INSTITUTE OF MATHEMATICAL ECONOMICS

Reference Functions and Solutions to Bargaining Problems with Claims

by

Anke Gerber

October 1997

University of Bielefeld

33501 Bielefeld, Germany

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Abstract

Following an idea due to Thomson (*Journal of Economic Theory* **25**, 1981, 431–441) we examine the role of reference functions in the axiomatic approach to the solution of bargaining problems with claims. A reference function is a means of summarizing essential features of a bargaining problem. Axioms like *Independence of Irrelevant Alternatives* and *Monotonicity* are then reformulated with respect to this reference function. Under some mild conditions on the reference function we obtain characterizations of different parametrized classes of solutions. We present several examples of reference functions and thereby recover most of the well known solutions to bargaining problems with claims.

1   Introduction

The formal introduction of the concept of a *reference function* is due to Thomson [10]. A reference function assigns to each bargaining problem a point in the utility space which can be interpreted as summarizing those features of the bargaining situation that are regarded as essential, either by the players or by some impartial arbitrator whose task it is to propose a fair solution to the bargaining problem at issue. By reformulating Nash's axiom of *Independence of Irrelevant Alternatives* with respect to the given reference function Thomson obtained a new class of solutions to bargaining problems. As special cases this class includes the Nash solution, where the reference point is identical to the status quo, and the solution proposed by Roth [9], where the reference point equals the *point of minimal expectations*.

In general, a reference point represents an origin from which relative utility gains or losses can be measured and therefore reference functions help to evaluate a proposed utility allocation. Thus, reference points like the status quo and the ideal point have played an important role in many axiomatizations of bargaining solutions long before they have been named like that. Although the status quo defines a natural reference function it is not obvious that it is always the appropriate point to measure utility gains or losses from since it completely ignores the geometry of the bargaining region.

In this paper we examine reference functions in the context of bargaining problems with claims. Solutions to this class of problems also make use of reference points like the status quo and the claims point. As Thomson [10] we consider very general reference functions and show how solutions to bargaining problems with claims can be classified apart from differences in the reference function. By formulating axioms like *Indepencence of Irrelevant Alternatives* and *Monotonicity* with respect to the given reference function g we obtain characterizations of three different classes of solutions, all parametrized by g: Nash solutions, egalitarian solutions and proportional solutions. In order to characterize the class of proportional solutions we extend Thomson's [10] definition of a reference function and consider functions that assign to each bargaining problem with claims not only a reference point but also a vector representing the relative bargaining strengths of the players. This type of reference function is in fact equivalent to a function which assigns to each bargaining problem with claims *two* reference points (a good example would be the status quo and the claims point). We have chosen the former type of reference functions mainly for technical reasons.

We present several examples of reference functions that fulfill the mild conditions needed for the characterization results but do not argue in favor of a particular reference function which would clearly go beyond the scope of this paper. For particular choices of the reference function we do not only recover well known solutions to bargaining problems with claims like the *proportional solution* proposed by Chun and Thomson [3] and the *claim-egalitarian solution* proposed by Bossert [1], but also find interesting new solution concepts.

The paper is organized as follows. Section 2 provides the basic definitions. In section 3, 4, and 5 we present the characterization results for Nash, egalitarian

and proportional solutions, respectively. Section 5 concludes the paper with some final remarks.

2   Notation and Definitions

In the following $\mathbf{R}^n$, $n \in \mathbb{N}$, will denote the n-dimensional euclidean space and the set $N = \{1, \ldots, n\}$, $n \geq 2$, will denote the *player set*. The notation for vector inequalities is $\geq, >, \gg$. By $x \cdot y$ we denote the scalar product of the vectors $x, y \in \mathbf{R}^n$. For $x \in \mathbf{R}^n$, $a \in \mathbf{R}$, and $i \in N$ the vector $(a, x_{-i}) \in \mathbf{R}^n$ is defined to be the vector $(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)$. Convergence of a sequence of subsets of $\mathbf{R}^n$ is defined in terms of the Hausdorff topology. A set $S \subseteq \mathbf{R}^n$ is called *symmetric* if $\pi(S) = S$ for all permutations $\pi : N \to N$. A set $S \subseteq \mathbf{R}^n$ is *comprehensive* if $x \in S$ and $y \leq x$ implies that $y \in S$. A set $S \subseteq \mathbf{R}^n$ is *strictly comprehensive* if $x \in S$ and $y \leq x$ implies that $y \in S$ and that there exists $z \in S$, $z \gg y$. The *comprehensive hull* of a set $A \subseteq \mathbf{R}^n$ is given by

$$\mathrm{Co}(A) = \{x \in \mathbf{R}^n \,|\, x \leq y \text{ for some } y \in A\}.$$

The *comprehensive convex hull* of the vectors $a^1, \ldots, a^k \in \mathbf{R}^n$ is given by

$$\mathrm{CoCon}\{a^1, \ldots, a^k\} = \left\{ x \in \mathbf{R}^n \,\middle|\, x \leq \sum_{i=1}^{k} \lambda_i a^i, \, \lambda_i \geq 0, \, i = 1, \ldots, k, \, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

For $S \subseteq \mathbf{R}^n$ let

$$\mathrm{WPO}(S) = \{x \in S \,|\, y \in \mathbf{R}^n, \, y \gg x \Rightarrow y \notin S\}$$

be the set of *weakly Pareto optimal* points in S and let

$$\mathrm{PO}(S) = \{x \in S \,|\, y \in \mathbf{R}^n, \, y > x \Rightarrow y \notin S\}$$

be the set of *Pareto optimal* points in S.

