INSTITUTE OF MATHEMATICAL ECONOMICS

Large Totally Balanced Games

by

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Abstract

We exhibit the structure of totally balanced games, discuss some conditions for a game to be extreme within the cone of totally balanced games and provide conditions for the core to be a von Neumann - Morgenstern stable set. Our emphasis is put on exact and 'orthogonal' games. The main message is that the unique representation of such games, the extremepoint property, and the existence of stable sets can be ensured if the game has 'sufficiently many players' of each of a finite set of types.

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0   Introduction

We consider cooperative **games**, represented by a triple $(N, \underline{P}, v)$. Here N is a finite set ('the set of players'), $\underline{P}$ is the power set ('the system of coalitions') and $v : \underline{P} \to I\!R_+, v(\emptyset) = 0$, is the 'coalitional' or 'characteristic' function (frequently also referred to as 'the game'); we prefer to consider nonnegative functions only. We always use $n = |N|$ and frequently assume tacitly $N = \{1, \cdots, n\}$.

The identification of a vector $m \in I\!R^n$ and an additive set function m on $\underline{P}$ via

$$m(S) := \sum_{i \in S} m_i \quad (S \in \underline{P})$$

will be common, we call m a *measure* if it is nonnegative ($\lambda \geq 0$) and *normalized* (a *probability*) if $m(N) = 1$ holds true. The set of additive set functions is denoted by $\mathbf{A}$, we use e.g. $\mathbf{A}^1_+$ in order to denote probabilities. The **carrier** of some $m \in \mathbf{A}$ is denoted by $C(m)$. For $S, T \in \underline{P}$ disjoint we write $S + T$ instead of $S \cup T$, thus additive set functions are characterized by $m(S+T) = m(S) + m(T), \quad (S, T \in \underline{P} \ disjoint)$.

Our discussion is focussed on a class of games which can be equivalently represented by one of the following types of set functions.

(1) *Totally balanced games:*
     no coalition improves by a balanced decomposition, i.e.,

$$\sum_{\underline{S}} c_S v(S) \leq v(T)$$

     for every $\underline{S}$ admitting a 'partition of the unit', i.e., a set of nonnegative coeficients $(c_S)_{S \in \underline{S}}$ such that

$$\sum_{\underline{S}} 1_S v(S) = 1_T,$$

     where 1_S denotes the indicator function (or vector) of a set S (see [2] , [16]).

(2) *Games with a nonempty core of every restriction:*

$$\mathcal{C}(v_T) \neq \emptyset$$

     for all $T \in N$. (see [2] , [16]).

(3) *Market Games:*
     generated by a TU.-market, that is, $v = v^{\mathcal{U}}$ is described via

$$v(S) = \max \left\{ \sum_{i \in S} u^i(x^i) \mid \sum_{i \in S} x^i \sum_{i \in S} a^i, \quad (S \in \underline{P}) \right\};$$

here $\mathcal{U} = (N, \mathbb{R}_+^m, (u^i)_{i \in N}, (a^i)_{i \in N})$ denotes a *market* or *pure exchange economy* specified by the set of players, the space of commodities, the (continuous, monotone, concave) utility functions and the initial asignments of the players (see [17], [20]).

(4) *Flow Games:* generated by generalized 'maximal-flow' / 'minimal cut' problems and written $v = v^\Gamma$. Here Γ is directed network allowing 'flows' from 'source' to 'sink'such that each edge is assigned to a player. $v^\Gamma(S)$ is the *max-flow / min-cut* value assigned to the subnetwork with edges restricted to the members of S and nodes beeing unchanged. (see [5]).

(5) *L.-P.-Games:*
generated by a linear programming setup and a vector-valued measure (distribution of raw factors). $v = v^{A,b,c}$ is given via

$$v(S) = \max \left\{ cx \mid x \in \mathbb{R}_+^m, Ax \leq b(S) \right\}, \quad (S \in \underline{\mathbb{P}})$$

here A is an $l \times m-$ matrix, $c \in \mathbb{R}_+^m$, and b is a 'vector-valued measure'on $\underline{\mathbb{P}}$, i.e., $b = (b^1, \cdots, b^l)$ with $b^j \in \mathbb{R}^n$, $b^j(S) = \sum_{i \in S} b_i^j$. (All quantities are assumed to be positive in order to avoid any existence problems) (see [6]).

(6) *Minima of finitely many measures:*
We shall write these functions as

(1) $$v = \bigwedge \left\{ \lambda^1, \cdots, \lambda^r \right\}.$$

Here $\lambda^1, \cdots, \lambda^r \in \mathbf{A}_+$ and $\bigwedge$ denotes the *minimum* taken in the space (lattice) of set functions, i.e. we mean v as defined via

(2) $$v(S) = \min \left\{ \lambda^1(S), \cdots, \lambda^r(S) \right\} \quad (S \in \underline{\mathbb{P}}).$$

(see [5]).

It is well known the the above description yields one and the same class, i.e., the totally balanced games. Since the last way of describing such games / set functions seems to be the most simple one, we shall consider functions described by item (6) above.

Among these we want to restrict ourselves to the *normalized* case; the class obtained this way is described by

$$\mathbb{E} = \left\{ v \mid v \text{ is totally balanced}, v \geq 0, \exists \lambda^1, \cdots, \lambda^r \in \mathbf{A}_+^1 : v = \bigwedge \left\{ \lambda^1, \cdots, \lambda^r \right\} \right\}$$

It is well known, that $\mathbb{E}$ consists precisely of the *exact* (totally balanced) games, i.e.,

$$\mathbb{E} = \left\{ v \mid v \geq 0, v \text{ is exact} \right\}$$

We discuss some topics concerning the possibly *unique* or *canonical* representation of some $v \in I\!\!E$ by means of some set $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$, the structure of the class $I\!\!E$ given by its extremepoints ($I\!\!E$ is a convex polyhedron) and solution concepts. The latter will deal with the question whether of presenting conditions such that the core is a stable set in the sense of VON NEUMANN − MORGENSTERN[19].

Within the framework of our disscussion a major tool will be be provided by the notion of *nondegeneracy* of a measure with respect to a system or family of sets. Basically this means that the measure is uniquely defined by its values on this system. For a start, let us recall what it means that a measure, say m is nondegenerate w.r.t. a set system $\underline{\underline{S}} \in \underline{\underline{P}}$:

Definition 0.1 *A (nonnegative) additive set function m is* **nondegenerate** *with respect to a system of sets $\underline{\underline{S}}$ if the system of linear equations in variables $(x_i)_{i \in \Omega}$*

$$(3) \qquad \sum_{i \in S} x_i = m(S) \quad (S \in \underline{\underline{S}})$$

*admits of the unique solution m only. In particular we say that m is nondegenerate w.r.t. α (**m n.d.** α) if m n.d. $\underline{\underline{Q}}_\alpha$ where the system $\underline{\underline{Q}}_\alpha$ is given by*

$$\underline{\underline{Q}}_\alpha := \{S \in \underline{\underline{P}} \mid m(S) = \alpha\} .$$

Applications of this concept have been found in various contexts:

(1) *CONVEX GAMES:*
Let $v = f \circ m$ such that f is a piecewise linear and monotone function having increasing first differences and m is a vectorvalued measure. Essentially it is true that m is n.d. with respect to the 'critical values' α of f, (the 'kinks' that is,) if and only if $f \circ m$ is extreme. This is the main result of extreme point theory of convex games as developed in ROSENMÜLLER − WEIDNER [14] and [15].

(2) *HOMOGENEOUS GAMES:* Let m me a probability and let $\alpha \in (0,1)$. Let $v^m_\alpha := 1_{[0,\alpha]} \circ m$ denote a (simple) homogenous game (see OSTMANN [7], ROSEN− MÜLLER [12], SUDHÖLTER [18]). Then v^m_α has no steps if and only if m n.d. α is true.

(3) *LP GAMES:*
These were introduced by OWEN([6] . See [11] for the following observations: Let $v = v^{A,b,c}$. Assume that the grand coalition has a unique *shadow price,* say $\bar{y}$ (i.e., assume some type of nondegeneracy in the LP sense). If b is n.d. with respect to $\underline{\underline{Q}} = \{S \mid S$ has shadow prices of $N\}$, then

$$\mathcal{C}(v) = \{b \cdot \bar{y}\},$$

here $\bar{y}$ denotes the shadow price of the grand coalition N. Note that the totally balanced games satisfying this kind of nondegeneracy in the L.P - sense are typically *not* the normalized elements of $\mathbb{E}$ - rather there is unique λ^ρ minimizing at N in the representation of formulas (1) or (2)

We consider nondegeneracy to be a 'finite surrogate' for the non-atomic property in the continuous case. in some well defined sense it has allways been the case that nondegeneracy could be obtained by requiring the presence of 'many' (small) players. Thus one is is usually concerned with 'large' games when dealing with versions of nondegeneracy. This view is supported by extreme-point results, convergence results or 'equivalence theorems' etc. as well as the fact that the 'n.d.' property can be sensibly established for nonatomic measures on a continuum.

We would like to study these methods as applied to totally balanced games, more precisely to the class $\mathbb{E}$. The 'critical system' in this case is foremost given by the 'diagonal sets' or for short the 'diagonal'. For $v \in \mathbb{E}$ the *diagonal* is given by

$$(4) \qquad \underline{\underline{\Delta}} = \{S \in \underline{\underline{P}} \mid v(S) + v(S^c) = v(N)\}$$

It is not hard to see that for any set $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ yielding $v = \wedge \{\lambda^1, \cdots, \lambda^r\}$ it follows that

$$(5) \qquad \underline{\underline{\Delta}} = \left\{S \in \underline{\underline{P}} \mid \lambda^1(S) = \cdots = \lambda^r(S)\right\}$$

holds true. I.e., the 'vector valued measure' $\lambda = (\lambda^1, \cdots, \lambda^r)$ throws diagonal sets into the diagonal of $[0, 1]^r$; this explains the notation.

Clearly this system is very decisive for the behavior of v, there is evidence in various papers already mentioned. Our first task will be to develop a suitable version of nondegeneracy within this context. It turns out that this is connected to the question of a 'canonical' representation of v (Section 1).

Next Section 2 exhibits the application of a type of n.d. requirement to the extreme-point problem in $\mathbb{E}$.

Finally, Section 3 shows that for some large games in $\mathbb{E}$ the core is stable in analogy to the results obtained for the nonatomic case in [3].

1   Complete Games.

Let $v \in I\!\!E$, i.e., let v be a nonnegative, totally balanced and exact set function. We call a set of measures $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ a *representation* of v if $v = \wedge \{\lambda^1, \cdots, \lambda^r\}$ holds true. The extreme points of the core of v allways constitute a representation - but some of them may be superfluous. Therefore we supply the following definition.

Definition 1.1 *Let $v \in I\!\!E$ and let $\lambda^1, \cdots \lambda^r$ be the extremepoints of $C(v)$. v is called* **complete** *if, for every $\rho \in \{1, \cdots, r\}$, there is $S \in \underline{P}$ such that*

$$v(S) = \lambda^\rho(S) < \lambda^\sigma(S) \quad (\sigma \in \{1, \cdots, r\}, \sigma \neq \rho)$$

holds true.

