C_0)$ and $p(C'_0)$ is easy to prove using any $j = 1, 2, \cdots, n$. q.e.d.

6 Relative Alienation

Lemma 5: For every $i < \infty$ there is only one common knowledge world of the model $\lambda_i = (\Omega_\infty; I_n; \mathcal{F}_i^1, \cdots, \mathcal{F}_i^n; X; \psi_\infty)$, namely all of Ω_∞.

Proof: Let C, C' be any two distinct members of $\mathcal{G}_i$. By Proposition 2, $p(C)$ and $p(C')$ are in the same minimal common knowledge world of the model $(\Omega_\infty; \mathcal{F}_\infty^1, \cdots, \mathcal{F}_\infty^n; X; \psi_\infty)$. Since for every $1 \le j \le n$, $\mathcal{F}_i^j$ is a coarser partition of Ω_∞ than $\mathcal{F}_\infty^j$, it follows that $p(C)$ and $p(C')$ are in the same minimal common knowledge world of the model $\lambda^i = (\Omega_\infty; \mathcal{F}_i^1, \cdots \mathcal{F}_i^n; X; \psi_\infty)$. The rest follows from the containment $p(D) \subseteq D$ for all $D \in \mathcal{G}_i$. q.e.d.

Let $\mathcal{P}_\infty(\mathbf{N_0})$ be the set of subsets of the whole numbers $\mathbf{N_0} = \{0, 1, 2, \cdots\}$ with infinite cardinality ($S \in \mathcal{P}_\infty(\mathbf{N_0}) \Rightarrow |S| = \infty$.) For any member S of $\mathcal{P}_\infty(\mathbf{N_0})$ and $C \in \mathcal{G}_i$ with $i \in S$ we will define a special point in C called the alienated extension of C with respect to S, labeled $p^S(C)$. If $i \in S \in \mathcal{P}_\infty(\mathbf{N_0})$ define $n_S(i) := \inf\{j \in \mathbf{N_0} \mid j > i, j \in S\}$. Define $p_{n(i)}^S := p_{n_S(i)}(\gamma_{n_S(i)-1}^{\lambda_i}(C))$, where $\gamma_j^{\lambda_i}$ is the mapping defined in Lemma 2 corresponding to the model λ_i (with $\gamma^{\lambda_i} = \cap_{j=1}^\infty \gamma_j^{\lambda_i}$.) For every $j \in S$ with $j \ge i$ and $p_j^S(C)$ already defined, define $p_{n(j)}^S(C)$ to be $p_{n(j)}^S(p_j^S(C))$. Lastly, for all $i \in S \in \mathcal{P}_\infty(\mathbf{N_0})$ and $C \in \mathcal{G}_i$ define

$$p^S(C) := \bigcap_{j \in S,\ j > i} p_j^S(C).$$

Notice from Lemmata 2 and 3 that if $C \in \mathcal{G}_i$ then $\gamma_i^{\lambda_i}(C) = C$, which means that $p^{\mathbf{N_0}} = p$. For any $i \in S \in \mathcal{P}_\infty(\mathbf{N_0})$ and $C \in \mathcal{G}_i$ we call the point $p^S(C)$ a point of alienation with respect to S.

Lemma 6: If $i \in S \in \mathcal{P}(\mathbf{N_0})$, $1 \le j \le n$, and C and C' are both members of $\mathcal{G}_i$ and both are contained in the same member A of $\mathcal{F}_i^j$, then $p^S(C)$ and $p^S(C')$ are both contained in the same member of $\mathcal{F}_\infty^j$.

Proof: For any model $\mu = (S; I_n; \mathcal{P}^1, \cdots, \mathcal{P}^n; X; \psi)$ and $l \ge 0$ it follows from the definition of γ_l^μ that $s, s' \in A' \in \mathcal{P}^j$ implies that $\gamma_l^\mu(s)$ and $\gamma_l^\mu(s')$ are contained in the same member of $\mathcal{F}_l^j$. Therefore it suffices to show that for any $l \ge 0$ and $D, D' \in \mathcal{G}_l$ that if $D, D' \subseteq A \in \mathcal{F}_l^j$ then $p_{l+1}(D)$ and $p_{l+1}(D')$ belong to the same member of $\mathcal{F}_{l+1}^j$. Since $l + 1 \not\equiv j (\mathrm{mod}\ n)$ implies that A is also a member of $\mathcal{F}_{l+1}^j$, it is sufficient to consider the case of $j = l + 1 (\mathrm{mod}\ n)$.

But as in the proof of Lemma 4, for any given $A' \in \mathcal{F}_{l-n+1}$ there is only one $A^* \in \mathcal{F}_{l+1}$ defining $p_{l+1}(D)$ for all $D \in \mathcal{G}_l$ satisfying $D \subseteq A'$. q.e.d.

Corollary 2: For every $S \in \mathcal{P}_\infty(\mathbf{N_0})$ all alienated extensions with respect to S share the same dense minimal common knowledge world.