3

Definition 2.1 *An n-person bargaining problem is a tuple (S, d), where*

1. $S \subseteq \mathbf{R}^n$ is convex, closed and comprehensive.

2. $d \in S$.

3. $\{x \in S | \ x \geq d\}$ is bounded.

Any bargaining problem is characterized by a set S of feasible utility allocations, measured in von Neumann–Morgenstern scales, and a point d, called threatpoint or disagreement point or status quo which is the outcome of the game if the players do not agree on a utility allocation in the feasible set. Thus, the status quo d can be unilaterally enforced by any player. Let Σ be the class of all n–person bargaining problems. For $(S, d) \in \Sigma$ the *utopia point* $u(S, d)$ is defined by

$$u_i(S, d) = \max\{x_i | \ x \in S, \ x \geq d\}, \ i = 1, \ldots, n.$$

A *solution* on a class of bargaining problems $\mathbf{D} \subseteq \Sigma$ is a mapping $f : \mathbf{D} \to \mathbf{R}^n$ such that $f(S, d) \in S$ for all $(S, d) \in \mathbf{D}$.

Imagine now a bargaining situation in which the players have claims that are not compatible with each other. Imagine further that the claims are credible or verifiable and that all players agree that they should be taken into account by any (fair) solution to the problem at issue. A good example for such a situation is a bankruptcy problem. While in the latter utility is transferable (utility is assumed to be linear in money) the following formal definition of a bargaining problem with claims refers to the general non–transferable utility case.

Definition 2.2 *An n–person bargaining problem with claims is a triple (S, d, c), where*

1. $(S, d) \in \Sigma$.

2. $c \in \mathbf{R}^n \setminus S, \ c > d$.

4

Let Σ^c be the class of all n-person bargaining problems with claims. A *solution* on a class of bargaining problems with claims $\mathbf{D}^c \subseteq \Sigma^c$ is a mapping $F : \mathbf{D}^c \to \mathbf{R}^n$ such that $F(S, d, c) \in S$ for all $(S, d, c) \in \mathbf{D}^c$.

For our characterization results we need some more notation. Let $a, b \in \mathbf{R}^n$, $a \gg 0$. The mapping $L^{a,b} : \mathbf{R}^n \to \mathbf{R}^n$ is called a *positive affine transformation* if for all $x \in \mathbf{R}^n$ and for all $i \in N$, $L_i^{a,b}(x) = a_i x_i + b_i$. For a set $A \subseteq \mathbf{R}^n$ let $L^{a,b}(A) = \{ L^{a,b}(x) | x \in A \}$. With a slight abuse of notation $L^{a,b}$ also induces a mapping $L^{a,b} : \Sigma^c \to \Sigma^c$ via

$$L^{a,b}(S, d, c) = (L^{a,b}(S), L^{a,b}(d), L^{a,b}(c)), \ (S, d, c) \in \Sigma^c.$$

3 Nash Solutions

The result and method of this section are an adaptation of Thomson [10] to our context of bargaining problems with claims.

We first consider a *reference function* g, given by a mapping $g : \Sigma^c \to \mathbf{R}^n$. Let $e = (1, \dots, 1) \in \mathbf{R}^n$ and define $\Delta = \{ \lambda e | \lambda \in \mathbf{R} \}$. We impose the following assumptions on the reference function $g : \Sigma^c \to \mathbf{R}^n$.

(A1) **Covariance with respect to positive affine transformations:** Let $(S, d, c) \in \Sigma^c$ and let $L^{a,b} : \mathbf{R}^n \to \mathbf{R}^n$ be a positive affine transformation. Then $g(L^{a,b}(S, d, c)) = L^{a,b}(g(S, d, c))$.

(A2) **Invariance with respect to symmetrization of almost symmetric problems:** Let $(S, d, c) \in \Sigma^c$ with $g(S, d, c) \in \Delta$ and $e \cdot x^* \geq e \cdot y$ for all $y \in S$ where $x^* \in \mathrm{WPO}(S) \cap \Delta$. Then there exists $(S', d', c') \in \Sigma^c$ with S' symmetric, $S' \supseteq S$, $e \cdot x^* \geq e \cdot y$ for all $y \in S'$, and $g(S', d', c') = g(S, d, c)$.

A1 is a natural assumption on a reference function given that we want relative utility gains over (or losses from) the reference point to be covariant under equivalent utility representations. On the other hand A2 is a completely technical assumption needed in the proof of Theorem 3.2. It requires that a bargaining problems with claims which already exhibits some symmetric structure can be replaced by a symmetric one with a larger set of feasible utility allocations without changing the essential features of the problem. In conjunction with axiom

$\mathbf{IA_g}$ below **A2** will be used to prove the uniqueness part of our characterization result.

Let $F : \mathbf{D^c} \to \mathbf{R}^n$ be a solution on the class $\mathbf{D^c} \subseteq \mathbf{\Sigma^c}$. Consider the following axioms.