Roughly speaking in a complete game all the extremepoints are necessary in order to supply a representation of v as the minimum. It is also seen at once that this way the representation is 'minimally unique'(in the set theoretical sense):

Theorem 1.2 *If $v \in I\!\!E^1$ is complete, then there is a unique minimal representation $v = \wedge \{\lambda^1, \cdots, \lambda^r\}$ with $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$. The λ^ρ $(\rho = 1, \cdots, r)$ are exactly the extreme points of $C(v)$.*

The **Proof** is obvious: Pick the extremes of the core, say $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$; as $v \in I\!\!E$, we know that $v = \wedge \{\lambda^1, \cdots, \lambda^r\}$.

Now let $v = \wedge \{\mu^1, \cdots, \mu^K\}$ constitute a further representation of v. All of the μ^κ are in the core. Hence all the μ^κ are convex combinations of the λ^σ $(\sigma \neq \rho)$. Thus should it occur for some ρ that $\lambda^\rho \notin \{\mu^1, \cdots, \mu^K\}$ holds true, then, if we choose S suitable according to Definition 1.1, we find that (with suitable sets of cefficients α.

$$\lambda^\rho(S) < \sum_{\sigma \neq \rho} \alpha_\sigma \lambda^\sigma(S) = \mu^\kappa(S)$$

would follow for all κ, which is clearly not possible.   **q.e.d.**

Example 1.3 *Let $\lambda^1 := (3, 4, 0, 0)$ and $\lambda^2 := (0, 0, 2, 5)$. The extreme points of $v = \wedge \{\lambda^1, \lambda^2\}$ apart from λ^1 and λ^2 are given by $(2, 4, 0, 1)$, $(0, 2, 2, 3)$, $(2, 2, 0, 3)$, $(3, 2, 0, 2)$, $(1, 0, 2, 4)$, and $(1, 2, 2, 2)$.*

Remark 1.4   *(1) Clearly extreme points of the core of satisfy the 'equivalence theorem', i.e., every element of the core is a convex combination of them and hence a solution of the dual program of the grand coalition in the LP-game sense. A non complete game might however have representations consisting of less than all extremes of the core. Completeness is a property of the game (of v), the equivalence theorem is a property of a representation.*

(2) If $v \in \mathbb{E}$ allows for a representation $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ with $\lambda^1, \cdots, \lambda^r \in \mathbf{A}_+^1$ **mutually orthogonal**, then the λ^ρ are extreme in $\mathcal{C}(v)$. For suppose we have $\lambda^1 = \frac{1}{2}(\mu + \nu)$ with $\mu, \nu \in \mathcal{C}(v)$, then the carriers $C(\mu)$ and $C(\nu)$ are contained in $C(\lambda^1)$, that is μ and ν are orthogonal to all the λ^ρ, $\rho = 2, \ldots, r$. Now choose i such that $\mu_i < \lambda_i$ is true and take $S := \{i\} \cup C^2 \cup \ldots \cup C^r$. Then we have

$$v(S) = \min\{\lambda_i^1, 1, \ldots 1\} = \lambda_i^1 > \mu_i^1 = \mu(S)$$

which contradicts $\mu \in \mathcal{C}(v)$. Hence $\mu = \nu = \lambda^1$.

(3) Again assume that $v \in \mathbb{E}$ allows for a representation $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ with $\lambda^1, \cdots, \lambda^r \in \mathbf{A}_+^1$ **mutually orthogonal**. Consider a second such representation given by probabilities $\mu^1, \ldots, \mu^s$ which are mutually orthogonal as well. Assume w.l.o.g $s \geq r$. Since $v((C^{\lambda^\rho})^c) = 0$, it follows that for every σ there is ρ with $\mu^\sigma((C^{\lambda^\rho})^c) = 0$, i.e., μ^σ is living on the carrier of λ^ρ. It is seen at once that the carrier of μ^σ cannot be strictly smaler than the one of λ^ρ, (v would be 0 on a larger complement), hence $r = s$. The μ^σ are extremes of the core as we have seen above, and the above consideration can be repeated in order to show that $\mu^\sigma = \lambda^\rho$ holds true. Hence the **orthogonal representation is unique**.

In order to proceed with our general study we now turn to some simple conclusions that can be inferred from separation or duality theorems. We choose to base these on the appropriate versions of the 'Theorem of the Alternative' or 'Farkas Lemma'. The first version is

Lemma 1.5 *Let A be a matrix and b a vector (with the appropriate dimensions). Then one and only one of the following statements is true:*

(A) *There exists a vector y satisfying*

$$Ay \leq b.$$

(B) *There exists a vector u satisfying*

$$uA = 0, ub < 0, u \geq 0.$$

In our present context this can be reformulated to yield

Corollary 1.6 *Let $\lambda^1, \cdots \lambda^r, \mu \in \mathbf{A}_+$ and let $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$. Then one and only one of the following statements is true:*

(AA) $\mu \in \mathcal{L}$. There exists $c \in \mathbb{R}^r$, $\sum_{\rho=1}^r c_\rho = 1$, such that

$$\sum_{\rho=1}^r c_\rho \lambda \rho = \mu$$

holds true.

(BB) There exists $x \in \mathbb{R}_+^n, x \neq 0, t \in \mathbb{R}, t > 0$ such that

$$\mu x < t = \lambda^\rho x \quad (\rho = 1, \cdots, r)$$

is satisfied.

Proof: Applying Lemma 1.5 we find that one and only one of the following two statements is true.

Either

(6)
$$\begin{pmatrix} -1, \cdots, -1 \\ \lambda^1, \cdots, \lambda^r \end{pmatrix} y \leq \begin{pmatrix} -1 \\ \mu \end{pmatrix}$$

or else

(7)
$$(t, x) \begin{pmatrix} -1, \cdots, -1 \\ \lambda^1, \cdots, \lambda^r \end{pmatrix} = 0, \ (t, p) \begin{pmatrix} -1 \\ \mu \end{pmatrix} < 0, \ (t, p) \geq 0.$$

This version is rewritten at once to yield either

(8)
$$(\lambda^1, \cdots \lambda^r) y \leq \mu, \ ey \geq 1,$$

or else

(9)
$$x(\lambda^1, \cdots \lambda^r) = te, \ \mu x < t, \ (x, t) \geq 0.$$

These alternatives are indeed equivalent to the ones of Corollary 1.6. E.g., as all the λ^ρ are nonnegative and normalized, equation (8) implies

$$1 = \mu(\Omega) \geq \sum_{\rho=1}^r y_\rho \lambda^\rho(\Omega) = ey \geq 1,$$

hence all the inequalities involved are indeed equations and this shows the first alternative of Corollary 1.6. The remaining equivalences are demonstrated accordingly.
q.e.d.

Corollary 1.6 admits for an obvious interpretation if we accept the notion of *profiles* or *generalized coalitions* Then μ is either an affine combination of the λ^ρ, or else for some diagonal profile x the value $\mu(x)$ is below the common value $\lambda^1 x = \cdots = \lambda^r x$ ($= v(x)$ with obvious interpretation). If x where to be the profile of a coalition or an indicator function, say $x = 1_S$, then 1.6 means that diagonal sets have to be affine combinations of the extreme core elements. Of course we would prefer an improved statement concerning the core. This is obtained by a slightly modified version of the 'Theorem of the Alternative' as follows:

Lemma 1.7 *Let A be a matrix and b a vector (with the appropriate dimensions). Then one and only one of the following statements is true:*

(C) *There exists a vector $y \geq 0$ satisfying*

$$Ay \leq b.$$

(D) *There exists a vector u satisfying*

$$uA \geq 0, ub < 0, u \geq 0.$$

Analogously to the development following Lemma 1.5 we obtain the following

Corollary 1.8 *Let $\lambda^1, \cdots \lambda^r, \mu \in \mathbf{A}_+$. Then one and only one of the following statements is true:*

(CC) $\exists\, \alpha \in \mathbb{R}^r_+,\ \sum_{\rho=1}^r \alpha_\rho = 1,\ \mu = \sum_{\rho=1}^r \alpha_\rho \lambda^\rho = \alpha\lambda.$

(DD) $\exists x \in \mathbb{R}^n_+, x \neq 0,$ *such that*

$$\mu x < \lambda^\rho x \quad (\rho = 1, \cdots, r)$$

is satisfied.

This clearly means that, in terms of profiles, a probability which weakly dominates $\wedge\{\lambda^1, \cdots, \lambda^r\}$ is a convex combination of the λ^ρ. Obviously this yields a kind of clue to completeness: again, if x were to be the profile of a coalition, $x = 1_S$, then μ cannot be extreme in the core unless it is the only measure attaining the value $v(S)$.

Of course this vague remarks do not really establish a precise statement. The traditional way would be to replace x by a rational vector (a continuity argument) and then by an integer vector (since Alternative (DD) is a linear relation). Thereafter the profile can be interpreted as the profile of a large coalition in a suitably large 'replica game' - and this would yield a statement of the type that large replicated games are complete. Essentially, this result is the one discussed in different context

in [6] as well as (for the nonatomic context) in [1]. We shall presently come back to this subject.

To proceed further with our general treatment we now define some affine spaces related to some set $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ or to some $v \in \mathbb{E}$.

To every finite set of measures $\lambda = (\lambda^1, \cdots, \lambda^r) \in (\mathbf{A}_+)^r$ we assign the affine manifold spanned of them and denote it by

$$(10) \qquad \mathcal{L} := \mathcal{L}^\lambda := \left\{ x \in \mathbb{R}^n \mid \exists\, c \in \mathbb{R}^r, \; \sum_{\rho=1}^r c_\rho = 1, \; x = \sum_{\rho=1}^r c_\rho \lambda^\rho = c\lambda \right\}.$$

Also, given $v \in \mathbb{E}$, we denote

$$(11) \qquad \mathcal{X} := \mathcal{X}^v \left\{ x \in \mathbb{R}^n : \mid x(S) = v(S) \; (S \in \underline{\underline{\triangle}}) \right\}.$$

Remark 1.9 *If λ is a representation of v, then clearly we have $\mathcal{L} \subseteq \mathcal{X}$ as well as $\mathcal{C}(v) \subseteq \mathcal{X}$. It would seem to constitute a generalized version of nondegeneracy to require that the λ^ρ span the affine manifold $\mathcal{X}$.*

To follow up this path, we introduce two further spaces which are in close connection to the diagonal $\underline{\underline{\triangle}}$ of some vector of measures $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$. or of some $v \in \mathbb{E}$.

