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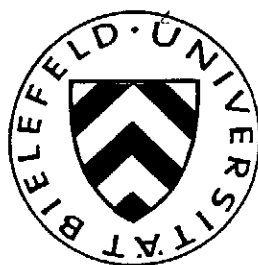
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**Existence of Generalized Walras Equilibria
for Generalized Economies**

by

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Existence of Generalized Walras Equilibria for Generalized Economies

Abstract

A generalized economy is a "usual" economy - i.e. a tuple consisting of initial endowments and preferences for each agent of a given (finite) set of agents - equipped with an additional net trade vector, which can be regarded as the foreign trade vector of the economy. Positive components of this vector correspond to the imports of a good, which can be distributed among the agents of the economy. Similarly, negative components can be interpreted as exports, which have to be brought up by the agents. The concept of Walras equilibrium is extended to this new setting in the way Thomson (1992) proposed. The main aim of the paper is to provide a quite general existence result.

1 Introduction

Recently, efforts have been undertaken to apply the game-theoretical consistency property to economies. Papers which have to be mentioned in this context are Thomson ([18], [20]), Dagan [7] and van den Nouweland, Peleg, Tijs [17]. A survey on general aspects of consistency was given by Thomson [19].¹ In analogy to reduced games in game theory, reduced economies have to be developed to define consistency properties for economies. Unless the agents are provided with equal initial endowments, a case Thomson treated in his paper in 1988, this causes severe problems, since the individual demands of agents in a reduced economy will in general not add to the sum of initial endowments in the reduced economy. Thus, a not necessarily positive bundle of goods - a net trade vector - remains. Therefore, the notion of economies has to be extended to the notion of generalized economies, which admit a net trade vector. For this bigger class of economies suitable extensions of the Walras correspondence have to be given. This can be done by varying budget constraints. One of the concepts resulting from this process is Thomson's proportional solution which we will focus on in the present paper.

Thomson's approach seems to be more meaningful than that used by both Dagan and van den Nouweland, Peleg, Tijs since it applies to a bigger class of generalized economies. Thomson's approach enables us to consider pure exports or pure imports vector, what cannot be done in the setting of Dagan or van den Nouweland et al., who are only able to deal with net trade vectors that are worth zero when evaluated with equilibrium prices. In effect, every equilibrium in the sense of van den Nouweland et al. is as well an equilibrium in Thomson's sense.

In the papers mentioned above existence proofs do not appear or do not play a dominant role. In the van den Nouweland et al. paper the existence of equilibrium can only be shown in a very technical setting and for very few cases. The present paper will provide an existence result for Thomson's proportional solution based only on standard assumptions on preferences.

¹For characterizations of the Walras correspondence without consistency properties the reader is referred, for example, to Nagahisa ([14], [15], [16]).

2 Generalized Economies and the Proportional Solution

A generalized economy E is a tuple $((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, T)$. Here, $N = \{1, \dots, n\}$ is the set of the agents of the economy, who are represented by their initial endowments $\omega_i \in \mathbf{R}_{++}^l$ and their preferences $\succeq_i \subset \mathbf{R}_+^l \times \mathbf{R}_+^l$. In addition, $T \in \mathbf{R}^l$ with $\sum_{j=1}^n \omega_j + T \in \mathbf{R}_+^l$ represents the net trade vector of this economy. Its components can be positive (indicating imports of the good in question) as well as negative (indicating exports of the good in question). Imports can be distributed among the economy's agents; exports have to be brought up by them. In this wider context a usual economy can be seen as a generalized economy E' with net trade vector $T' = 0$. An **allocation** of an economy $E = ((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, T)$ is a vector $z = (z_1, \dots, z_n) \in \mathcal{A} := \{\zeta \in (\mathbf{R}_+^l)^n \mid \sum_{j=1}^n \zeta_j = \sum_{j=1}^n \omega_j + T\}$. $\mathcal{A} = \mathcal{A}(E)$ is called the set of allocations. A price system is a vector P in the $(l-1)$ -dimensional unity simplex Δ^l . Very often boundary solutions will be excluded from consideration. Then we will use the notation $P \in \overset{\circ}{\Delta}^l$, where $\overset{\circ}{\Delta}^l$ is the interior of the price simplex Δ^l .

Throughout the paper, preferences are assumed to be reflexive, transitive, complete, continuous, monotonic and strictly convex. Sometimes consumers' tastes will be described by means of utility functions instead of preferences. Furthermore, generalized economies will be denoted by E or E_T . Economies with same initial endowments and preferences but with net trade vector 0 will be called **corresponding usual economies** and will be denoted by E_0 .

The most important aspects of the Walras equilibrium are the market clearing condition and the preference maximization of the agents as regards their budget constraints. The following definition is made to emphasize these aspects, based on which several generalizations of the Walras correspondence can be obtained by varying only the amount of the budget constraints.