(WPO) **Weak Pareto optimality:** $F(S, d, c) \in \mathrm{WPO}(S)$ for all $(S, d, c) \in \mathbf{D^c}$.

(SY) **Symmetry:** If $(S, d, c) \in \mathbf{D^c}$ is such that S is symmetric and $g(S, d, c) \in \Delta$, then $F(S, d, c) \in \Delta$.

(COV) **Covariance with respect to positive affine transformations:** For all $(S, d, c) \in \mathbf{D^c}$ if $L^{a,b} : \mathbf{R}^n \to \mathbf{R}^n$ is a positive affine transformation and if $L^{a,b}(S, d, c) \in \mathbf{D^c}$, then $F(L^{a,b}(S, d, c)) = L^{a,b}(F(S, d, c))$.

($\mathbf{IA_g}$) **Independence of alternatives other than $\mathbf{g(S, d, c)}$:** If (S, d, c), $(S', d', c') \in \mathbf{D^c}$ with $S \subseteq S'$, $g(S, d, c) = g(S', d', c')$ and $F(S', d', c') \in S$, then $F(S, d, c) = F(S', d', c')$.

For any reference function $g : \mathbf{\Sigma^c} \to \mathbf{R}^n$ let $\mathbf{\Sigma_g^c} = \{(S, d, c) \in \mathbf{\Sigma^c} \mid \exists x \in S, \, x \gg g(S, d, c)\}$.

Definition 3.1 *Let $g : \mathbf{\Sigma^c} \to \mathbf{R}^n$ be a reference function. The **Nash solution with respect to g** is defined to be the function $N^g : \mathbf{\Sigma_g^c} \to \mathbf{R}^n$, given by*

$$N^g(S, d, c) = \operatorname{argmax}\left\{ \prod_{i=1}^{n} (x_i - g_i(S, d, c)) \,\middle|\, x \in S, \, x \geq g(S, d, c) \right\},$$

$(S, d, c) \in \mathbf{\Sigma_g^c}$.

N^g is well defined since by assumption for all $(S, d, c) \in \mathbf{\Sigma_g^c}$ there exists $x \in S$, $x \gg g(S, d, c)$.

Theorem 3.2 *If $g : \mathbf{\Sigma^c} \to \mathbf{R}^n$ satisfies assumptions **A1** and **A2**, then N^g is the unique solution on $\mathbf{\Sigma_g^c}$ which satisfies **WPO**, **SY**, **COV** and $\mathbf{IA_g}$.*

6

Proof: It is obvious that N^g satisfies the axioms. To prove uniqueness let $F : \Sigma_g^c \to \mathbf{R}^n$ be a solution which satisfies **WPO**, **SY**, **COV**, **IA$_g$**, and let $(S, d, c) \in \Sigma_g^c$. Let $x^* = N^g(S, d, c)$. By **COV** and **A1** we can assume that $g(S, d, c) = 0$ and $x^* = e$.[1] (Observe that $N^g(S, d, c) \gg g(S, d, c)$ for all $(S, d, c) \in \Sigma_g^c$.) By definition of N^g we have $e \cdot y \leq e \cdot x^* = n$ for all $y \in S$ and therefore by **A2** there exists $(S', d', c') \in \Sigma^c$ with S' symmetric, $S' \supseteq S$, $e \cdot y \leq n$ for all $y \in S'$, and $g(S', d', c') = g(S, d, c)$. Since $S' \supseteq S$ and $g(S', d', c') = g(S, d, c)$ it is true that $(S', d', c') \in \Sigma_g^c$. By **WPO** and **SY** we get $F(S', d', c') = x^*$ and by **IA$_g$** we conclude that $F(S, d, c) = x^*$.

<div align="right">Q.E.D.</div>

In the following we examine some examples of reference functions which fulfill **A1** and **A2**.

3.1 Examples

1. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by

$$g(S, d, c) = d \text{ for all } (S, d, c) \in \Sigma^c.$$

This is the trivial case where g is completely independent of the claims point and thus N^g treats any bargaining problem with claims as if it were a bargaining problem without claims. If we identify $\{(S, d) \in \Sigma | \exists x \in S, x \gg d\}$ with the class Σ_d^c, then Theorem 3.2 recovers the characterization of the *Nash solution* (Nash [8]) on the class of bargaining problems without claims.

g fulfills the assumptions **A1** and **A2**. To see **A2** let $(S, d, c) \in \Sigma^c$ with $d \in \Delta$ and $e \cdot x^* \geq e \cdot y$ for all $y \in S$ where $x^* \in \text{WPO}(S) \cap \Delta$. Define $S' = \{y \in \mathbf{R}^n | e \cdot x^* \geq e \cdot y\}$, $d' = d$, $c' \in \{y \in \mathbf{R}^n | y \notin S', y \geq d'\}$ arbitrary. Then $(S', d', c') \in \Sigma^c$, $S' \supseteq S$, S' symmetric, and $g(S', d', c') = d' = d = g(S, d, c)$.

2. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by $g(S, d, c) = t(S, d, c)$, where for all $(S, d, c) \in \Sigma^c$

$$t_i(S, d, c) = \max\{d_i, \max\{x_i | (x_i, c_{-i}) \in S\}\}, \quad i = 1, \ldots, n.[2]$$

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[1] By 0 we denote the vector $(0, \ldots, 0) \in \mathbf{R}^n$.

t fulfills assumptions **A1** and **A2**. **A1** is straightforward. To see **A2** let $(S, d, c) \in \Sigma^c$ with $t(S, d, c) \in \Delta$ and $e \cdot x^* \geq e \cdot y$ for all $y \in S$ where $x^* \in \text{WPO}(S) \cap \Delta$. Define $S' = \{y \in \mathbf{R}^n \,|\, e \cdot x^* \geq e \cdot y\}$, $d' = t(S, d, c)$, $c' \in \Delta$, $c' \gg x^*$, large enough so that $t(S', d', c') = d'$. Then $(S', d', c') \in \Sigma^c$ fulfills the conditions required in **A2**.

The function t is a natural reference function in the context of bargaining problems with claims. No player can expect someone else to settle with less than what is necessary to satisfy the claims of the other players and no rational player will accept any payoff below his disagreement utility. Thus, $t(S, d, c)$ represents a *minimally equitable agreement* (see Herrero [5]) and it seems natural to measure the players' relative utility gains over this point.

4 Egalitarian Solutions

In this section we will impose a different set of axioms on our solution. It will turn out that under this set of axioms, the players are doing interpersonal comparisons of utility.

We impose the following assumptions on the reference function $g : \Sigma^c \to \mathbf{R}^n$.

(B1) Covariance with respect to translations: Let $(S, d, c) \in \Sigma^c$ and let $L^{e,b} : \mathbf{R}^n \to \mathbf{R}^n$ be a positive affine transformation. Then $g(L^{e,b}(S, d, c)) = L^{e,b}(g(S, d, c))$.[3]

(B2) Invariance with respect to the restriction to a symmetric subset: Let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta$. Then there exists $(S', d', c') \in \Sigma^c$, $S' \subseteq S$ symmetric, such that $S' \cap \Delta = S \cap \Delta$ and $g(S', d', c') = g(S, d, c)$.

(B3) Invariance with respect to approximation: Let $(S, d, c) \in \Sigma^c$, $g(S, d, c) \in \Delta$. Then there exists a sequence $((S^n, d^n, c^n))_n \subset \Sigma^c$ such that $S^n \to S$ and $S \subseteq S^n$, $\text{WPO}(S^n) \cap \Delta = \text{PO}(S^n) \cap \Delta$, $g(S^n, d^n, c^n) = g(S, d, c)$ for all n.

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[2]By definition $\max(\emptyset) = -\infty$.

[3]Observe that $e = (1, \ldots, 1) \in \mathbf{R}^n$, i.e. $L^{e,b}$ defines a translation.

Assumption **B1** needs no further explanation. As **A2** in the previous section assumptions **B2** and **B3** are of technical nature. **B3** allows for an approximation of the feasible set by supersets that are strictly comprehensive on the diagonal without changing the reference point.

Let $F : \mathbf{D}^c \to \mathbf{R}^n$ be a solution on the class $\mathbf{D}^c \subseteq \mathbf{\Sigma}^c$. Consider the following axioms.

(**TRANS**) **Covariance with respect to translations:** For all $(S, d, c) \in \mathbf{D}^c$ if $L^{e,b} : \mathbf{R}^n \to \mathbf{R}^n$ is a positive affine transformation and if $L^{e,b}(S, d, c) \in \mathbf{D}^c$, then $F(L^{e,b}(S, d, c)) = L^{e,b}(F(S, d, c))$.

(**RMON$_g$**) **Restricted monotonicity with respect to g(S,d,c):** Let $(S, d, c), (S', d', c') \in \mathbf{D}^c$ with $g(S', d', c') = g(S, d, c)$ and $S \subseteq S'$. Then $F(S, d, c) \le F(S', d', c')$.

Definition 4.1 *Let* $g : \mathbf{\Sigma}^c \to \mathbf{R}^n$ *be a reference function. The **egalitarian solution with respect to g** is defined to be the function* $E^g : \mathbf{\Sigma}^c \to \mathbf{R}^n$, *given by*

$$E^g(S, d, c) = g(S, d, c) + \bar{\lambda}e,$$

where $\bar{\lambda} = \max\{\lambda \in \mathbf{R} \mid g(S, d, c) + \lambda e \in S\}$, $(S, d, c) \in \mathbf{\Sigma}^c$.

Since for all $(S, d, c) \in \mathbf{\Sigma}^c$ the set S is closed and comprehensive and the set $\{x \in S \mid x \ge d\}$ is bounded the solution E^g is well defined on $\mathbf{\Sigma}^c$. If $g(S, d, c) \in S$ $(g(S, d, c) \notin S)$, then E^g equalizes the gains (losses) from the reference point.

Theorem 4.2 *If* $g : \mathbf{\Sigma}^c \to \mathbf{R}^n$ *satisfies assumptions* **B1**, **B2**, **B3**, *then* E^g *is the unique solution on* $\mathbf{\Sigma}^c$ *which satisfies* **WPO**, **SY**, **TRANS** *and* **RMON$_g$**.