Given $v \in \mathbb{E}$, define

$$(12)\, \mathcal{S} := \mathcal{S}^v := \left\{ s \in \mathbb{R}^n \mid s = \sum_{S \in \underline{\underline{\triangle}}} d_S 1_S \;\; \text{for some set of coefficients } (d_S)_{S \in \underline{\underline{\mathbf{S}}}} \right\}.$$

Also, given $\lambda = (\lambda^1, \cdots, \lambda^r) \in (\mathbf{A}_+)^r$ define

$$(13) \qquad \mathcal{D} := \mathcal{D}^\lambda := \left\{ s \in \mathbb{R}^n \mid \lambda^1 s = \cdots = \lambda^r s \right\}.$$

Remark 1.10 *Clearly, $\mathcal{D}$ is a generalized version of the diagonal (again thinking in terms of profiles). Also, $\mathcal{S}$ is the span of the indicators of diagonal sets. Generally, we have $\mathcal{S} \subseteq \mathcal{D}$ provided λ is representing v. . If $\mathcal{S} = \mathcal{D}$, i.e., if the indicators of the diagonal span the full space $\mathcal{D}$, then there are 'many' diagonal sets which would mean that again some kind of nondegeneracy prevails.*

The fact that these notions are useful is also corroborated by our first

Lemma 1.11 *Let* $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ *and let* $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$. *If* $\mathcal{S} = \mathcal{D}$, *then* $\mathcal{X} \subseteq \mathcal{L}$. *In particular , as* $\mathcal{C}(v) \subseteq \mathcal{X}$ *it follows that* $\mathcal{S} = \mathcal{D}$ *implies* $\mathcal{C}(v) \subseteq \mathcal{L}$, *i.e., the core is contained in the affine span of* $\lambda^1, \cdots, \lambda^r$.

Proof: Let $\mu \in \mathcal{X}$. For any $s \in \mathcal{D}$ there is a set of cefficients $(d_S)_{S \in \underline{\mathbf{S}}}$ such that

$$s = \sum_{S \in \underline{\underline{\Delta}}} d_S 1_S$$

is satisfied. Hence we obtain for all $\rho = 1, \cdots, r$ the following set of equations:

$$
\begin{aligned}
\mu s &= \mu \sum_{S \in \underline{\underline{\Delta}}} d_S 1_S &= \sum_{S \in \underline{\underline{\Delta}}} d_S \mu(S) \\
&= \sum_{S \in \underline{\underline{\Delta}}} d_S \lambda^\rho(S) &= \lambda^\rho(S) \sum_{S \in \underline{\underline{\Delta}}} d_S 1_S \\
&= \lambda^\rho s.
\end{aligned}
$$

That is, alternative **(BB)** of Corollary 1.6 does not occur and therefore alternative **(AA)** , i.e., $\mu \in \mathcal{L}$ is the true one. Thus we have shown that $\mathcal{X} \subseteq \mathcal{L}$ is satisfied.

However, a slightly more comprehensive statement can be directly drawn from some rank considerations. To this end, given $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ we denote by Λ the matrix with rows $\lambda^1, \cdots \lambda^r$. Also, let us denote by E the matrix the rows of which are the indicators of all coalitions $S \in \underline{\underline{\Delta}}$. Thus we define

$$(14) \qquad \Lambda := \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^r \end{pmatrix} \quad E := (1_S)_{S \in \underline{\underline{\Delta}}}$$

Then we have :

Theorem 1.12 *Let* $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ *and let* $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ *be represented by* λ. *Then the following statements are equivalent.*

(1) $\mathcal{L} = \mathcal{X}$.

(2) $\mathcal{S} = \mathcal{D}$

(3) $(n+1) - \operatorname{rank} \Lambda = \operatorname{rank} E$.

Proof:

1st STEP : For $S \in \underline{\underline{\Delta}}$ and $t = v(S) = \lambda^1(S)$ we have clearly

$$\sum_{i \in S} \lambda_i = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}$$

from which it follows that

$$(15) \qquad rank\ \Lambda\ =\ rank\ \left(\Lambda \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \right).$$

Analogously, in view of the equation

$$\lambda^1(S) = \sum_{i \in S} 1_S(i)\lambda_i^1 = v(S)$$

which holds true for every $S \in \mathcal{S}$, we derive the equation

$$rank\ E\ =\ rank\ (1_S)_{S \in \underline{\underline{\Delta}}}\ =\ rank\ (1_S, v(S))_{S \in \underline{\underline{\Delta}}}\ =\ rank\ (E, v(\cdot)).$$

This we are going to use as follows.

2ndSTEP : The spaces

$$\mathcal{D} = \left\{ x \in \mathbb{R}^n \mid \lambda^1 x = \cdots = \lambda^r x \right\}$$

and

$$\begin{aligned} \mathcal{D}' : &=\ \{(x,t) \in \mathbb{R}^{n+1} \mid \lambda^1 x - t = 0, \cdots, \lambda^r x - t = 0\} \\ &=\ \left\{ (x,t) \in \mathbb{R}^{n+1} \mid \left(\Lambda \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \right)(x,t) = 0 \right\} \end{aligned}$$

have the same dimension, since the mapping $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $x \to (x, \lambda^1 x)$ throws $\mathcal{D}$ bijectively onto $\mathcal{D}'$. Therefore the dimensions are given by

$$\begin{aligned} dim\ \mathcal{D}\ &=\ dim\ \mathcal{D}' \\ (16) \qquad &=\ (n+1) - rank\ \left(\Lambda \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \right) \\ &=\ (n+1) - rank\ \Lambda. \end{aligned}$$

Now the indicators $(1_S)_{S \in \underline{\underline{\Delta}}}$ will spann the space $\mathcal{D}$ (i.e. $\mathcal{S} = \mathcal{D}$ holds true) if and only if the vectors $(1_S, \lambda^1(S))_{S \in \underline{\underline{\Delta}}}$ span $\mathcal{D}'$ and this is the case if and only if we have

$$(17) \qquad rank\ E\ =\ rank\ (E, v(\cdot))\ =\ (n+1) -\ rank\ \Lambda.$$

3rdSTEP :

3rd STEP :

Similarly, consider the space

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid x(S) = v(S)\}$$
$$= \{x \in \mathbb{R}^n \mid Ex = v(\cdot)\}$$

This time the vectors λ^ρ will span the affine manifold $\mathcal{X}$ ('affinely') if and only if the dimension of $\mathcal{X}$ satisfies $dim\ \mathcal{X} = rank\Lambda - 1$; however this dimension is obviously given by

$$dim\ \mathcal{X} = n - rank\ E.$$

That is, the vectors λ^ρ span $\mathcal{X}$ if and only if $n - rank\ E = rank\ \Lambda - 1$ or

(18) $$(n+1) - rank\ \Lambda = rank\ E$$

holds true. Now comparing equations (17) and (18) we obtain the desired result. **q.e.d.**

We believe that the conditions of Theorem 1.12 are a constituting a version of non-degeneracy. This is supported by a study of the case $r = 1$. In this situation λ^1 spans the space $\mathcal{X}$ which means λ^1 is the only solution of the linear system (3) required by Definition 0.1 w.r.t $\underline{\underline{\Delta}}$. And simultaneously, the indicators $(1_S)_{S \in \underline{\underline{\Delta}}}$ must spann the full space, which means that the coefficient matrix of the linear system (3) of Definition 0.1 is nonsingular. This motivates the following definition.

Definition 1.13 *A set of measures* $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ *is said to be* **weakly non degenerate** *(weakly n.d.) if either one of the conditions of Theorem 1.12 is satisfied.*

As a first simple consequence we note:

Theorem 1.14 *Let* $v \in \mathbb{E}$. *Then the extremepoints of the core are weakly n.d.*

Proof: We are going to show that the extremepoints of the core, say $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$, span the space $\mathcal{X}$. Now let $\bar{x} \in \mathcal{X}$. Define $\bar{\mu} := \frac{1}{r} \sum_{\rho=1}^r \lambda^\rho > 0$. We claim that, for small positive ϵ

$$\mu^\epsilon := (1 - \epsilon)\bar{\mu} + \epsilon\bar{x} \in \mathcal{C}(v).$$

Indeed, for any $S \in \underline{\underline{\Delta}}$ we have $\mu^\epsilon(S) = v(S)$ as $\bar{\mu}, \bar{x} \in \mathcal{X}$. And for any S not in the diagonal there is at least one λ^σ such that $\lambda^\sigma(S) > v(S)$, hence we have $\bar{\mu}(S) > v(S)$ and $\mu^\epsilon(S) > v(S)$. So $\mu^\epsilon \in \mathcal{C}(v)$ is indeed true.

Since the λ^ρ constitute the extremes of the core, we find that μ^c is a convex combination of the λ^ρ. But then $\bar{x}$ is a linear combination of the λ^ρ, **q.e.d.**

Thus the extremepoints of the core allways span the corresponding space $\mathcal{L}$. The main question is, whether ther is a canonical representation of a balanced game by means of a well specified set $\lambda^1, \cdots, \lambda^r$. This is the case if v is complete and Theorem 1.15 provides a clue to completeness: it can be said that large orthogonal games are complete. A first and easy step into this direction is provided by the following theorem.

Theorem 1.15 *Let* $\lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+$ *be orthogonal and weakly n.d. Then* $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ *is complete and* $\lambda^1, \cdots, \lambda^r$ *constitute exactly the extremepoints of the core of* v.

Proof: The $\lambda^1, \cdots, \lambda^r$ are core elements of v. If μ happens to be any further core element then, because of weak non degeneracy, there are affine coefficients c_ρ, $r = 1, \cdots, r$, $\sum_{\rho=1}^r c_\rho = 1$, such that $\mu = \sum_{\rho=1}^r c_\rho \lambda^\rho$ holds true. Since the λ^ρ are orthogonal and nonnegative and μ is nonnegative, it is seen at once that the coefficients c_ρ have to be *nonnegative* as well. This means that μ is indeed a *convex* combination of the λ^ρ. Again since the λ^ρ are orthogonal they have to be extremepoints of the core and there are no further extreme points (cf. Remark 1.4). Now, to every ρ the set $S := (C^\rho)^c$ has exactly the property that $v(S) = 0 = \lambda^\rho(S) < 1 = \lambda^\sigma(S)$ $(\sigma \neq \rho)$, which is required by Definition 1.1, **q.e.d.**

Another simple observation is

Remark 1.16 *If* $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ *is weakly n.d., then equal treatment prevails. Indeed, as* $\mathcal{C}(v) \subseteq \mathcal{X} = \mathcal{L}$, *we know that any element of the core satisfies* $\mu = \sum_{\rho=1,\ldots,r}^r c_\rho$ *with suitable 'affine' coefficients. If two players have the same* λ_i^ρ *then they have the same* μ_i.

So far we have more or less exploited duality theorems. N.D. Theory however strives to imitate the nonatomic result not by replica objects but by describing the distributions of players over the types such that a 'limit theorem' or 'equivalence theorem' holds true.This is done by exploiting the linear relations with the aid of combinatorial methods. For the nondegenerate version the discussion is found in [11]. For the space $I\!\!E$ we shall present a formulation which is not bound to a replica version but also not as comprehensive as [11]. To this end we first introduce 'types'

Remark 1.17. *When assigning 'types' to the players we have to leave the range of probabilities as the representing measures are concerned. Instead, we assume that the measures are integer valued.*

To this end, let $g^1, \ldots, g^T$ be positive integer valued vectors, i.e., elements of $I\!\!N^r$. Also let $k_1, \ldots k_T \in I\!\!N$ be integers such that $\sum_{\tau=1}^T k_\tau = n$ holds true. Fix a decomposition

of N into T disjoint sets $K_1, \ldots K_T$ (the sets of 'types') with $N = \sum_{\tau=1}^{T} K_\tau$. Now define $\lambda^\rho \in \mathbf{A}^+$ by

$$(19) \qquad \lambda^\rho(\bullet) = \sum_{\tau=1}^{T} \mid K_\tau \cap \bullet \mid g_\tau^\rho.$$

We always assume that $\lambda^1(N) = \ldots \lambda^r(N)$ so as to have $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\} \in \mathbb{E}$. In this context we shall say that $\lambda = (\lambda^1, \cdots, \lambda^r) \in \mathbf{A}^{+r}$ is **type patterned**. Also, for any $S \in \underline{\mathbf{P}}$ we call the vector $s = (|S \cap K_1|, \ldots, |S \cap K_T|)$ the **profile** of S.