Definition 2.1 $(z, P) \in \mathcal{A}(E) \times \Delta^l$ is called an **equilibrium of E relative to the budget constraints $v_i(P)$** , if and only if

1. $\sum_{j=1}^n z_j = \sum_{j=1}^n \omega_j + T$ (market clearing condition)
2. $z_i \in B_i(P) := \{x \in \mathbf{R}_+^l \mid \langle P, x \rangle \leq v_i(P)\} \quad \forall i$
3. $\forall x_i \in B_i(P) : z_i \succeq_i x_i \quad \forall i.$

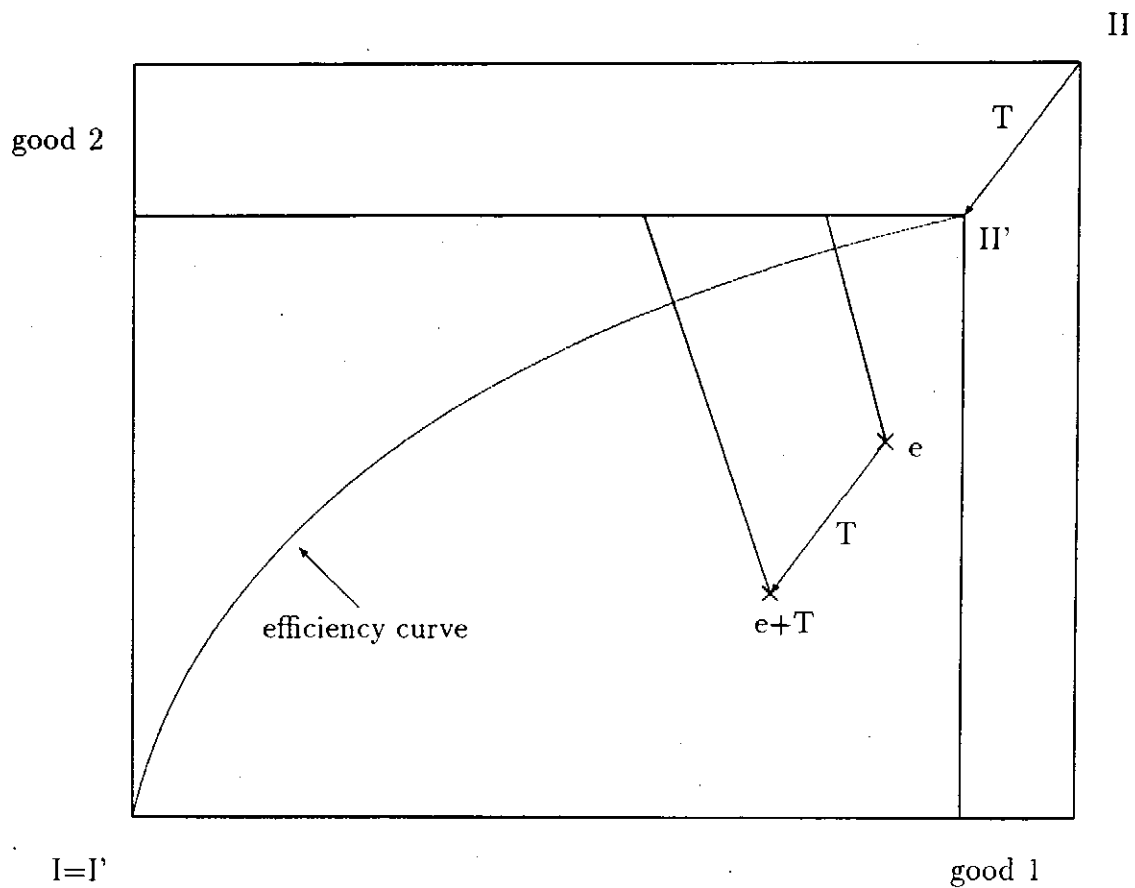


Figure 1: Edgeworth-Box: The point e represents the initial endowments of the agents I and II in the economy E_0 without net trade vector T . The agents in economy E_T with net trade vector T are named I' and II' . The drawn efficiency curve is that of economy E_T . In that economy there is no point representing initial endowments. Special points are e resp. $e+T$ where agent II' resp. agent I' has to bring up all T .