Proof: It is straightforward to see that E^g satisfies the axioms. Let $F : \mathbf{\Sigma}^c \to \mathbf{R}^n$ be a solution which satisfies **WPO**, **SY**, **TRANS**, **RMON$_g$**, and let $(S, d, c) \in \mathbf{\Sigma}^c$. By **TRANS** and **B1** we can assume that $g(S, d, c) = 0$. Then $E^g(S, d, c) = x^* \in \Delta$. By **B2** there exists $(S', d', c') \in \mathbf{\Sigma}^c$, $S' \subseteq S$ symmetric, such that $S \cap \Delta = S' \cap \Delta$ and $g(S', d', c') = g(S, d, c)$. Therefore, $x^* \in S'$ and by

9

RMON$_g$ $F(S', d', c') \leq F(S, d, c)$. By **SY** and **WPO** we have $F(S', d', c') = x^*$. If $x^* \in PO(S)$ this implies $F(S, d, c) = x^*$.

If $x^* \in WPO(S) \setminus PO(S)$ by **B3** there exists a sequence $((S^n, d^n, c^n))_n \subset \Sigma^c$ such that $S^n \to S$ and $S \subseteq S^n$, $WPO(S^n) \cap \Delta = PO(S^n) \cap \Delta$, and $g(S^n, d^n, c^n) = g(S, d, c)$ for all n. Therefore, $E^g(S^n, d^n, c^n) \in PO(S^n) \cap \Delta$ and by the above we conclude that $F(S^n, d^n, c^n) = E^g(S^n, d^n, c^n)$ for all n. Therefore, by **RMON$_g$** for all n

$$x^* = F(S', d', c') \leq F(S, d, c) \leq F(S^n, d^n, c^n) = E^g(S^n, d^n, c^n).$$

Since $g(S^n, d^n, c^n) = g(S, d, c)$ for all n and $S^n \to S$ we have $E^g(S^n, d^n, c^n) \to x^*$ which implies that $F(S, d, c) = E^g(S, d, c)$.

<div align="right">Q.E.D.</div>

We conclude this section by presenting some examples of reference functions satisfying **B1**, **B2**, **B3**.

4.1 Examples

1. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by

$$g(S, d, c) = d \text{ for all } (S, d, c) \in \Sigma^c.$$

As in Example 3.1(1) g is independent of the claims point c. If we identify Σ with Σ^c then Theorem 4.2 recovers the characterization of the *egalitarian solution* (Kalai [7]) on the class of bargaining problems without claims.

It is obvious that g satisfies **B1**. To see **B2** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta$. Let $x \in WPO(S) \cap \Delta$ and define $S' = Co\{x\}$, $d' = d$, $c' = c$. Then $(S', d', c') \in \Sigma^c$ (observe that $d' = d = g(S, d, c) \in S \cap \Delta$ and therefore $x \geq d'$), $S' \subseteq S$ symmetric, $S' \cap \Delta = S \cap \Delta$, and $g(S', d', c') = g(S, d, c)$.

To see **B3** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta$ and let $x \in WPO(S) \cap \Delta$, i.e. $x = \lambda e$ for some $\lambda \in \mathbf{R}$. For $\delta > 0$ define $S_\delta = CoCon(S \cup \{(\lambda + \delta)e\})$ and let $\bar{\delta} > 0$ be small enough so that $c \notin S_{\bar{\delta}}$. Let $(\delta_n)_n \subset]0, \bar{\delta}[$ be a sequence with $\delta_n \to 0$. For all n define $S^n = S_{\delta_n}$, $d^n = d$, $c^n = c$. Then for all n we have $(S^n, d^n, c^n) \in \Sigma^c$, $S \subset S^n$, $WPO(S^n) \cap \Delta = (\lambda + \delta_n)e = PO(S^n) \cap \Delta$, $g(S^n, d^n, c^n) = g(S, d, c)$, and $S^n \to S$. Thus, g satisfies **B3**.

2. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by

$$g(S, d, c) = u(S, d) \text{ for all } (S, d, c) \in \Sigma^c.$$

Again g is independent of the claims point c and if we identify Σ with Σ^c then Theorem 4.2 recovers the characterization of the equal-loss solution (Chun [2]) on the class of bargaining problems without claims.

Assumption **B1** is straightforward to see. To see **B2** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) = u(S, d) \in \Delta$. Let $x \in \text{WPO}(S) \cap \Delta$ and for all $i \in N$ let $\lambda_i^* = \max\{\lambda | ((\lambda e)_{-i}, u_i(S, d)) \in S\}$. Define $\lambda^* = \min_{i \in N} \lambda_i^*$ and let $y^i = ((\lambda^* e)_{-i}, u_i(S, d)), i \in N$. Let $S' = \text{CoCon}\{y^1, \dots, y^n, x\}, d' = \lambda^* e, c' > d', c' \notin S'$ arbitrary. Then $(S', d', c') \in \Sigma^c$, $S' \subseteq S$ symmetric, $g(S', d', c') = u(S', d') = u(S, d) = g(S, d, c)$, and $S' \cap \Delta = S \cap \Delta$. Thus, g satisfies **B2**.

To see that g satisfies **B3** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) = u(S, d) \in \Delta$. If $u(S, d) \in S$ then $(S^n, d^n, c^n) = (S, d, c)$ for all n fulfills the conditions given in **B3**. If $u(S, d) \notin S$ define $((S^n, d^n, c^n))_n \subset \Sigma^c$ as in the previous example where $\bar{\delta} > 0$ small enough such that $u(S, d) \gg (\lambda + \bar{\delta})e$. Then it is easy to see that $((S^n, d^n, c^n))_n$ fulfills the conditions of **B3**.

3. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by

$$g(S, d, c) = c \text{ for all } (S, d, c) \in \Sigma^c.$$

Then E^g is identical to the claim-egalitarian solution proposed by Bossert [1]. Again it is obvious that g satisfies **B1**. To see **B2** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta$. Let $x \in \text{WPO}(S) \cap \Delta$ and as in Example 4.1(1) define $S' = \text{Co}\{x\}$. Let $c' = c$ and $d' \in \{y \in S' | y \le c'\}$ arbitrary. Then $(S', d', c') \in \Sigma^c$, $S' \subseteq S$ symmetric, $S \cap \Delta = S' \cap \Delta$ and $g(S', d', c') = g(S, d, c)$.

The proof for **B3** is exactly the same as in Example 4.1(1).

4. Let $g : \Sigma^c \to \mathbf{R}^n$ be given by

$$g(S, d, c) = t(S, d, c) \text{ for all } (S, d, c) \in \Sigma^c.$$

(For a definition of t see Example 3.1(1).) Clearly, t satisfies **B1**. To see **B2** let $(S, d, c) \in \Sigma^c$ be such that $t(S, d, c) \in \Delta$. As before let $x \in \text{WPO}(S) \cap \Delta$

and define $S' = \text{Co}\{x\}$. Let $d' = t(S, d, c)$ and $c' \gg x$ arbitrary. Then $(S', d', c') \in \Sigma^c$, $S' \subseteq S$ symmetric, $S \cap \Delta = S' \cap \Delta$, and $t(S', d', c') = t(S, d, c)$.

t also satisfies **B3**. Let $(S, d, c) \in \Sigma^c$ be such that $t(S, d, c) \in \Delta$. Define $(S^n)_n$ as in Example 4.1(1) and for all n let $d^n = t(S, d, c)$, $c^n \gg u(S_{\bar{\delta}}, t(S, d, c))$ large enough so that $t(S^n, d^n, c^n) = d^n$. Then for all n we have $(S^n, d^n, c^n) \in \Sigma^c$, $S \subset S^n$, $\text{WPO}(S^n) \cap \Delta = \text{PO}(S^n) \cap \Delta$, $t(S^n, d^n, c^n) = t(S, d, c)$, and $S^n \to S$.

5   Proportional Solutions

In order to characterize proportional solutions we consider the following type of reference function: Let $g : \Sigma^c \to \mathbb{R}^n \times \mathbb{R}^n_+$, where $g(S, d, c) = (g^r(S, d, c), g^p(S, d, c))$ and $\|g^p(S, d, c)\| = 1$ for all $(S, d, c) \in \Sigma^c$.[4] Thus, the reference function g assigns to any bargaining problem with claims not only a reference point $g^r(S, d, c)$ but also a vector of proportions $g^p(S, d, c)$. We interpret g^p as a vector reflecting the bargaining strenghts of the players.

Let $\overline{\Sigma^c_g} = \{(S, d, c) \in \Sigma^c \mid g^p(S, d, c) \in \mathbb{R}^n_{++}\}$. We impose the following assumptions on the reference function $g : \Sigma^c \to \mathbb{R}^n \times \mathbb{R}^n_+$.

(C1) **Covariance with respect to positive affine transformations:** Let $(S, d, c) \in \Sigma^c$ and let $L^{a,b} : \mathbb{R}^n \to \mathbb{R}^n$ be a positive affine transformation. Then

$$
\begin{aligned}
g^r(L^{a,b}(S, d, c)) &= L^{a,b}(g^r(S, d, c)), \\
g^p(L^{a,b}(S, d, c)) &= \lambda L^{a,0}(g^p(S, d, c)),
\end{aligned}
$$

where $\lambda = \|L^{a,0}(g^p(S, d, c))\|^{-1}$.

(C2) **Invariance with respect to the restriction to a symmetric subset:** Let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta \times \Delta$. Then there exists $(S', d', c') \in \Sigma^c$, $S' \subseteq S$ symmetric, such that $S' \cap \Delta = S \cap \Delta$ and $g(S', d', c') = g(S, d, c)$.

---

[4]$\|\cdot\|$ denotes some norm on $\mathbb{R}^n$.

(C3) **Invariance with respect to approximation:** Let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta \times \Delta$. Then there exists a sequence $((S^n, d^n, c^n))_n \subset \Sigma^c$ such that $S^n \to S$ and $S \subseteq S^n$, $WPO(S^n) \cap \Delta = PO(S^n) \cap \Delta$, $g(S^n, d^n, c^n) = g(S, d, c)$ for all n.

Observe that $(S, d, c) \in \Sigma^c$ and $g^p(S, d, c) \in \Delta$ already imply that $(S, d, c) \in \overline{\Sigma^c_g}$. We will use this fact throughout in the proof of Theorem 5.2.

Let $F : \mathbf{D^c} \to \mathbf{R}^n$ be a solution on the class $\mathbf{D^c} \subseteq \Sigma^c$. We reformulate the axioms **SY** and **RMON$_g$** since they now refer to a different type of reference function.