A first consequence of this change of setup is presented by the following simple observation.

Remark 1.18 Let $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\} \in \mathbb{E}$ such that λ is type patterned. Assume that there is a diagonal set S which properly cuts into each type, i.e.,

$$(20) \qquad \exists S \in \underline{\underline{\Delta}} : \emptyset \neq S \cap K_\tau \neq N \quad (\tau = 1, \ldots, T).$$

Then equal treatment prevails in the core.

Indeed, let $\bar{x} \in \mathcal{C}(v)$ and choose $S \in \underline{\underline{\Delta}}$ satisfying (20) such that S contains of each type the players that are 'best off' at $\bar{x}$. Then $\bar{x}(S) = v(S)$ as S is diagonal. Choose S' to have the same profile as S hence satisfying (20) as well but containing the players worst off of each type. Since S and S' have the same profile, we have $\bar{x}(S') = v(S) = x(S)$ but if equal treatment does not prevail we must have $\bar{x}(S) > \bar{x}(S')$, which cannot happen.

The following is the well known equivalence theorem for L.-P.-games slightly disguised and improved with respect to the quantifiers. It is traditional (cf. [6]) and we provide the present version in order to contrast it with the final result of this section, i.e. Theorem 1.22 .

Theorem 1.19 Let $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ with λ integer valued, orthogonal, and type patterned (cf. Remark 1.17), i.e., $\lambda^\rho(\bullet) = \sum_{\tau=1}^{T} \mid K_\tau \cap \bullet \mid g_\tau^\rho$. Then there are constants and $\hat{k}_\tau$ $(\tau = 1, \ldots, T)$ **depending on** $g_{\bullet}^{\bullet}$ **only** with the following property: whenever $k_\tau \geq \hat{k}_\tau (\tau = 1, \ldots, T)$, then $\lambda^1, \cdots, \lambda^r$ are the extremepoints of the core of v. i.e., the representation satisfies the equivalence theorem.

Proof: First of all choose $\hat{k}_\tau$ such that equal treatment is ensured (e.g. by choosing every one of them to be even). Next consider $v^0 := \bigwedge\{g^1, \ldots, g^r\}$ and let $H = \{h^1, \ldots, h^s\}$ denote the sets of extremepoints of the core of v^0 different from the g^ρ.

We may assume that $H \neq \emptyset$, for every μ in the core of v, with $\mu \neq \lambda^\rho$ for all ρ, implies an element in H in view of equal treatment.

To every $h = h^\sigma \in H$ choose $\hat{x} = \hat{x}^\sigma$ with $h\hat{x} \leq g^\rho \hat{x}$ according to Corollary 1.8 $(\rho = 1, \ldots, r)$. Next choose $\bar{x} = \bar{x}^\sigma$ **rational** close to $\hat{x}$ such that $h\bar{x} \leq g^\rho \bar{x}$ is true for $(\rho = 1, \ldots, r)$. Multiplying by the common denominator yields an integer vector $\bar{s} = bars^\sigma$ which finally yields

$$(21) \qquad \sum_{\tau=1}^{T} h_\tau s_\tau \leq \sum_{\tau=1}^{T} g_\tau^\rho \bar{s}_\tau$$

again for $(\rho = 1, \ldots, r)$. All these quantities vary with the elements in H, i.e., they depend on σ. Since we have finitely many h^σ, we may now choose $\hat{k}_\tau$ exceeding all $\bar{s}_\tau = \bar{s}_\tau^\sigma$ for $\sigma = 1, \ldots s$. Next, defining $\mu^\sigma := \sum_{\tau=1}^{T} h_\tau^\sigma |K_\tau \cap \bullet|$ for $\sigma = 1, \ldots, s$, we see at once that (21) reads

$$(22) \qquad \mu^\sigma(\bar{S}) \leq \lambda^\rho(\bar{S}) \quad \sigma = 1, \ldots, s, \quad \rho = 1, \ldots, r.$$

Now return to the study of $v = \wedge \{\lambda^1, \cdots, \lambda^r\}$. If μ is extreme in $\mathcal{C}(v)$ and *not* one of the $\lambda^1, \ldots, \lambda^\rho$, then it is not a convex combination of the $\lambda^1, \cdots, \lambda^r$. Because of equal treatment, μ has the form $\mu := \sum_{\tau=1}^{T} h_\tau |K_\tau \cap \bullet|$ with suitable h in the core of v^0 and it is easily seen that h is no convex combination of the g^ρ constituting v^0. As μ is extreme, so is h, that is $h \in H$. Hence there is some $\bar{S}$ satisfying (22), a contradiction to the fact that $\mu \in \mathcal{C}(v)$,   **q.e.d.**

For the remainder of this section we focus on the orthogonal case. We will come up with an improvement of the last theorem within this context.

Remark 1.20 *Given a set of measures $\lambda^1, \cdots, \lambda^r$ with carriers $C^\rho = C(\lambda^\rho)$ $(\rho = 1, \ldots, r)$ and the resulting diagonal $\underline{\triangle}$, let us define the systems*

$$(23) \qquad \underline{\triangle}^\rho := \{S \cap C^\rho \mid S \in \underline{\triangle}\} \quad (\rho = 1, \ldots, r)$$

If an orthogonal set of measures $\lambda^1, \cdots, \lambda^r$ is weakly n.d., then each λ^ρ is n.d. with respect to $\underline{\triangle}^\rho$ $(\rho = 1, \ldots, r)$. This definition refers to Definition 0.1, the relevant system of linear equations in variables $(x_i)_{i \in C^\rho}$ can also be written as

$$(24) \qquad \sum_{i \in S, i \in C^\rho} x_i = \lambda^\rho(S) = v(S), \quad (S \in \underline{\triangle}).$$

Indeed, if for some ρ the system (24) has at least two different solutions, say λ^ρ and μ^ρ, then because of orthogonality there are at least $r + 1$ (affinely independent) elements $\lambda^1, \cdots, \lambda^\rho, \mu^\rho, \cdots, \lambda^r$ which will render the dimension of $\mathcal{X}$ to be

$$\dim \mathcal{X} = r > r - 1 = \dim \mathcal{L},$$

meaning that the $\lambda^1, \cdots, \lambda^r$ are not weakly n.d.

Generally the opposite direction is not necessarily true: if each λ^ρ considered as a measure living on C^ρ is n.d. with respect to the the system $\underline{\underline{\Delta}}^\rho$ $(\rho = 1, \ldots, r)$ then it does *not* necessarily follow that $\lambda^1, \cdots, \lambda^r$ is weakly nondegenerate. However, if we require slightly more,then we can indeed state a converse relation and this has important consequences.

Theorem 1.21 *Let $\lambda^1, \cdots, \lambda^r$ be orthogonal,$v = \lambda^1, \cdots, \lambda^r \in \mathbf{A}_+^1$, and let δ be a value of the diagonal, i.e., there is $S \in \underline{\underline{\Delta}}$ such that $\delta = v(S) = \lambda^\rho(S)$ for all $\rho = 1, \cdots, r$. Assume that each λ^ρ on C^ρ is n.d. w.r.t. δ (i.e. w.r.t. $\underline{\underline{\mathbf{Q}}}_\delta = \underline{\underline{\mathbf{Q}}}_\delta^\rho$, see Definition (0.1) of Section 0 - this is slightly more than to ask for n.d. w.r.t. the system (24)). Then $\lambda^1, \cdots, \lambda^r$ is weakly nondegenerate. Consequently, by Theorem 1.15 ,v is complete and the $\lambda^1, \cdots, \lambda^r$ constitute all the extremepoints. Besides, all the λ^ρ are rational (hence can be renormalized to be integer-valued).*

Proof: 1stSTEP : The rank of each matrix

$$(25) \qquad\qquad (1_S)_{S \subseteq C^\rho, \lambda^\rho(S)=\delta} = (1_S)_{S \in \underline{\underline{\mathbf{Q}}}_\delta^\rho}$$

is $|C^\rho|$, this is actually the meaning of the original version of n.d - or a special (trivial) version of Theorem 1.12.

2ndSTEP : The rank of Λ on the other hand is clearly r since the λ^ρ are assumed to be orthogonal. According to Theorem 1.12 it suffices, therefore, to show that *rank* $E = (n+1) - r$ holds true. Here E is the matrix of indicators of the diagonal sets (in all of N.)

3rdSTEP : Now E contains (among others) at least those rows which are obtained by combining all the indicators of some $S \in \underline{\underline{\mathbf{Q}}}_\delta^\rho$ for all $\rho = 1, \cdots, r$. That is, the matrix indicated by

$$(26) \qquad \begin{pmatrix} 1_{S^1} & 1_{S^2} & \cdots & 1_{S^r} \\ 1_{S^1} & 1_{S^2} & \cdots & 1_{S^{r\cdot}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & 1_{S^{r\cdot\cdots\cdot}} \\ 1_{S^1} & 1_{S^{2\cdot}} & \cdots & 1_{S^r} \\ 1_{S^1} & 1_{S^{2\cdot}} & \cdots & 1_{S^{r\cdot}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & 1_{S^{r\cdot\cdots\cdot}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1_{S^{1\cdot\cdots\cdot}} & 1_{S^{2\cdot\cdots\cdot}} & \cdots & 1_{S^{r\cdot\cdots\cdot}} \end{pmatrix}$$

is a submatrix of E. We now have to convince ouselves that the matrix indicated by (26) has indeed rank $n - (r-1)$.

4$^{\text{th}}$STEP : E.g., if a $k \times k$-matrix

$$A = \begin{pmatrix} a^1 \\ \vdots \\ a^k \end{pmatrix}$$

of rank k and an $l \times l$-matrix

$$B = \begin{pmatrix} b^1 \\ \vdots \\ b^l \end{pmatrix}$$

are combined in order to construct a matrix

$$C = \begin{pmatrix} a^1 & b^1 \\ \vdots & \vdots \\ a^1 & b^l \\ a^2 & b^1 \\ \vdots & \vdots \\ a^2 & b^l \\ \vdots & \vdots \\ a^k & b^1 \\ \vdots & \vdots \\ a^k & b^l \end{pmatrix}$$

with $n = k + l$ columns, then C is easily seen to have rank $k + l - 1 = n - 1$, and this is generalized at once.