That is, agent i chooses his consumption bundle z_i within his budget $B_i(P) := \{x \in \mathbf{R}_+^l \mid \langle P, x \rangle \leq v_i(P)\}$ such that his preferences are maximized. Given monotonicity of preferences $\langle P, z_i \rangle = v_i(P)$ is satisfied for all $i \in N$. Thus

$$\begin{aligned} \sum_{j=1}^n v_j(P) &= \langle P, \sum_{j=1}^n \omega_j + T \rangle = \langle P, \sum_{j=1}^n \omega_j \rangle + \langle P, T \rangle \\ &= \sum_{j=1}^n \langle P, \omega_j \rangle + \langle P, T \rangle = \sum_{j=1}^n w_j(P) + \langle P, T \rangle \end{aligned} \quad (1)$$

where $w_j(P) := \langle P, \omega_j \rangle$ are the budget constraints of the corresponding usual economy. Since the value $\langle P, T \rangle$ of the net trade vector does not have to be zero - think for example of $T \in \mathbf{R}_{++}^l$ -, one cannot always expect that v_j and w_j are equal.

2.1 The Proportional Equilibrium

We are now looking for a new concept which generalizes the concept of Walras equilibrium. As it will, in general, not be possible to choose budget constraints $v_i = w_i$ for all $i \in N$, one has to think about how to deviate from equality without causing too much damage (and without violating equation (1), of course). One way to solve the problem is to do it proportionally, i.e. to choose budget constraints v_i such that v_i/w_i is independent of i . This ensures equality $v_i(P) = w_i(P)$ in the case that $\langle P, T \rangle = 0$, and especially for $T = 0$ we get $v_i(P) = w_i(P)$ for all $P \in \overset{\circ}{\Delta}^l$, and thus, the new concept coincides with the concept of Walras correspondence for usual economies.

Definition 2.2 (z, P) is called **proportional equilibrium**, if it is an equilibrium relative to the budget constraints $v_i(P) := \lambda_i \langle P, \sum_{j=1}^n \omega_j + T \rangle$ with $\lambda_i := \langle P, \omega_i \rangle / \langle P, \sum_{j=1}^n \omega_j \rangle$.

Since monotonicity of preferences is assumed, j 's share of the value of total endowments is the same in $E_T := ((\omega_i)_{i \in N}, (\sum_i)_{i \in N}, T)$ and $E_0 := ((\omega_i)_{i \in N}, (\sum_i)_{i \in N}, 0)$, i.e.

$$\frac{v_i(P)}{\sum_{j=1}^n v_j(P)} = \frac{v_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T \rangle} = \lambda_i = \frac{w_i(P)}{\langle P, \sum_{j=1}^n \omega_j \rangle} = \frac{w_i(P)}{\sum_{j=1}^n w_j(P)}.$$

The foregoing concept is the same as the one defined by the budget constraints

$$\bar{v}_i(P) := \langle P, \omega_i \rangle + \lambda_i \langle P, T \rangle \text{ with } \lambda_i := \frac{\langle P, \omega_i \rangle}{\langle P, \sum_{j=1}^n \omega_j \rangle}.$$

Here, agents get their budget constraints w_i plus a share λ_i of the value of the net trade vector, where λ_i is proportional to w_i .² Both ways lead to the same concept because the budget constraints are equal as can be seen from the following chain of equations.

$$\begin{aligned} \frac{\bar{v}_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T \rangle} &= \frac{\langle P, \omega_i \rangle + \lambda_i \langle P, T \rangle}{\langle P, \sum_{j=1}^n \omega_j + T \rangle} \\ &= \frac{\langle P, \omega_i \rangle}{\langle P, \sum_{j=1}^n \omega_j \rangle} \frac{\langle P, \sum_{j=1}^n \omega_j \rangle + \langle P, T \rangle}{\langle P, \sum_{j=1}^n \omega_j + T \rangle} \\ &= \lambda_i = \frac{v_i(P)}{\langle P, \sum_{j=1}^n \omega_j + T \rangle} \end{aligned}$$

In the remainder of the paper we will use the proportional representation of this solution concept.

3 Remarks on the Price Correspondence

Let $E_T := ((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, T)$ be a generalized economy and $\mathcal{A} := \mathcal{A}(E_T)$. Then each $a \in \mathcal{A}$ defines a usual economy $E^a := (a, (\succeq_i)_{i \in N})$. For each $a \in \mathcal{A}$ let $P_{\mathcal{A}}(a)$ be the set of all equilibrium prices of E^a . This defines the Walras price correspondence

$$P_{\mathcal{A}} : \mathcal{A} \rightrightarrows \overset{\circ}{\Delta}^l$$

of the economy E_T . Given monotonicity of preferences (resp. of utility functions), boundary prices in $\partial \Delta^l$ need not be considered. Moreover, let \mathcal{E} be the set of all usual economies $E^x := (x, (\succeq_i)_{i \in N})$ where preferences from E_T are kept fixed and $x \in (\mathbf{R}_+^l)^n$ is varied (thus, $\mathcal{E} \cong (\mathbf{R}_+^l)^n$). Let P_W be the Walras price correspondence on \mathcal{E}

$$P_W : \mathcal{E} \rightrightarrows \overset{\circ}{\Delta}^l.$$

Then $P_{\mathcal{A}} = P_W|_{\mathcal{A}}$. We will need both results on $P_{\mathcal{A}}$ and on P_W to show existence of the proportional equilibrium.