Proof: Given any $i, k \in S$ with $C \in \mathcal{G}_i$ and $D \in \mathcal{G}_k$ consider $p^S_{max(i,k)}(C)$, $p^S_{max(i,k)}(D) \in \mathcal{G}_{max(i,k)}$. The result follows from Lemma 5 and repeated application of Lemma 6. q.e.d.

For every $0 \leq i < \infty$, $1 \leq j \leq n$, $C \in \mathcal{G}_i$ and $C \subseteq A^j_i \in \mathcal{F}^j_i$ define a formula

$$f_{i,C} := f_C \Rightarrow (\bigwedge_{j=1}^{n}(\bigwedge_{D \in \mathcal{G}_i,\ D \subseteq A^j_i} \neg k_j(\neg f_D) \quad \bigwedge_{D \in \mathcal{G}_i,\ D \cap A^j_i = \emptyset} k_j(\neg f_D))),$$

where f_C and f_D are the formulae defined in Lemma 3. Then define

$$f_i := \bigwedge_{C \in \mathcal{G}_i} f_{i,C}.$$

For all $i > n$ define $k_i := k_j$ where $1 \leq j \leq n$ and $j = i (\mathrm{mod}\ n)$.

Lemma 7: If $i \geq 1$, $i \in S \in \mathcal{P}_\infty(\mathbf{N_0})$ and $C \in \mathcal{G}_i$, then $\gamma^{\lambda'}_{i+n+l}(C) \subseteq \phi_\infty(k_{i+l} k_{i+l-1} \cdots k_{i+1} k_i f_i)$.

Proof: Notice that $\phi^{\lambda'}(f_i) = \phi_i(f_i) = \Omega_\infty$, which implies by Lemma 2 and Lemma 0 in Section 2 that $\gamma^{\lambda_i}(\Omega_\infty) \subseteq \phi_\infty(k_{a(m)} \cdots k_{a(1)} f_i)$ for all $m \geq 1$ and all $a : \{1, \cdots, m\} \to \{1, \cdots, n\}$. Notice that by Lemma 1 $\phi_{i+n-1+m}(f_i) = \phi_\infty(f_i)$ and $\phi_{i+n+l+m}(k_{i+l} \cdots k_i f_i) = \phi_\infty(k_{i+l} \cdots k_i f_i)$ for all $m \geq 0$. Suppose for the sake of contradiction that $\gamma^{\lambda'}_{i+n+l}(C) \not\subseteq \phi_\infty(k_{i+l} \cdots k_i f_i) = \phi_{i+n+l}(k_{i+l} \cdots k_i f_i)$. Because $\phi_{i+n+l}(k_{i+l} \cdots k_i f_i) \in \overline{\mathcal{G}_{i+n+l}} \cup \{\emptyset\}$, we have that $\gamma^{\lambda'}_{i+n+l}(C) \cap \phi_{i+n+l}(k_{i+l} \cdots k_i f_i) = \gamma^{\lambda'}_{i+n+l}(C) \cap \phi_\infty(k_{i+l} \cdots k_i f_i) = \emptyset$, a contradiction since $\gamma^{\lambda_i}(C) \in \gamma^{\lambda_i}(\Omega_\infty) \cap \gamma^{\lambda_i}_{i+n+l}(C)$. q.e.d.

Define functions $z : \mathbf{N} \to \mathbf{N} = \{1, 2, \cdots\}$ and $h : \{0, \cdots, n\} \to \mathbf{N}$ by $h(0) := 2^{|X|}$, if $1 \leq i \leq n$ then $z(i) := 2^{h(i-1)-1}$ and $h(i) := z(i)h(i-1)$, and if $i > n$ then

$$z(i) := 2^{-1+\prod_{j=1}^{n-1} z(i-j)}.$$

Lemma 8: If $i \geq 1$ and $C \in \mathcal{G}_{i-1}$ then $|\{D \in \mathcal{G}_i \mid D \subseteq C\}| \geq z(i)$.

Proof: $h(0)$ is the number of members of $\mathcal{G}_0$. One can show by induction that if $i \leq n$ then $h(i)$ is the number of members of $\mathcal{G}_i$ and that the inequality is an equality.

Let us assume that $i > n$ and that the statement is true for every $j < i$. Consider the $B \in \mathcal{G}_{i-n-1}$ and $A \in \mathcal{F}_{i-n}$ with $C \subseteq A \cap B \in \mathcal{G}_{i-n}$. By the induction hypothesis the number of members of $\mathcal{G}_{i-1}$ in $A \cap B$ is at least $\prod_{j=1}^{n-1} z(i-j)$. Therefore for the construction of $\mathcal{G}_i$ we have $\rho(C) \geq z(i)$, where ρ is the function defined in Section 3. q.e.d.

Lemma 9: If $i \geq 1$ and $C \in \mathcal{G}_i$ then $p_{i+n}(C) \cap \phi_\infty(k_i f_i) = \emptyset$.