(SY) **Symmetry:** If $(S, d, c) \in \mathbf{D^c}$ is such that S is symmetric and $g(S, d, c) \in \Delta \times \Delta$, then $F(S, d, c) \in \Delta$.

(RMON$_g$) **Restricted monotonicity with respect to g(S, d, c):** If (S, d, c), $(S', d', c') \in \mathbf{D^c}$ are such that $g(S', d', c') = g(S, d, c)$ and $S \subseteq S'$, then $F(S, d, c) \leq F(S', d', c')$.

Definition 5.1 *Let $g : \Sigma^c \to \mathbf{R}^n \times \mathbf{R}^n_+$ be a reference function. The **proportional solution with respect to g** is defined to be the function $P^g : \overline{\Sigma^c_g} \to \mathbf{R}^n$, given by*

$$P^g(S, d, c) = g^r(S, d, c) + \bar{\lambda} g^p(S, d, c),$$

where $\bar{\lambda} = \max\{\lambda \in \mathbf{R} \mid g^r(S, d, c) + \lambda g^p(S, d, c) \in S\}$, $(S, d, c) \in \overline{\Sigma^c_g}$.

Given the assumptions on the class $\overline{\Sigma^c_g}$ it is straightforward to see that P^g is well defined.

Theorem 5.2 *Let $g : \Sigma^c \to \mathbf{R}^n \times \mathbf{R}^n_+$ satisfy assumptions **C1**, **C2**, **C3**. Then P^g is the unique solution on $\overline{\Sigma^c_g}$ which satisfies **WPO**, **SY**, **COV** and **RMON$_g$**.*

Proof: Obviously, P^g satisfies the axioms. To see that it is unique let $F : \overline{\Sigma^c_g} \to \mathbf{R}^n$ satisfy **WPO**, **SY**, **COV** and **RMON$_g$** and let $(S, d, c) \in \overline{\Sigma^c_g}$. By **COV** and assumption **C1** we can assume that $g^r(S, d, c) = 0$ and $g^p(S, d, c) \in$

13

Δ. Thus, $P^g(S,d,c) = x^* \in \Delta$. By assumption **C2** there exists $(S',d',c') \in \overline{\Sigma}^c_g$, $S' \subseteq S$ symmetric, such that $S \cap \Delta = S' \cap \Delta$ and $g(S',d',c') = g(S,d,c)$. Therefore, $x^* \in S'$ and by **RMON$_g$** $F(S',d',c') \leq F(S,d,c)$. By **SY** and **WPO** $F(S',d',c') = x^*$. If $x^* \in \text{PO}(S)$ this implies $F(S,d,c) = x^*$.

If $x^* \in \text{WPO}(S) \setminus \text{PO}(S)$ by assumption **C3** there exists a sequence $((S^n,d^n,c^n))_n \subset \overline{\Sigma}^c_g$ such that $S^n \to S$ and $S \subseteq S^n$, $\text{WPO}(S^n) \cap \Delta = \text{PO}(S^n) \cap \Delta$, and $g(S^n,d^n,c^n) = g(S,d,c)$ for all n. Therefore, $P^g(S^n,d^n,c^n) \in \text{PO}(S^n)$ and by the above $F(S^n,d^n,c^n) = P^g(S^n,d^n,c^n)$ for all n. Thus,

$$x^* = F(S',d',c') \leq F(S,d,c) \leq F(S^n,d^n,c^n) = P^g(S^n,d^n,c^n).$$

Since $P^g(S^n,d^n,c^n) \to x^*$ we conclude that $F(S,d,c) = P^g(S,d,c)$.

<div align="right">Q.E.D.</div>

Again we conclude the section by presenting some examples of reference functions which satisfy **C1**, **C2**, **C3**.

5.1 Examples

1. Let $g : \Sigma^c \to \mathbf{R}^n \times \mathbf{R}^n_+$ be given by

$$g^r(S,d,c) = d, \quad g^p(S,d,c) = (c-d)\|c-d\|^{-1}, \quad (S,d,c) \in \Sigma^c.$$

   Observe that g is well defined since $c > d$ for all $(S,d,c) \in \Sigma^c$. For this choice of g the solution P^g is identical to the *proportional solution* proposed by Chun and Thomson [3]. It is straightforward to see that g satisfies **C1**. In order to show that g satisfies the assumptions **C2** and **C3** we can use exactly the same construction as in Example 4.1(1).

2. Let $g : \Sigma^c \to \mathbf{R}^n \times \mathbf{R}^n_+$ be given by

$$g^r(S,d,c) = t(S,d,c), \quad g^p(S,d,c) = \lambda(c - t(S,d,c)),$$

   where $\lambda = \|c - t(S,d,c)\|^{-1}$, $(S,d,c) \in \Sigma^c$. Since $t(S,d,c) \in S$ and $c \notin S$ the function g is well defined. For this choice of g the solution P^g is the *adjusted proportional solution* proposed by Herrero [6]. It is easy to see that g satisfies **C1**. To see **C2** let $(S,d,c) \in \Sigma^c$ be such that $g(S,d,c) \in \Delta \times \Delta$.