In this section results will be presented for cases in which consumers have continuous demand functions. A demand function is a function

$$f : [0, \infty) \times \overset{\circ}{\Delta}^l \rightarrow \mathbf{R}_+^l$$

²This concept is due to Thomson, see [20].

with $\langle P, f_i(w, P) \rangle = w$. If one requires the following desirability assumption on demand functions, non-emptiness of P_W and P_A can be guaranteed.

If the sequence $(w^q, p^q) \in (0, \infty)^n \times \overset{\circ}{\Delta}^l$ converges to $(w^0, p^0) \in (0, \infty)^n \times \partial\Delta^l$, then $\sum_{i=1}^n \|f_i(w_i^q, p^q)\|$ converges to infinity. (D₀)

In our case (D₀) is fulfilled because of strict monotonicity of preferences.

Proposition 3.1 *If the agents have continuous demand functions f_i satisfying (D₀), then an equilibrium exists for all $a \in (\mathbf{R}_{++}^l)^n$. Equivalently, one can say that the equilibrium price correspondence P_W is non-empty on $(\mathbf{R}_{++}^l)^n$.*

Of course, there are much more direct ways to ensure existence of equilibria.³ But since we will anyway work with demand functions later on, we will formulate the results of this section in terms of demand functions.⁴

Proposition 3.2 *If the agents have continuous demand functions f_i , then the equilibrium price correspondence P_W is upper hemi-continuous (u.h.c.) on $(\mathbf{R}_{++}^l)^n$. P_W is a continuous function where it is single valued.*

The graph Γ of the equilibrium price correspondence P_W in $(\mathbf{R}_{++}^l)^n \times \overset{\circ}{\Delta}^l$ is called equilibrium price manifold.⁵ If the demand of the agents is given by continuous functions

$$f_i : [0, \infty) \times \overset{\circ}{\Delta}^l \rightarrow \mathbf{R}_+^l$$

satisfying Walras' Law, then one can embed the space of common budget situations $B := [0, \infty)^n \times \overset{\circ}{\Delta}^l$ into the equilibrium price manifold Γ .⁶ To see this, let $f : B \rightarrow \Gamma$ be defined by

$$f(w_1, \dots, w_n; P) := (f_1(w_1, P), \dots, f_n(w_n, P); P) \quad .$$

$f(B)$ is the set of no-trade equilibria. In addition, let $g : f(B) \rightarrow B$ be defined by

$$g(e_1, \dots, e_n; P) := (\langle P, e_1 \rangle, \dots, \langle P, e_n \rangle; P) \quad .$$

³See for example Debreu [8] and [10]. He provides existence results based on assumptions on the preferences of agents only.

⁴See for example Dierker [12].

⁵Indeed, if one requires suitable differentiability assumptions, Γ can be shown to be a manifold.

⁶See also Balasko [2].

Since both functions are obviously continuous, it remains to show that $g \circ f$ and $f \circ g$ are the identities on B resp. on $f(B)$. We have

$$\begin{aligned} g(f(w, P)) &= g(f_1(w_1, P), \dots, f_n(w_n, P); P) \\ &= (\langle P, f_1(w_1, P) \rangle, \dots, \langle P, f_n(w_n, P) \rangle); P \\ &= (w_1, \dots, w_n; P) \end{aligned} \quad (2)$$

and

$$\begin{aligned} f(g(e, P)) &= f(\langle P, e_1 \rangle, \dots, \langle P, e_n \rangle; P) \\ &= f(w_1, \dots, w_n; P) \quad \text{with } w_i := \langle P, e_i \rangle \\ &= (f_1(w_1, P), \dots, f_n(w_n, P); P) \\ &= (e_1, \dots, e_n; P) \end{aligned} \quad (3)$$

The last equation of (3) holds for the following reason: For all $(e, P) \in f(B)$ we get $e_i = f_i(\bar{w}, P)$ for some $\bar{w} \in \mathbf{R}_+$. Now, $w_i = \langle P, e_i \rangle = \langle P, f_i(\bar{w}, P) \rangle = \bar{w}$ by monotonicity. Hence $e_i = f_i(w_i, P)$. Therefore, we have proved

Proposition 3.3 *The set of no-trade equilibria $f(B)$ is homeomorphic to the set B of common budget situations.*