As for the final comment (the λ^ρ are rational, note that this is a consequence of n.d. as stated in [14],   **q.e.d.**

As a final consequence we are now in the position to state

Theorem 1.22 *(Large orthogonal games are n.d.,complete, and uniquely represented.)*

Let $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ *with* λ *integer valued, orthogonal, and type patterned (cf. Remark 1.17), i.e.,* $\lambda^\rho(\bullet) = \sum_{\tau=1}^{T} \mid K_\tau \cap \bullet \mid g_\tau^\rho$. *Then there are nice constants* R *and* l_τ *(*$\tau = 1, \ldots, T$*)* **depending on** $g_\bullet^\bullet$ **only** *with the following property: if* $k_\tau \geq l_\tau$ *(*$\tau = 1, \ldots, T$*) and* $\lambda^\rho(N) \geq R$ *(*$\rho = 1, \ldots, r$*), then* v *is complete and the* λ^ρ *constitute exactly the extremepoints of* $C(v)$.

Proof: Denote the greatest common divisor of $g_1^\rho, \ldots, g_T^\rho$ by g.c.d. g^ρ and let $D = \prod_{\rho=1}^{r}$ g.c.d. g^ρ. Choose the quantities R^ρ and l_τ^ρ as defined by g^ρ via Theorem 3.5 of [14] such that for $k_\tau^\rho \geq l_\tau^\rho$, $\lambda^\rho(N) - R^\rho \geq s \geq R^\rho$ and s within the ideal spanned by g^ρ (i.e. s a multiple of g.c.d. g^ρ), it follows that

(27) $\qquad$ λ^ρ is n.d. w.r.t. $\underline{\underline{\mathbf{Q}}}_s = \{S \in \underline{\underline{\mathbf{P}}} \mid \lambda^\rho(S) = s\}$

holds true. Next define

$$(28) \qquad C: = 1 + \max_{\rho=1,\dots,r} \left[\frac{R^\rho}{D}\right],$$

CD will play the rôle of δ in Theorem 1.21, i.e., we will show that each l^ρ is n.d. w.r.t. δ. Indeed, we have for all ρ

$$
\begin{aligned}
CD &= D + D \max_{\sigma=1,\dots,r} \left[\frac{R^\sigma}{D}\right] \\
&\geq D + D \left[\frac{R^\rho}{D}\right] \\
&\geq R^\rho
\end{aligned}
$$
(29)

as well as

$$
\begin{aligned}
\lambda\rho(N) &\geq D + R^\sigma + R^\rho \\
&\geq D + \left[\frac{R^\sigma}{D}\right] D + R^\rho \\
&= \left(1 + \left[\frac{R^\sigma}{D}\right]\right) D + R^\rho
\end{aligned}
$$
(30)

for all σ, meaning

$$(31) \qquad \lambda^\rho(N) \geq \max_{\sigma=1,\dots,r} \left(1 + \left[\frac{R^\sigma}{D}\right]\right) D + R^\rho = CD + R^\rho.$$

Writing (29) and (31) in one line we have

$$(32) \qquad \lambda^\rho(N) - R^\rho \geq CD \geq R^\rho$$

while clearly CD is a multiple of each g.c.d. g^ρ. These are just the conditions to apply Theorem 3.5. of [14] which says that now indeed *each* λ^ρ is n.d. w.r.t. $\delta := CD$. The present Theorem therefore follows now from Theorem 1.21, $\qquad$ **q.e.d.**

2   Extreme Games.

The set $I\!\!E$ is a closed, convex, and polyhedral cone when seen, say, as a subset of $I\!\!R^{2^n-1}$. This is a consequence of an appropriate modification of **SHAPLEY'S** proof in [16]. We shortly discuss a class of extremal rays of this cone that can be obtained by combinatorial methods quite analogously to those employed in the previous section.

Remark 2.1 *Given a set of measures* $\lambda^1, \cdots, \lambda^r$ *with carriers* $C^\rho = C(\lambda^\rho)$ $(\rho = 1, \ldots, r)$, *we may consider systems of 'subdiagonals' via*

$$(33) \quad \underline{\underline{\triangle}}{}^\rho_\sigma \; = \; \{S \cap C^\rho \mid S \in \underline{\mathbf{P}}, \; \lambda^\rho(S) = \lambda^\sigma(S) < \lambda^\tau(S)(\tau \neq \rho, \sigma)\} \quad (\sigma \neq \rho)$$

We also want to introduce

$$(34) \qquad\qquad \underline{\underline{\triangle}}{}_\rho := \bigcup_{\sigma \neq \rho} \underline{\underline{\triangle}}{}^\rho_\sigma,$$

which in view of the following results will be called the system **defining** λ^ρ.

Theorem 2.2 *Let* $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\} \in I\!\!E$ *and assume the* λ^ρ *to be mutually orthogonal. If* v *is* **extremal** *in* $I\!\!E$, *then every* λ^ρ *is nondegenerate with respect to* $\underline{\underline{\triangle}}{}_\rho$.

Proof: Assume that v is extremal in $I\!\!E$. Suppose that for some ρ there is a solution $\bar{x} \neq \lambda^\rho$. Then, for real ϵ define

$$(35) \qquad\qquad \lambda^{\rho,\epsilon} = (1 - \epsilon)\lambda^\rho + \epsilon\bar{x}.$$

Now for small ϵ clearly $\lambda^{\rho,\epsilon}$ is nonnegative since $\lambda^\rho_i > 0$ $(i \in C^\rho)$ holds true; moreover $\lambda^{\rho,\epsilon}(N) = 1$ is obviously true.

Hence, if we define

$$(36) \qquad\qquad v^\epsilon := \bigwedge\left\{\lambda^1, \cdots, \lambda^{\rho,\epsilon}, \cdots, \lambda^r\right\},$$

then, for sufficiently small ϵ, we find that $v^\epsilon \in I\!\!E$ holds true. Note that $\lambda^{\rho,\epsilon} \neq \lambda^\rho$ follows immediately from the fact that $\bar{x}$ differs from λ^ρ while the argument that $v^\epsilon \neq v$ uses orthogonality: one has to consider a set $S^\rho \subseteq C^\rho$, $S^\rho \neq C^\rho$ on which $\lambda^{\rho,\epsilon}(S^\rho) \neq \lambda^\rho(S^\rho)$ and construct $S := C^1 \cup \ldots \cup S^\rho \cup \ldots \cup C^r$ such that $v^\epsilon(S) \neq v(S)$ holds true. Next observe the equation

$$(37) \qquad\qquad \frac{\lambda^{\rho,\epsilon} + \lambda^{\rho,-\epsilon}}{2} = \lambda^\rho.$$

Now, whenever $\lambda^\rho(S) < \lambda^\sigma(S)$ holds true for some $S \in \underline{P}$ and all $\sigma \neq \rho$, then $\lambda^{\rho,\epsilon}, \lambda^{\rho,-\epsilon} < \lambda^(S)$ for all σ provided ϵ is sufficiently small. From this observation and (37) it follows that indeed

$$\frac{v^\epsilon + v^{-\epsilon}}{2} = v$$

holds true and this shows that v is not extremal in $\mathbb{E}$.   **q.e.d.**

As for the converse direction we have to require that v is complete in order to prove the following theorem. Also we initially want to slightly modify the version of non-degeneracy as given in definition (0.1) of Section 0. To this end we restrict ourselves for the moment to probabilities. Instead of requiring an additive m to be the unique solution of (3) in Section 0, we want a slightly more general version given by

Definition 2.3 *A probability m is* **strongly non-degenerate** *w.r.t.* $\underline{\underline{S}}$ *if the system of linear equations in variables* $(x_i)_{i \in \Omega}, (\alpha_S)_{S \in \underline{\underline{S}}}$

$$
\begin{aligned}
\sum_{i \in S} x_i - \alpha_S &= 0 \quad (S \in \underline{\underline{S}}) \\
\sum_{i \in \Omega} x_i &= 1 \quad (S \in \underline{\underline{S}})
\end{aligned}
$$
(38)

admits of the unique solution $m, (m(S))_{S \in \underline{\underline{S}}}$ *only.*

Theorem 2.4 *Let* $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\} \in \mathbb{E}^1$ *be complete. If every* λ^ρ *is strongly nondegenerate with respect to* $\underline{\underline{\Delta}}^\rho$ *then* v *is* **extremal** *in* $\mathbb{E}^1$.

Proof:

1ˢᵗSTEP :

Assume v to be a convex combination of two elements of $\mathbb{E}^1$, say

$$v = \frac{v^1 + v^2}{2},$$
(39)

we have to prove that $v^1 = v^2 = v$ holds true.

To this end assume that both the v^k are represented as members of $\mathbb{E}$, say

$$v^k = \bigwedge \left\{ \lambda^{k,1}, \cdots, \lambda^{k,r_k} \right\}.$$

Now it is seen at once that

$$\lambda^{\sigma,\tau} := \frac{\lambda^{1,\sigma} + \lambda^{2,\tau}}{2} \in \mathcal{C}(v)$$

holds true. Moreover, for every S there has to be a pair σ, τ such that

$$v(S) = \frac{v^1(S) + v^2(S)}{2} = \frac{\lambda^{1,\sigma}(S) + \lambda^{2,\tau}(S)}{2} = \lambda^{\sigma,\tau}(S)$$

is satisfied. The consequence is clearly that we a given a representation of v via

(40)
$$v = \bigwedge_{\sigma,\tau} \lambda^{\sigma,\tau}.$$

2$^{\text{nd}}$STEP : Because of the completeness we have required we can now apply Theorem 1.2. Accordingly, we know that for every ρ there is a pair σ, τ such that

(41)
$$\lambda^\rho = \lambda^{\sigma,\tau} = \frac{\lambda^{1,\sigma} + \lambda^{2,\tau}}{2}$$

holds true.

Now, whenever $\lambda_i^\rho = 0$ is the case then necessarily $\lambda_i^{1,\sigma} = \lambda_i^{2,\tau} = 0$ is a consequence as all quantities involved are nonnegative.

Next, observe that the diagonal of v is contained in the diagonals of both v^1 and v^2; for if $S \in \underline{\underline{\Delta}}$ the $v(S) + v(S^c) = 1$, hence $v^1(S) + v^1(S^c) = 1$ for otherwise v^2 would yield an inequality contradicting superadditivity. Therefore, the pair $\lambda^1, v^1(S)|_{S \in \underline{\underline{\Delta}}}$ constitutes a solution of the linear system of equations

(42)
$$\begin{aligned} \sum_{i \in S} x_i - \alpha_S &= 0 \quad (S \in \underline{\underline{\Delta}}) \\ \sum_{i \in \Omega} &= 1 \\ x_i &= 0 \quad (i \notin C^\rho). \end{aligned}$$

However, by the assumption of strong nondegeneracy, λ^ρ and the corresponding values of v are the only solution of this system, hence we conclude that in particular $\lambda^{1,\sigma} = \lambda^\rho$ and analogously $\lambda^{2,\tau} = \lambda^\rho$ is indeed the case. (Already at this stage it follows also that $v^1(S) = v^2(S) = v(S)$ is true for $S \in \underline{\underline{\Delta}}$ - but we want this relation to be true for all $S \in \underline{\underline{\Delta}}$.)