Proof: Consider the $A \in \mathcal{F}_{i+1-n}$ containing C, the $A' \in \mathcal{F}_{i+1}$ with $p_{i+1}(C) = A' \cap C$, the $A^* \in \mathcal{F}_i$ containing C, and the $B \in \mathcal{G}_{i-n}$ containing C. Consider $A \cap B \in \mathcal{G}_{i+1-n}$, or $A \cap B = \Omega_\infty$ if $i \leq n-1$. By Lemma 8 there is some $C' \in \mathcal{G}_i$ with $C' \neq C$ and $C' \subseteq A \cap B$. (We needed from Lemma 8 only that $z(i) \geq 2$ for all $i \geq 1$!) Therefore there is an $A'' \in \mathcal{F}_{i+1}$ with $A'' \neq A'$, $A'' \cap C \neq \emptyset$, and $A'' \cap C' = \emptyset$. Let $C^* := A'' \cap C \in \mathcal{G}_{i+1}$. Since $C^* \subseteq A^*$ we have $A^{**} \cap C^* \neq \emptyset$ where $p_{i+n}(C) = p_{i+n-1}(C) \cap A^{**}$ and $A^{**} \in \mathcal{F}_{i+n}$. It follows that $A^{**} \cap A'' \neq \emptyset$, which means that $p_{i+n}(C) \subseteq \phi_{i+n}(\neg k_i(\neg(k_{i+1} \neg f_{C'})))$, or equivalently $p_{i+n}(C) \cap \phi_{i+n}(k_i(\neg k_{i+1} \neg f_{C'})) = \emptyset$. $C' \subseteq A$ shows that $\phi_{i+n}(k_i f_i) \subseteq \phi_{i+n}(k_i(\neg k_{i+1} \neg f_{C'}))$. Finally Lemma 1 implies that $\phi_{i+n}(k_i f_i) = \phi_\infty(k_i f_i)$. q.e.d.

Notice that the statements of Lemmata 7 and 9 must be altered slightly for the case of $i = 0$.

Define a mapping $\beta : \mathcal{P}(\mathbf{N_0}) \to \mathcal{P}_\infty(\mathbf{N_0})$ by $\beta(S) := \{0, 1, 2, 4, 8, \cdots\} \cup \{2^i + 1, \cdots, 2^{i+1} - 1 \mid i \in S\}$.

Define an equivalence relation on $\mathcal{P}(\mathbf{N_0})$ by $S \sim T$ if and only if there exists an $m \in \mathbf{N_0}$ such that $S - \{0, 1, 2, \cdots m\} = T - \{0, 1, 2, \cdots, m\}$. The cosets of this equivalence relation is an uncountable set.

Proposition 3: There is an uncountable set of dense minimal common knowledge worlds of Ω_∞.

Proof: Due to Corollary 2, if suffices to show that if S and T are both subsets of $\mathbf{N_0}$ with $S \not\sim T$ then $p^{\beta(S)}(C)$ does not share the same minimal common knowledge world as $p^{\beta(T)}(C)$ for some $C \in \mathcal{G}_1$. For the sake of contradiction, let us suppose that adjacency-distance $(p^{\beta(S)}(C),\ p^{\beta(T)}(C)) = l < \infty$. Because $S \not\sim T$ there exists an $i > log_2((l+1)n)$ such that $i \in S$ and $i \notin T$, or vice versa. By symmetry, let us assume that $i \in S$

and $i \notin T$. By Lemma 7 applied to $p_{2^i}^{\beta(T)}(C)$ it follows that $p^{\beta(T)}(C) \in \phi_\infty(k_{2^{i+1}-n-1} \cdots k_{2^i} f_{2^i})$. But adjacency-distance $(p^{\beta(S)}(C), p^{\beta(T)}(C)) = l$ implies that $p^{\beta(S)}(C) \in \phi_\infty(k_{2^{i+1}-(l+1)n-1} \cdots k_{2^i} f_{2^i}) \subseteq \phi_\infty(k_{2^i} f_{2^i})$, a contradiction to Lemma 9. q.e.d.

7 Acknowledgements

The inquiry into the cardinality of dense minimal common knowledge worlds was inspired by communication with Robert Aumann and Dov Samet, and later with Aviad Heifetz. Among other things, Mamoru Kaneko helped me to improve the presentation of this article. Joseph Halpern informed me of existing related literature and discoveries, as well as explore open problems related to the results of this article.

8 References

[Au1] Aumann, R. (1976), Agreeing to Disagree, Annals of Statistics 4, pp. 1236-1239.

[Au2] Aumann, R. (1989), Notes on Interactive Epistemology, Cowles Foundation for Research in Economics working paper.

[Fa-Va] Fagin, R., Vardi, M.Y. (1985), Proceedings of the 17th Annual A. C. M. Symposium on Theory of Computing, Providence, R. I., May 6-8, 1985, pp. 305-315.

[Fa-Ha-Va] Fagin, R., Halpern, Y.J. and Vardi, M.Y. (1991), A Model-Theoretic Analysis of Knowledge, Journal of the A.C.M. 91 (2), pp. 382-428.

[He-Sa] Heifetz, A., Samet, D., Universal Partition Spaces, in preparation.

[Hu-Cr] Hughes, G.E., Cresswell, M.J. (1968), An Introduction to Modal Logic, Routledge.

[Le] Lewis, D. (1969), Convention: A Philosophical Study, Harvard University Press.