We will be especially interested in those no-trade equilibria which belong to a fixed generalized economy E_T , i.e. those equilibria whose endowments sum up to the total endowments of E_T . The set will be called

$$PO := PO(E_T) := f(B) \cap (\mathcal{A}(E_T) \times \overset{\circ}{\Delta}^l)$$

because it is the set of all Pareto optimal allocations of E_T together with their supporting price vectors obtained by the second welfare Theorem. For the next result we will have to exclude boundary solutions by

$$f_i((0, \infty) \times \overset{\circ}{\Delta}^l) \subset \mathbf{R}_{++}^l \quad (D_1)$$

Lemma 3.4 *If the agents have continuous demand functions satisfying (D₁) and their preferences are monotonic, continuous, smooth and strictly convex, then PO is simply connected.*

Proof: Consider $U : \mathcal{A} \rightarrow \mathbf{R}_+^n$ defined by the vector of utility functions

$$U(a) := (u_1(a_1), \dots, u_n(a_n))$$

Let \mathcal{V} be the set of Pareto optimal allocations. Then \mathcal{V} is the projection of PO onto the set of allocations \mathcal{A} . Moreover, let $\mathcal{U} := U(\mathcal{V})$ be the set of utility tuples belonging to Pareto optimal allocations. Then $U|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{U}$ is a homeomorphism. Obviously, it is continuous and surjective. It is an open mapping because the u_i are open. To see injectivity, let $x, y \in \mathcal{V}$ with $U(x) = U(y)$. Then $(x + y)/2$ is an allocation as well. Assume that there is an $i \in N$ such that $x_i \neq y_i$. Because of i 's strictly convex preferences $u_i((x_i + y_i)/2) > u_i(x_i) = u_i(y_i)$. Hence, neither x nor y can be efficient allocations - a contradiction. Since \mathcal{U} is homeomorphic to the unit simplex Δ^n and thus simply connected,⁷ \mathcal{V} is simply connected as well.

It remains to show that PO is simply connected. But PO is the graph of the equilibrium price correspondence $P_{\mathcal{A}}$ restricted on the set of Pareto allocations \mathcal{V} . As preferences are smooth there is exactly one equilibrium price vector belonging to every Pareto allocation. Hence, by Proposition 3.2, $P_{\mathcal{A}}$ is a continuous function on \mathcal{V} . This yields the desired result since graphs of continuous functions on simply connected domains are themselves simply connected. \square

4 Existence of the Proportional Equilibrium

Apart from the usual assumptions on preferences we will need the following desirability assumption on preferences.

$$\forall x \in \mathbf{R}_{++}^I, y \in \mathbf{R}_+^I [y \succeq_i x \Rightarrow y \in \mathbf{R}_{++}^I] \quad (D_2)$$

It ensures that consumers demand bundles of goods in the interior of their consumption sets and therefore implies assumption (D_1) .

Theorem 4.1 *Let $E_T := ((\omega_i)_{i \in N}, (\succeq_i)_{i \in N}, T)$ be a generalized economy with reflexive, complete, transitive, continuous, monotonic and strictly convex preferences such that (D_2) is satisfied. Then a proportional equilibrium exists.*

Proof: We will first show that the results of the last section can be applied. Owing to the continuity of preferences there exist continuous utility functions. The resulting demand correspondences $f_i : (0, \infty) \times \overset{\circ}{\Delta}^I \rightarrow \mathbf{R}_+^I$ are also

⁷See for example the book of Arrow and Hahn [1].

continuous.⁸ As preferences are strictly convex, they are even functions. In addition, they can be continuously extended to $\{0\} \times \overset{\circ}{\Delta}^l$ by 0. Because of monotonicity of preferences they fulfill Walras' Law, i.e.

$$\langle P, f_i(w_i, P) \rangle = w_i \quad .$$

Furthermore, (D_2) yields (D_1) , and (D_0) follows from strict monotonicity.

For the concrete proof we will consider the space $\Delta^n \times \overset{\circ}{\Delta}^l$, where $\overset{\circ}{\Delta}^l$ is, as before, the open price simplex. Δ^n is called the space of shares of total endowments, i.e. if $\lambda \in \Delta^n$, then $v_i(P) = \lambda_i \langle P, \sum_{j=1}^n \omega_j + T \rangle$ for every price vector P . Remember that $\sum_{j=1}^n \omega_j + T$ is the vector of the total endowments of the economy E_T . We shall call $\lambda \in \Delta^n$ a share vector. In $\Delta^n \times \overset{\circ}{\Delta}^l$ we will focus on two sets. On the one hand, we will consider the set of tuples (λ, P) where $P \in P_{\mathcal{A}}$ is a equilibrium price systems belonging to an allocation $a \in \mathcal{A}$ and

$$\lambda = \frac{1}{\langle P, \sum_{i=1}^n a_i \rangle} (\langle P, a_1 \rangle, \dots, \langle P, a_n \rangle) \in \Delta^n \quad .$$

This set could be called the set of Pareto efficient common budget situations⁹ and will later on be denoted by C . On the other hand, we will discuss the set of tuples (λ, P) where P is an arbitrary price vector and

$$\lambda = \frac{1}{\langle P, \sum_{j=1}^n \omega_j \rangle} (\langle P, \omega_1 \rangle, \dots, \langle P, \omega_n \rangle) \in \Delta^n \quad .$$

This set is the graph of a continuous function $P \rightarrow \lambda(P)$ and will later on be denoted by D .