3$^{\text{rd}}$STEP : Thus we see that all the λ^ρ involved in the representation of v appear also in the representation of v^1 as well as in the representation of v^2 . As a consequence v^1 and v^2 are dominated by v (we take minima !). But strict domination of either one of them would result in strict domination of the convex combination, which is v - and this cannot occur. Thus we finally come up with $v = v^1 = v^2$, meaning that v is indeed extreme in $\mathbb{E}$; q.e.d. .

Remark 2.5 *Some simple facts concerning the various types of nondegeneracy are now being collected.*

(1) *To require (strong) nondegeneracy w.r.t.* $\underset{\rho}{\triangle}$ *is less than to require it w.r.t.* $\triangle^{\rho}$ *since the former system is larger.*

(2) *Nondegeneracy w.r.t. some value δ of a measure (see Definition 0.1 in Section 0) is the same as strong nondegeneracy.*

(3) *The relation towards weak n.d. has been pointed out in Remark 1.20 .*

Similarly to the procedure in Section 1 and in view of the n.d. results obtained in [14] we may now construct large extreme games of $I\!\!E$ as follows:

Theorem 2.6 *Let $\lambda^1, \cdots, \lambda^r$ be orthogonal, $v = \lambda^1, \cdots, \lambda^r \in \mathbf{A}^1_+ \in I\!\!E^1$, and let δ be a value of the diagonal, i.e., there is $S \in \triangle$ such that $\delta = v(S) = \lambda^{\rho}(S)$ for all $\rho = 1, \cdots, r$. Assume that each λ^{ρ} on C^{ρ} is n.d. w.r.t. δ. Then v is extremal in $I\!\!E^1$. Also, all the λ^{ρ} are rational (hence can be renormalized to be integer-valued).*

Proof:

This follows clearly from Remark 2.5 and Theorem 2.4 For, since we have a set of pairwise orthogonal measures, each λ^{ρ} beeing nondegenerate w.r.t. a value on the diagonal, we know that each λ^{ρ} is (strongly) n.d. w.r.t. $\triangle^{\rho}$, that is, the n.d. condition of Theorem 2.4 holds true. Completeness follows from Theorem 1.21, hence all conditions of Theorem 2.4 are ensured.   **q.e.d.**

Finally we have

Theorem 2.7 *(Large orthogonal games are extremal.)*

*Let $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ with λ integer valued, orthogonal, and type patterned (cf. Remark 1.17), i.e., $\lambda^{\rho}(\bullet) = \sum_{\tau=1}^{T} |K_{\tau} \cap \bullet| g_{\tau}^{\rho}$. Then there are nice constants R and l_{τ} $(\tau = 1, \ldots, T)$ **depending on** $g_{\bullet}^{\bullet}$ **only** with the following property: if $k_{\tau} \geq l_{\tau}$ $(\tau = 1, \ldots, T)$ and $\lambda^{\rho}(N) \geq R$ $(\rho = 1, \ldots, r)$, then v constitutes an extremal ray in $I\!\!E$.*

The **Proof** of course runs quite analogously to the one of Theorem 1.22.

3 The Core is stable.

Again we consider a totally balanced game represented as

$$v = \bigwedge \left\{ \lambda^1, \cdots, \lambda^r \right\}$$

which during this section is assumed to be normalized and *orthogonal*. The (mutually disjoint) carriers of the measures λ^ρ will be denoted by C^ρ $(\rho = 1, \cdots, r)$. In what follows ξ will denote an imputation of v.

Definition 3.1 *(1) Let $S \subseteq C^\rho, S \neq \emptyset$. Then*

$$\underline{\alpha}_S := \min_{i \in S} \frac{\xi_i}{\lambda_i^\rho}$$

*is the **minimal rate** of S; similarly*

$$\bar{\alpha}_S := \max_{i \in S} \frac{\xi_i}{\lambda_i^\rho}$$

*is the **maximal rate**. A player $k \in S$ having the property that*

$$\underline{\alpha}_S := \frac{\xi_k}{\lambda_k^\rho}$$

*is satisfied will be called a player **worst off** in S since his share at ξ compared to his weight is minimal. Similarly there are players **best off** in S.*

(2) Let $\prec$ be an ordering on a subset T of Ω. For disjoint sets $S, S' \subseteq T$ we write $S \prec S'$ if and only if $i \prec i'$ holds true for all $i \in S, i' \in S'$.

*(3) Let $\underline{\mathbf{S}} \subseteq \underline{\mathbf{P}}$ be a decomposition of Ω. For every $S \in \underline{\mathbf{S}}$ we abbreviate $S^\rho := S \cap C^\rho$ (the **partners** constituting S. Also we use $\underline{\mathbf{S}}^\rho := \{S \cap C^\rho \mid S \in \underline{\mathbf{S}}\}$ in order to denote the decomposition **induced** on C^ρ. If it so happens that $\underline{\mathbf{S}} \subseteq \underline{\triangle}$ (this is the case we are interested in), then clearly $\sum_{\rho=1}^r S^\rho = S$ and $\lambda^1(S^1) = \cdots \lambda^\rho(S^\rho)$.*

*(4) Now consider a binary relation $\prec$ on Ω such that every C^ρ is ordered as well as a decomposition $\underline{\mathbf{S}}$. We shall say that $\underline{\mathbf{S}}$ is **well ordered by** $\prec$ if every $\underline{\mathbf{S}}^\rho$ is ordered and this is done in a consistent manner, i.e., if it is true that for $S, T \in \underline{\mathbf{S}}$ the relation $S^\rho \prec T^\rho$ or the relation $T^\rho \prec S^\rho$ is simultaneously satisfied for all ρ - in which case we shall of course write $S \prec T$ or $T \prec S$. Thus, while $\prec$ orders within each C^ρ only, the induced ordering acting on the C^ρ - elements of the decomposition can globally be viewed as an ordering of the decomposition as such.*

Eventually the ordering we have in mind will be closely related to the relative share of the players w.r.t. an imputation ξ of v. Our first attempt to link both concepts is as follows.

Definition 3.2 *Let ξ be an imputation of v. A decomposition $\underline{S} \subseteq \underline{\Delta}$ of Ω is said to be ξ - **consistent** if there exists a binary relation $\prec$ on Ω such that $\underline{S}$ is well ordered and in addition the following conditions are satisfied.*

(1) *For every $S^\rho \in \underline{S}^\rho$ there is $\underline{S}^\rho \subseteq C^\rho$ with the following properties:*

    (a) *$i \preceq j$ whenever $i \in \underline{S}^\rho, j \in S^\rho$ is the case.*

    (b) *$\bar{\alpha}_{\underline{S}^\rho} \leq \underline{\alpha}_{S^\rho}$*

    (c) *$\lambda^\rho(\underline{S}^\rho) = \lambda^\rho(S^\rho)$*

(2) *For every $S \in \underline{S}$ **having a predecessor** there exists a ρ such that $\underline{S}^\rho \prec S^\rho$ holds true.*

Thus, if we look at a member S^ρ of the (induced) decomposition, then there is a coalition preceeding it (strictly, up to possibly one common member) which has the same total weight. The members of this preceeding coalition all are worse off than the members of the original coalition. And at least in the territory of one λ^ρ the precedence relation is strict with the possible exemption of the first elements of $\underline{S}$.

The following remark contains some observations which are more or less obvious but may be helpful to have in mind.

Remark 3.3 (1) *$\underline{S}^\rho$ is not necessarily an element of the decomposition $\underline{S}$.*

(2) *However, because of condition (1c) it follows clearly that $\sum_{\rho=1}^r \underline{S}^\rho \in \underline{\Delta}$ holds true.*

(3) *If S^ρ is a singleton, then trivially $\bar{\alpha}_{S^\rho} = \underline{\alpha}_{S^\rho}$ holds true, hence it may well occur that $\underline{S}^\rho = S^\rho$ is the case.*

(4) *$\underline{S}^\rho$ and S^ρ do have at most one common element: this is why condition (1a) is formulated the way it is and not via $\underline{S}^\rho \prec S^\rho$. Recall however that $\prec$ is an ordering, so both coalitions cannot have two players in common.*

(5) *Therefore, if $S_0 \in \underline{S}$ is the first element of the decomposition according to $\prec$, then necessarily all S_0^ρ have to be singletons and it follows that $S_0^\rho = \underline{S}_0^\rho$ is the case. It is then reasonable to write*

$$\alpha_\rho^0 := \bar{\alpha}_{S_0^\rho} = \underline{\alpha}_{S_0^\rho}.$$

Theorem 3.4 *Let ξ be an imputation of v which admits of a ξ-consistent decomposition $\underline{S} \subseteq \underline{\Delta}$ of Ω. If $\xi \notin C(v)$, then there exist $\mu \in C(v)$ and $S \in \underline{P}$ such that $\mu \text{ dom}_S \xi$. In other words, imputations outside the core are being dominated from inside the core given a consistent decomposition.*

Proof: **1st STEP** : Fix $S \in \underline{\mathbf{S}}$ and consider the partner-sets S^ρ ,i.e., we have $S = \sum_{\rho=1}^r S^\rho$. For every S^ρ choose $\underline{S}^\rho$ according to condition (1).

Let us first assume

$$(43) \qquad \sum_{\rho=1}^r \bar{\alpha}_{\underline{S}^\rho} < 1.$$

Then, in view of the definition of $\bar{\alpha}_{\underline{S}^\rho}$ we have for each ρ :

$$(44) \qquad \frac{\xi_i}{\lambda_i^\rho} \le \bar{\alpha}_{\underline{S}^\rho} \ , \ \ \xi_i \le \bar{\alpha}_{\underline{S}^\rho} \lambda_i^\rho \ \ (i \in S^\rho)$$

Now choose $\epsilon_\rho > 0$ $(\rho = 1, \cdots, r)$ such that

$$(45) \qquad \sum_{\rho=1}^r \bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho = 1,$$

Then it follows at once that for each ρ :

$$(46) \qquad \xi_i < (\bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho)\lambda_i^\rho \ \ (i \in S^\rho).$$

Hence, if we define $\bar{\mu}^\epsilon \in \mathcal{C}(v)$ by

$$(47) \qquad \bar{\mu}^\epsilon := \sum_{\rho=1}^r \left(\bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho \right) \lambda^\rho \ ,$$

then (46) means just

$$(48) \qquad \bar{\mu}_i^\epsilon > \xi_i \ \ (i \in S^\rho \ , \ \ \rho = 1, \cdots, r).$$

On the other hand we observe that $\underline{S} := \sum_{\rho=1}^r \underline{S}^\rho \in \underline{\underline{\Delta}}$ is true and hence we infer the equations

$$(49) \qquad \begin{aligned} \bar{\mu}^\epsilon(\underline{S}) &= \sum_{\rho=1}^r \left(\bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho \right) \lambda^\rho(\underline{S}^\rho) \\ &= \sum_{\rho=1}^r \left(\bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho \right) v(\underline{S}) \\ &= \left(\sum_{\rho=1}^r (\bar{\alpha}_{\underline{S}^\rho} + \epsilon_\rho) \right) v(\underline{S}) \\ &= v(\underline{S}). \end{aligned}$$

Now combining (48) and (49) we have indeed

(50)
$$\bar{\mu}^\epsilon \ dom_{\underline{S}} \ \xi.$$

Hence the case indicated by the inequality (43) has been delt with; henceforth we may assume that the opposite is true,i.e., we now assume