<div align="center">14</div>

Let $x \in \text{WPO}(S) \cap \Delta$ and define $S' = \text{Co}\{x\}$. Let $d' = t(S, d, c)$ and $c' \in \{y \in \Delta \,|\, y \gg x\}$ arbitrary. Then $(S', d', c') \in \Sigma^c$ fulfills the conditions given in **C2** .

To see **C3** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta \times \Delta$. Define $(S^n)_n$ as in Example 4.1(1). For all n let $d^n = t(S, d, c)$ and $c^n \in \{y \in \Delta \,|\, y \gg u(S_{\bar{\delta}}, t(S, d, c))\}$ large enough so that $t(S^n, d^n, c^n) = d^n$. Then $(S^n, d^n, c^n) \in \Sigma^c$ and $g^r(S^n, d^n, c^n) = g^r(S, d, c)$ for all n. Further, $S^n \to S$ and for all n we have $S \subseteq S^n$ and $\text{WPO}(S^n) \cap \Delta = \text{PO}(S^n) \cap \Delta$. Since $c^n - t(S^n, d^n, c^n) \in \Delta$ it is true that $g^p(S^n, d^n, c^n) = g^p(S, d, c)$ and therefore $g(S^n, d^n, c^n) = g(S, d, c)$ for all n. Thus, g fulfills **C3** .

3. Let $g : \Sigma^c \to \mathbf{R}^n \times \mathbf{R}^n_+$ be given by

$$g^r(S, d, c) = t(S, d, c), \quad g^p(S, d, c) = \lambda(u(S, t(S, d, c)) - t(S, d, c)),$$

where $\lambda = \|u(S, t(S, d, c)) - t(S, d, c)\|^{-1}$, $(S, d, c) \in \Sigma^c$. Again it is straightforward to see that $u(S, t(S, d, c)) > t(S, d, c)$ for all $(S, d, c) \in \Sigma^c$ so that g is well defined. For this choice of g the solution P^g is the Raiffa–Kalai–Smorodinsky solution to the bargaining problem with feasible set S and adjusted threatpoint $t(S, d, c)$. Gerber [4] proposed this solution under the name *extended Raiffa–Kalai–Smorodinsky solution.*

Again **C1** is straightforward. To see **C2** let $(S, d, c) \in \Sigma^c$ be such that $g(S, d, c) \in \Delta \times \Delta$. Let $x \in \text{WPO}(S) \cap \Delta$ and define $S' = \text{Co}\{x\}$. Let $d' = t(S, d, c)$, $c' \in \{y \in \Delta \,|\, y \gg x\}$ arbitrary. Then, $t(S', d', c') = d' = t(S, d, c)$ and $u(S', t(S', d', c')) = x$ which implies $g^p(S', d', c') \in \Delta$ and therefore $g(S', d', c') = g(S, d, c)$. Thus, $(S', d', c') \in \Sigma^c$ fulfills the conditions given in **C2**.

To see **C3** let $(S, d, c) \in \Sigma^c$ be given with $g(S, d, c) \in \Delta \times \Delta$. Define $(S^n)_n$ as in Example 4.1(1). For all n let $d^n = t(S, d, c)$ and let $c^n \in \{y \in \Delta \,|\, y \gg u(S_{\bar{\delta}}, t(S, d, c))\}$ be large enough so that $t(S^n, d^n, c^n) = d^n$. Then $(S^n, d^n, c^n) \in \Sigma^c$ and $g^r(S^n, d^n, c^n) = g^r(S, d, c)$ for all n. By construction for all n either $u(S^n, t(S^n, d^n, c^n)) = u(S, t(S, d, c))$ or $u(S^n, t(S^n, d^n, c^n)) = (\lambda + \delta_n)e$. In any case $u(S^n, t(S^n, d^n, c^n)) \in \Delta$ which together with $t(S^n, d^n, c^n) \in \Delta$ implies that $g^p(S^n, d^n, c^n) \in \Delta$ and therefore $g^p(S^n, d^n, c^n) = g^p(S, d, c)$ for all n. Further, $\text{WPO}(S^n) \cap \Delta = \text{PO}(S^n) \cap \Delta$ and $S^n \to S$.

Thus, g fulfills **C3** .

6 Conclusion

Our paper shows that there are two main features which distinguish different so-
lutions to bargaining problems with claims: One is the type of reference function
the solution is related to and the other is the way in which the solution reacts
to changes in a bargaining problem with claims that do not move the reference
point. Concerning the latter we have concentrated on the axioms of *Independence
of Irrelevant Alternatives* and *Monotonicity*, being the most prominent ones in
classic axiomatizations of bargaining solutions. Three different classes of solu-
tions are obtained, all parametrized by the reference function g, namely Nash,
egalitarian and proportional solutions. For particular choices of the reference
function we recover well known solutions to bargaining problems with claims and
also propose new solution concepts involving the minimally equitable agreement
$t(S, d, c)$.

The concept of a reference function allows for a systematic classification of
solutions to bargaining problems with claims and also provides the framework
within which new solutions can be developed. A necessary next step that we
have disregarded in this paper is, of course, to examine the choice of the reference
function. This will clearly depend on properties of the reference function in
addition to those needed for the characterization results. For example, we will
certainly require the reference function to depend on the claims point. Otherwise,
as can be seen from our examples, the claims are irrelevant for the solution to
the bargaining problem and we obtain solutions to bargaining problems without
claims.

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