If the intersection of these sets is non-empty, the Theorem is proved. To prevent sets from penetrating each other without intersecting we will set great store by simple connectedness of the sets.

We will make the shape of the last set more pleasant by showing that without loss of generality it can be assumed as an affine linear subspace of $\Delta^n \times \overset{\circ}{\Delta}^l$.

⁸ f_i has the representation $f_i(w, P) = \{x \in B_i(w, P) | u_i(x) = \max_{y \in B_i(w, P)} u_i(y)\}$ where the budget correspondence B_i is continuous and the utility function u_i is continuous (remember that preferences are continuous). See for example Debreu [9].

⁹The set can be compared to the equilibrium price manifold (replace economies $E \in \mathcal{E}$ by share vectors $\lambda \in \Delta^n$).

Let $E = ((\omega_i)_{i \in N}, (u_i)_{i \in N}, T)$ be a generalized economy. Let $\mu * z$ with $\mu \in \mathbf{R}_{++}^l$ and $z \in \mathbf{R}^l$ be defined by $(\mu * z)_i := \mu_i z_i$ and $\mu^{-1} \in \mathbf{R}_{++}^l$ by $(\mu^{-1})_i := \mu_i^{-1}$. Furthermore, let $E' := E'(\mu) := ((\omega'_i)_{i \in N}, (u'_i)_{i \in N}, T')$ be given by $\omega'_i := \mu * \omega_i$, $T' := \mu * T$ and u'_i by $u'_i(x) := u_i(\mu^{-1} * x)$. Then E' is a generalized economy.

Proposition 4.2 *Let E be a generalized economy and $E' := E'(\mu)$ with $\mu \in \mathbf{R}_{++}^l$.*

- a) *(P, x) is a proportional equilibrium of E , if and only if (P', x') is a proportional equilibrium of E' , with $P' = \mu^{-1} * P$ and $x' = \mu * x$.*
- b) *If the agents of economy E have continuous demand functions f_i , the agents of economy E' have continuous demand functions f'_i , with $f'_i = \mu * f_i$. Properties such as (D_0) , (D_1) and (D_2) are retained.*

Now we can assume $\sum_{j=1}^n \omega_j + T = (1, \dots, 1)$ without loss of generality (choose $\mu := (\sum_{j=1}^n \omega_j + T)^{-1}$). The homeomorphism $g : f(B) \rightarrow B$ can be used to embed PO into B . The image $C := g(PO) = f^{-1}(PO)$ is not only a subset of B , but also a subset of $\Delta^n \times \overset{\circ}{\Delta}^l$ as proved in the following chain of equations:

$$\begin{aligned} \sum_{i=1}^n \omega_i &= \sum_{i=1}^n \langle P, e_i \rangle = \langle P, \sum_{i=1}^n e_i \rangle \\ &= \langle P, \sum_{j=1}^n \omega_j + T \rangle = \langle P, (1, \dots, 1) \rangle = 1 \end{aligned} \quad (4)$$

Yet, PO is simply connected and so is $C = g(PO)$.

The projection of C onto Δ^n is surjective. Thus, C can be seen as the graph of a correspondence $\Delta^n \Rightarrow \overset{\circ}{\Delta}^l$. To see surjectivity let $\lambda \in \Delta^n$ be arbitrarily chosen and $x := \lambda * (\sum_{j=1}^n \omega_j + T) := (\lambda_1(\sum_{j=1}^n \omega_j + T), \dots, \lambda_n(\sum_{j=1}^n \omega_j + T)) \in \mathcal{A}$. For x exists an equilibrium $(y, P) \in PO$. Then there exists $g(y, P) \in C$ with

$$\begin{aligned} g(y, P) &= (\langle P, y_1 \rangle, \dots, \langle P, y_n \rangle; P) \\ &= (\langle P, x_1 \rangle, \dots, \langle P, x_n \rangle; P) \\ &= (\lambda_1(\sum_{j=1}^n \omega_j + T), \dots, \lambda_n(\sum_{j=1}^n \omega_j + T); P) \\ &= (\lambda_1, \dots, \lambda_n; P) \end{aligned} \quad (5)$$

The other set mentioned in the introductory comments to this proof is

$$D := \left\{ (\lambda, P) \in \Delta \times \overset{\circ}{\Delta}^l \mid \lambda_i = \frac{\langle P, \omega_i \rangle}{\langle P, \sum_{j=1}^n \omega_j \rangle} \right\} \quad (6)$$

$$= \text{Gr } k$$

with $k : \overset{\circ}{\Delta}^l \rightarrow \Delta^n$ defined by

$$P \mapsto \frac{1}{\langle P, \sum_{j=1}^n \omega_j \rangle} (\langle P, \omega_1 \rangle, \dots, \langle P, \omega_n \rangle)$$

Since k is continuous and $\overset{\circ}{\Delta}^l$ simply connected, D is simply connected.