(51)
$$\sum_{\rho=1}^{r} \bar{\alpha}_{\underline{S}^\rho} \geq 1.$$

2ⁿᵈ STEP : Because of condition (1b) it is clear that equation (51) implies another one which reads

(52)
$$\sum_{\rho=1}^{r} \underline{\alpha}_{S^\rho} \geq 1.$$

Again compare definitions: the way $\underline{\alpha}_\bullet$ is defined (c.f. 1 of Definition 3.1), we know that for each ρ

(53)
$$\xi_i \geq \underline{\alpha}_{S^\rho} \lambda_i^\rho \quad (i \in S^\rho)$$

holds true. Since $\underline{S}$ is a decomposition of Ω, we have therefore the following chain of inequalities and equations:

(54)
$$\begin{aligned}
\xi(S) &\geq \sum_{\rho=1}^{r} \underline{\alpha}_{S^\rho} \lambda^\rho(S) \\
&= \sum_{\rho=1}^{r} \underline{\alpha}_{S^\rho} v(S) \\
&= v(S) \sum_{\rho=1}^{r} \underline{\alpha}_{S^\rho} \\
&\geq v(S) \\
&= \xi(S).
\end{aligned}$$

Now, observe that all inequalities involved in (54) must necessarily be equations. This yields

(55)
$$\sum_{\rho=1}^{r} \underline{\alpha}_{S^\rho} = 1.$$

(in view of (52)) as well as

(56)
$$\xi_i \geq \underline{\alpha}_{S^\rho} \lambda_i^\rho \quad (i \in S^\rho)$$

(in view of (53)), which is true for all ρ. The last equation (56) renders the quotient ξ_i/λ_i^ρ to be equal for all $i \in S^\rho$ (for any ρ), that is , all players in some S^ρ are equally well off. Thus it follows that

$$(57) \qquad \underline{\alpha}_{S^\rho} = \bar{\alpha}_{S^\rho} =: \alpha_{S^\rho} \quad (S \in \underline{S})$$

is the case. Also, we obtain from (51), (55) and the condition (1b) the additional information that

$$(58) \qquad \bar{\alpha}_{\underline{S}^\rho} = \underline{\alpha}_{S^\rho} = \alpha_{S^\rho}$$

holds true for all ρ (and all $S \in \underline{S}$).

This completes the second step: all players in an arbitrary S^ρ are equally well off and the player best off in the predecessor set also has the same wealth .

3$^{\text{rd}}$STEP : Now let S_0 be the first element of $\underline{S}$ and let

$$(59) \qquad \alpha^0 := (\alpha_1^0, \cdots, \alpha_r^0) := (\alpha_{S_0^1}, \cdots, \alpha_{S_0^\rho}),$$

c.f. (5) in Remark 3.3 . Within this step we are now going to show by induction that all αs are the same, i.e., we prove the equation

$$(60) \qquad \alpha_{S'} = \alpha^0 \quad (S' \in \underline{S}) .$$

For $S = S^0$ there is nothing to show. Let $S \in \underline{S}$ be arbitrary and assume that all $\prec$ −preceeding $S' \in \underline{S}$ satisfy (60).

Choose S^ρ such that $\underline{S}^\rho \prec S^\rho$ holds true, this may be performed by condition 2 of Definition 3.2. By induction hypothesis we have $\xi_i = \alpha_\rho^0 \lambda_i^\rho$ $(i \in S'^\rho, S'^\rho \prec S^\rho$. Because of

$$\underline{S}^\rho \subseteq \bigcup_{S'^\rho \prec S^\rho} S'^\rho$$

it follows therefore that we have

$$\underline{\alpha}_{S^\rho} = \bar{\alpha}_{\underline{S}^\rho} = \alpha_\rho^0$$

As well as

$$\xi_i = \alpha_\rho^0 \lambda_i^\rho \quad (i \in \underline{S}^\rho)$$

holds true. It follows then from (58) that indeed the equation

$$\alpha_{S^\rho} = \bar{\alpha}_{\underline{S}^\rho} = \alpha_\rho^0$$

prevails. This completes the induction step, hence (60) is verified.

4ᵗʰSTEP : The final step is now quite obvious: indeed (60) shows that ξ is a convex combination of the λ^ρ, i.e.

$$(61) \qquad\qquad\qquad \xi = \sum_{\rho=1}^{r} \alpha_\rho^0 \lambda^\rho.$$

This means that ξ is located in the core of v . We have however assumed that we are dealing with an element outside the core. Therefore, equation (51) cannot prevail, instead the only alternative is equation (43). In the first step of our proof we have already established that this implies the domination of ξ from inside the core. **q.e.d.**

In order to prove that the core is stable we can now, given an imputation outside the core, try to construct a consistent decomposition of Ω by diagonal sets. Clearly the potential of admitting such a decomposition for every imputation not contained in the core should be a universal property of the game, hence in a next step we must somehow get of the specific attachment of $\underline{\underline{S}}$ to ξ. This attempt is prepared by the following decomposition.

Definition 3.5 *Let $\underline{\underline{S}} \subseteq \underline{\underline{\Delta}}$ be a partition of Ω and let $\prec$ be a binary relation on Ω. We shall say that is **universally ordered** by $\prec$ if $\underline{\underline{S}}$ is well ordered (c.f. Definition 3.1) and in addition the following conditions are satisfied.*

(1) *For every $S \in \underline{\underline{S}}$ and $\rho \in \{1, \cdots, r\}$ such that S^ρ is not a singleton, there exists , for any $i \in S^\rho$ some $T^\rho \subseteq C^\rho$ with the following properties:*

   (a) $T^\rho - i \prec S^\rho$.

   (b) *T^ρ contains players with the same weight as i only.*

   (c) $\lambda(T^\rho) = \lambda(S^\rho)$.

(2) *For every $S \in \underline{\underline{S}}$ except the $\prec$ −first one, there exists $\rho \in \{1, \cdots, r\}$ such that $T^\rho \neq S^\rho$ holds true.*

Theorem 3.6 *If $v = \bigwedge \{\lambda^1, \cdots, \lambda^r\}$ admits of a universally ordered partition $\underline{\underline{S}} \in \underline{\underline{\Delta}}$, then , for very imputation $\xi \in \mathcal{C}(v)$, there exists a $\xi -$ consistent partition.*

Proof: **1ˢᵗSTEP** : Let $\underline{\underline{\hat{S}}}$ be universaly ordered by means of some binary relation $\hat{\prec}$ and let ξ be an arbitrary imputation not contained in the core. We are going to define some partition $\underline{\underline{S}} \subseteq \underline{\underline{\Delta}}$ as well as some binary relation $\prec$ such that the ordering $\hat{\prec}$ is not changed across the types (i.e. players of equal weight) but within the types satisfies

$$(62) \qquad i \prec j \Longrightarrow \frac{\xi_i}{\lambda_i^\rho} \leq \frac{\xi_i}{\lambda_j^\rho} \ .$$

Or, in other words, within some S^ρ we exchange players of the same type until equation (62) is satisfied. This defines the new relation $\prec$ when leaving $\hat{\prec}$ unchanged otherwise.

More precisely, define π to be a permutation of Ω which leaves every $C^\rho \cap K_\tau^\rho$ unchanged and within the types of each C^ρ satisfies

$$(63) \qquad \pi(i) \hat{\prec} \pi(j) \Longrightarrow \frac{\xi_i}{\lambda_i^\rho} \leq \frac{\xi_i}{\lambda_j^\rho} \ \ (i, j \in C^\rho \cap K_\tau^\rho).$$

Then define the new binary relation $\prec$ by

$$(64) \qquad i \prec j \Longleftrightarrow \pi(i) \hat{\prec} \pi(j) \ \ r(i, j \in \Omega).$$

2ndSTEP :

Next the partition $\underline{\mathbf{S}}$ is given by

$$\underline{\mathbf{S}} := \pi(\hat{\underline{\mathbf{S}}}) := \left\{ \pi(\hat{\underline{\mathbf{S}}}) \mid \hat{S} \in \hat{\underline{\mathbf{S}}} \right\}$$

Again let us be more precise: $\hat{S}_0$ is the coalition that for each ρ yields the smallest $\hat{S}_0^\rho$ which in addition is a singleton. Hence, if $\hat{S}_0^\rho = \{i_0\}$, then it follows that $\pi(i_0) \hat{\prec} \pi(j) \ (j \in C^\rho)$, i.e., $i_0 \prec j \ (j \in C^\rho)$ meaning that , for each ρ, the coalition S_0^ρ is the $\prec$ −smallest one. Assume that for each $\hat{S}$ we have already constructed $S' = \pi(\hat{S}')$ for all $\hat{S}'$ which are $\hat{\prec}$−preceeding $\hat{S}$. As the types of $\hat{S}'^\rho$ are exactly copied there are of each type exactly as many players available in

$$\left(\bigcup \left\{ \hat{S}'^\rho \mid S'^\rho \hat{\prec} \hat{S}^\rho \right\} \right)^c$$

as in $\hat{S}^\rho$. Of these we take the $\prec$ −minimal ones and collect of the first ones so many as to render the profile to be exactly the one of $\hat{S}^\rho$. This then yields S^ρ.

Now given some $i \in \hat{S}^\rho$ we put $T^\rho := \pi(\hat{T}^\rho)$ and observe that T^ρ with respect to $\pi(i)$ satisfies the conditions (1) of Definition 3.5. That is, $\underline{\mathbf{S}}$ is universally ordered with respect to $\prec$.

3rdSTEP : We are now going to show that $\underline{\mathbf{S}}$ in addition is ξ−consistent in view of $\prec$. To this end we have to define, for any given ρ, the predecessor set $\underline{S}^\rho$ for any S^ρ. This task is performed as follows.

- If S^ρ is a singleton, the we put $\underline{S}^\rho := S^\rho$.

- If S^ρ is no singleton, then let i_0 be the player worst off in this coalition. Let T^ρ be the chosen corresponding to S^ρ and i_0 as given by condition (1) in Definition 3.5 . Define $\underline{S}^\rho := T^\rho$.

Clearly we have constructed the data such that $\lambda^\rho(\underline{S}^\rho) = \lambda^\rho(\underline{T}^\rho) = \lambda^r(S^\rho)$ as T^ρ and S^ρ have equal measure according to condition (1) of Definition 3.5, hence (1a) and (1c) of Definition 3.2 are satisfied. Moreocer, since all players of T^ρ are of the same type as i_0 and as $\prec$ respects the quotients $\frac{\xi_i}{\lambda_i^r}$, we may conclude that

$$(65) \qquad \bar{\alpha}_{S^\rho} = \bar{\alpha}_{T^\rho} = \frac{\xi_{i_0}}{\lambda_{i_0}^\rho} = \underline{\alpha}_{S^\rho}$$

holds true. Hence , all of Definition 3.2 is satisfied,   q.e.d.

The consequence is immediately given by the following corollary.

Corollary 3.7 *Let* $v = \wedge\{\lambda^1, \cdots, \lambda^r\} \in \mathbb{E}$. *Suppose there exists a universally ordered partition* $\underline{S} \in \underline{\Delta}$. *Then the core* $\mathcal{C}(v)$ *is stable.*

Proof: By Theorem 3.4 and Theorem 3.6.