The problem is solved once $C \cap D \neq \emptyset$ is shown. We will construct a homeomorphism

$$\Phi : \Delta^n \times \overset{\circ}{\Delta}^l \rightarrow \Delta^n \times \overset{\circ}{\Delta}^l$$

such that $D' := \Phi(D) = \{(\frac{1}{n}, \dots, \frac{1}{n})\} \times \overset{\circ}{\Delta}^l$ and such that the projection of $C' := \Phi(C)$ onto Δ^n is surjective. Then $C' \cap D'$ contains at least one point $(1/n, \dots, 1/n, P)$ and thus $C \cap D = \Phi^{-1}(C') \cap \Phi^{-1}(D') = \Phi^{-1}(C' \cap D')$ is non-empty.

The construction of Φ is as follows: Let $\tilde{\phi}$ be a homeomorphism from Δ^n to $[0, 1]^{n-1}$ and

$$\phi := (\tilde{\phi}, \text{proj}_{\overset{\circ}{\Delta}^l}) : \Delta^n \times \overset{\circ}{\Delta}^l \rightarrow [0, 1]^{n-1} \times \overset{\circ}{\Delta}^l,$$

where $\text{proj}_{\overset{\circ}{\Delta}^l}$ is the projection onto $\overset{\circ}{\Delta}^l$. Then it is sufficient to find an appropriate homeomorphism Ψ of $[0, 1]^{n-1} \times \overset{\circ}{\Delta}^l$. Let $\alpha(P) := \tilde{\phi}(k(P))$ and $\beta := \tilde{\phi}(\frac{1}{n}, \dots, \frac{1}{n})$. Then $\alpha(P)$ and β are in the interior of $[0, 1]^{n-1}$. Now, let Ψ be defined by

$$\Psi := (\tilde{\Psi}, \text{proj}_{\overset{\circ}{\Delta}^l}) : [0, 1]^{n-1} \times \overset{\circ}{\Delta}^l \rightarrow [0, 1]^{n-1} \times \overset{\circ}{\Delta}^l$$

with

$$\tilde{\Psi}_i(x, P) := \begin{cases} \frac{x_i}{\alpha_i(P)} \beta_i & \text{if } x_i \leq \alpha_i(P) \\ \frac{(1-\beta_i)x_i + \beta_i - \alpha_i(P)}{1-\alpha_i(P)} & \text{if } x_i > \alpha_i(P) \end{cases}$$

for all $i \in N$. Now, Ψ is continuous and bijective. The inverse function of Ψ has the same shape and is therefore continuous as well. Thus, Ψ is a homeomorphism and the identity on $\partial[0, 1]^{n-1} \times \overset{\circ}{\Delta}^l$. Since the homeomorphisms ϕ and ϕ^{-1} map boundaries onto boundaries, Φ is the identity on $\partial\Delta^n \times \overset{\circ}{\Delta}^l$.

Now, let $C' := \Phi(C)$ and $D' := \Phi(D)$. For $(k(P), P) \in D$ one obtains

$$\begin{aligned}\Phi(k(P), P) &= \phi^{-1} \circ \Psi \circ \phi(k(P), P) \\ &= \phi^{-1} \circ \Psi(\alpha(P), P) \\ &= \phi^{-1}(\beta(P), P) \\ &= \left(\frac{1}{n}, \dots, \frac{1}{n}, P\right),\end{aligned}$$

that is, $D' = \left(\frac{1}{n}, \dots, \frac{1}{n}\right) \times \overset{\circ}{\Delta}^l$.

It remains to show that the projection $\Pi(C')$ of C' onto Δ^n is surjective. This is obvious since C' is simply connected and has the same boundary as C has. \square

5 Outlook

With the existence proof presented here, the theoretic basis for the consideration of consistency properties in equilibrium theory is provided. The setting of generalized economies will enable and enforce us to discuss sharing problems such as how to divide the net trade vector among the agents and how to do it in a simple and decentralized way. Thomson's proportional concept seems to be an appropriate answer to both questions since the consumption bundle of each agent depends only on his own initial endowments, total endowments and the price system. One can therefore try to characterize the proportional solution by means of its consistency and its sharing properties.

Up to now, generalized economies have been used only to provide a framework for the discussion of consistency properties. But probably they will also have a value of their own.

- One can, for example, think of a fixed foreign trade vector, i.e. T is not due to free foreign trade (where every agent is free to exchange his goods with the goods of an arbitrary foreigner) but due to political reasons. This may arise in the case of foreign claims or subsidies such as
 - reparations (reparation payments) of a country after a lost war (for example Germany after World War I). Then T will be a pure exports vector with negative components only.

- humanitarian aid for countries with famine (then T will be a pure imports vector with strictly positive components only for goods like wheat, medicine, etc.)
- technical help for under- or less developed countries.
- States have to plan their expenses and revenues in advance. Hence, they have to fix a vector T to be brought up by the agents of the state. Positive components of T are for example public transport and social contributions; negative components are for example bureau equipment for state administration.

May be, the discussion of solution concepts for generalized economies can help understand how surpluses or deficiencies an economy is confronted with should be distributed among the agents.

References

- [1] ARROW, K. AND F. HAHN: *General Equilibrium Analysis*, Holden Day, San Francisco, Oliver & Boyd, Edinburgh, 1971.
- [2] BALASKO, Y.: Some Results on Uniqueness and on Stability of Equilibrium in General Equilibrium Theory, *Journal of Mathematical Economics* 2, 1975, pp. 95-118.
- [3] BALASKO, Y.: The Graph of the Walras Correspondence, *Econometrica* 43, 1975, pp. 907-912.
- [4] BARTEN, A. AND V. BÖHM: Consumer Theory, in *Handbook of Mathematical Economics, Vol. 2*, K. Arrow and M. Intriligator (eds.), North-Holland, Amsterdam · New York · Oxford · Tokyo 1982.
- [5] BEGLE, E.: A Fixed Point Theorem, *Annals of Mathematics* 51, 1950, pp. 544-550.
- [6] BERGE, C.: *Espaces topologiques et fonctions multivoques*, deuxième édition, Dunod, Paris 1966.

- [7] DAGAN, N.: Consistency, Decentralization and the Walrasian Allocations Correspondence, Mimeo, *Department of Economics*, The Hebrew University of Jerusalem, Jerusalem, Israel.
- [8] DEBREU, G.: *Theory of Value*, John Wiley and Sons, New York 1959, translated *Werttheorie*, Springer, Berlin · Heidelberg · New York 1976.
- [9] DEBREU, G.: Economies with a Finite Set of Equilibria, *Econometrica* 38, 1970, pp. 387-392.
- [10] DEBREU, G.: Existence of Competitive Equilibrium, in *Handbook of Mathematical Economics*, Vol. 2, K. Arrow and M. Intriligator (eds.), North-Holland, Amsterdam · New York · Oxford · Tokyo 1982.
- [11] DIERKER, E.: Topological Methods in Walrasian Economics, *Lecture Notes in Economics and Mathematical Systems*, Vol. 92, Springer, Berlin · Heidelberg · New York 1974.
- [12] DIERKER, E.: Regular Economies, in *Handbook of Mathematical Economics*, Vol. 2, K. Arrow and M. Intriligator (eds.), North-Holland, Amsterdam · New York · Oxford · Tokyo 1982.
- [13] EILENBERG, S. AND D. MONTGOMERY: Fixed Point Theorem for Multivalued Functions, *American Journal of Mathematics* 68, 1946, pp. 214-222.
- [14] NAGAHISA, R.: A Local Independence Condition for Characterization of Walrasian Allocations Rule, *Journal of Economic Theory* 54, 1991, pp. 106-123.
- [15] NAGAHISA, R.: Walrasian Social Choice in a Large Economy, *Mathematical Social Sciences* 24, 1992, pp. 73-78.
- [16] NAGAHISA, R.: A Necessary and Sufficient Condition for Walrasian Social Choice, *Journal of Economic Theory* 62, 1994, pp. 186-208.
- [17] NOUWELAND, A. VAN DEN, B. PELEG AND S. TIJS: Axiomatic Characterization of the Walras Correspondence for Generalized Economies, *CentER Discussion Paper No. 9458*, Tilburg University, The Netherlands.

- [18] THOMSON, W.: A Study of Choice Correspondences in Economies with a Variable Number of Agents, *Journal of Economic Theory* 46, 1988, pp. 237-254.
- [19] THOMSON, W.: The Consistency Principle, in *Game Theory and its Applications*, Ichiishi et al., Academic Press, San Diego, California 1990.
- [20] THOMSON, W.: Consistency in Exchange Economies, Mimeo, *Department of Economics*, University of Rochester, Rochester, New York 1992.
- [21] WALRAS, L.: *Elements d'économie politique pure*, L. Corbaz, Lausanne 1874, translated *Elements of Pure Economics*, Allen and Unwin, London 1954.