The important point is that universally ordered partitions can be constructed without reference to imputations, hence it is indeed a property of the game (or rather of $\lambda^1, \cdots, \lambda^r$) to admit such a decomposition.

The construction of a universally ordered decomposition can be viewed as a sucessive introduction of larger weights into a sequence of diagonal sets which eventually contains all players, hence is a decomposition. To clear the foggy picture we consider some examples. We restrict ourselves to the case $r = 2$, hence a coalition consists of two partners of equal weight. The coalitions of $\underline{S}$ are listet parewise. Players are indicated by their weights and the ordering is from left to right.

Example 3.8

$$
\begin{array}{cccccccc}
1 & 11 & 11 & 111111 & 111111 & 4 & 4 & 12 \\
1 & 2 & 2 & 6 & 6 & 22 & 22 & 6222
\end{array}
$$

The first coalition consists of two singleton partners. Then successively larger weights can be introduced. E.g. the weight 6 (located in C^2) can be introduced by the coalition

$$S = \begin{array}{c} 111111 \\ 6 \end{array}$$

since 5 players with weight 1 are preceding S - this is condition (1) of Definition 3.5.

Similarly, weight 12 (in C^1) can be introduced since its partners 6 2 2 2 do have the required preceeding coalitions of the same type for every member: there are 2 players of weight 6 (which yields 12 - one 6 would be sufficient !) and there are 6 players of weight 2 (5 would be sufficient since one can always be supplied from S^2 !).

the next example suggests how to introduce large weights provided there are sufficiently many players of small weight:

Example 3.9

$$1 \quad 1 \quad 1 \quad 111 \quad 111 \quad 6 \quad 6 \quad 9 \quad 66 \quad 66 \quad 24$$
$$1 \quad 1 \quad 1 \quad 3 \quad 3 \quad 33 \quad 33 \quad 333 \quad 12 \quad 12 \quad 12 \; 3333$$

In a simple context the concept of sufficiently many small players available is exhibited as follows:

Example 3.10 *We take again $r = 2$ and assume λ^1, λ^2 to be type patterned as in Remark 1.17 , but with two types only. In each carrier the first type has weight 1. other than in Remark 1.17 we admit the size of each type to depend on ρ, i.e. we have for $\rho = 1, 2$:*

$$(66) \qquad \begin{aligned} \lambda^\rho(\bullet) \;&=\; \sum_{\tau=1}^{2} \mid K_\tau^\rho \cap \bullet \mid g_\tau^\rho. \\ &=\; \mid K_1^\rho \cap \bullet \mid + \mid K_2^\rho \cap \bullet \mid g_2^\rho. \end{aligned}$$

How many players of weight 1 will be necessary in order to admit for the construction of a universally ordered partition? Let d denote the greatest common divisor of the two nontrivial weights, i.e., $d = g.c.d. \,(g_2^1, g_2^2)$.

Assuming we have a large reservoir of players of weight 1 available, we construct a universally ordered partition as follows: First we take pairs of singletons with weight 1

$$(67) \qquad \begin{aligned} 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad &\cdots \quad 1 \\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad &\cdots \quad 1 \end{aligned}$$

the number being at least $\max_\rho(g_2^\rho - 1)$. Next we introduce g_2^1 by collecting pairs consisting of this weight and the corresponding number of weight 1:

$$(68) \qquad \begin{aligned} g_2^1 \quad\;\; g_2^1 \quad\;\; g_2^1 \quad\;\; g_2^1 \quad\;\; g_2^1 \quad\;\; g_2^1 \quad\;\; &\cdots \quad g_2^1 \\ 1\cdots 1 \;\; 1\cdots 1 \;\; 1\cdots 1 \;\; 1\cdots 1 \;\; 1\cdots 1 \;\; 1\cdots 1 \;\; &\cdots \;\; 1\cdots 1, \end{aligned}$$

the number of these blocks we choose to be $\min(k_2^1, \frac{g_2^2}{d} - 1)$. Similarly we choose $\min(k_2^2, \frac{g_2^1}{d} - 1)$ blocks of the analogous shape

$$(69) \qquad \begin{matrix} 1\cdots 1 & 1\cdots 1 & 1\cdots 1 & 1\cdots 1 & 1\cdots 1 & 1\cdots 1 & \cdots & 1\cdots 1 \\ g_2^2 & g_2^2 & g_2^2 & g_2^2 & g_2^2 & g_2^2 & \cdots & g_2^2. \end{matrix}$$

After we have taken all these blocks away, the remaining set, if any, is a diagonal one which can be decomposed into blocks having shape similar to those indicated in (67),(68), (69) or else the shape

$$(70) \qquad \begin{matrix} g_2^1 \cdots g_2^1 \\ g_2^2 \cdots g_2^2. \end{matrix}$$

All other blocks being introduced, the remainder (if any) is of shape (70) - and it has to be diagonal. Let p and q denote the number of the remaining players, i.e., the number of weights appearing in the upper and lower row of (70) respectively. Then necessarily $pg_2^1 = qg_2^2$ or $p\frac{g_2^2}{d} = q\frac{g_2^2}{d}$ holds true, meaning that g_2^2 is a divisor of p and g_2^1 is a divisor of q. Thus we can arrange the remainder in blocks of shape (70) such that in the upper row the number of weights is $\frac{g_2^2}{d}$ and in the lower row the number of weights is $\frac{g_2^1}{d}$. But this kind of block can be introduced under the criterion for universally ordered decompositions into the present construction in view of number of blocks of shape (68) and (69) already constructed.

This way we have indeed constructed a universally ordered decomposition which can be imagined by lining up all blocks of the 4 shapes described above. We can also exactly indicate the number of players of weight 1 in each carrier which was necessary for the construction, this is for $\rho = 1, 2$

$$(71) \qquad N_\rho := \max_\rho (g_2^\rho - 1) + g_2^\rho \min(k_2^\rho, \frac{g^{3-\rho}}{d} - 1).$$

Hence we have the following insight:

Remark 3.11 *In the situatuin described by example 3.10 there are nice numbers N_ρ ($\rho = 1, 2$), such that, whenever $k_1^\rho \geq N_\rho$ ($\rho = 1, 2$) holds true, it follows that there exists a universally ordered decomposition $\underline{\mathbf{S}}$ and hence for $v = \wedge\{\lambda^1, \lambda^2\}$ the core $\mathcal{C}(v)$ is stable.*

We call the numbers N_ρ 'nice' because they are sharp lower bounds. In general it is now easy to see that there exist lower bounds for the number of players of weight 1 in order to construct universally ordered decompositions. We restrict ourseves to

the case of two (orthogonal) measures in order to simplyfy the notation. A slightly modified argument would do the job for arbitrary r.

We assume that the λ^ρ $(r = 1, 2)$ are type patterned according to

$$(72) \qquad \lambda^\rho(\bullet) \;=\; \mid K_1^\rho \cap \bullet \mid \;+\; \sum_{\tau=2}^{T_\rho} \mid K_\tau^\rho \cap \bullet \mid g_\tau^\rho.$$

with integer g_τ^ρ. Let the greatest common divisors be denoted by

$$(73) \qquad d_{\tau\sigma}^\rho : \;=\; \text{g.c.d } (g_\tau^\rho, g_\sigma^{3-\rho}).$$

The relevant numbers are given by

$$(74) \qquad N_\rho : \;=\; \max_{\rho=1,2,\tau=2,\ldots,T_\rho} (g_2^\rho - 1) \;+\; \sum_{\tau=2}^{T_\rho} g_\tau^\rho \sum_{\sigma=2}^{T_\sigma} \min(k_\tau^\rho, \frac{g_\sigma^{3-\rho}}{d_{\tau\sigma}^\rho} - 1).$$

Indeed, we have

Theorem 3.12 *Let* λ^ρ *$(\rho = 1, 2)$ be given by (72) and* N_ρ *$(\rho = 1, 2)$ by (74). Then, whenever* $k_1^\rho \geq N_\rho$ *$(\rho = 1, 2)$ holds true, it follows that there exists a universally ordered decomposition* $\underline{\mathbf{S}}$ *and hence for* $v = \wedge\{\lambda^1, \cdots, \lambda^r\}$ *the core* $\mathcal{C}(v)$ *is stable.*

Proof: In order to construct the universally ordered decomposition, we start out by taking blocks of shape (67), their number is taken to be

$$(75) \qquad \max_{\rho=1,2,\tau=2,\ldots,T_\rho} (g_2^\rho - 1)$$

and the number of players of weight necessary for this procedure is the same. Next, for fixed ρ, we buildt up blocks of shape

$$(76) \qquad \begin{array}{c} g_\tau^\rho \\ 1 \cdots 1 \end{array}$$

corresponding to shape (68) above; the amount of such blocks required is for each τ given by

$$(77) \qquad \min\left(k_\tau^\rho, \left(\sum_{\sigma=1}^{T_{3-\rho}} \frac{g_\sigma^{3-\rho}}{d_{\tau\sigma}^\rho} \right) - 1 \right)$$

and hence the number of weights 1 necessary to construct these blocks is

$$(78) \qquad g_\tau^\rho \cdot \min\left(k_\tau^\rho, \left(\sum_{\sigma=1}^{T_{3-\rho}} \frac{g_\sigma^{3-\rho}}{d_{\tau\sigma}^\rho} \right) - 1 \right).$$

Summing these terms up over τ we obtain the second part in (74). Suppose now we have introduce all blocks of the first and second shape, consider a remaining block which may look like

$$
\begin{aligned}
(79) \qquad & g_2^1, \ldots, g_2^1, \quad \cdots \quad g_{T_1}^1, \ldots, g_{T_1}^1 \\
& g_2^2, \ldots, g_2^2, \quad \cdots \quad g_{T_2}^2, \ldots, g_{T_2}^2
\end{aligned}
$$

since the set is diagonal, the numbers of weights appearing in each C^ρ (i.e. on top and on the bottom of 79) have to satisfy

$$
(80) \qquad \sum_{\tau=2}^{T_1} p_\tau g_\tau^1 \;=\; \sum_{\tau=2}^{T_2} q_\tau g_\tau^2.
$$

Now, if it so happens that, for some τ, we have

$$
(81) \qquad p_\tau \geq \sum_{\sigma=2}^{T_2} \frac{g_\sigma^2}{d_{\tau\sigma}^1},
$$

then it would follow that

$$
(82) \qquad g_\tau^1 p_\tau \geq \sum_{\sigma=2}^{T_2} g_\tau^1 \frac{g_\sigma^2}{d_{\tau\sigma}^1} \;=\; \sum_{\sigma=2}^{T_2} \frac{g_\tau^1}{d_{\tau\sigma}^1} g_\sigma^2
$$

is the case. Hence, in view of (80), not all the q_σ could satisfy

$$
(83) \qquad \frac{g_\tau^1}{d_{\tau\sigma}^1} \geq q_\sigma.
$$

Therefore, we could find $\bar\sigma$ with

$$
(84) \qquad p_\tau > \sum_{\sigma=2}^{T_2} \frac{g_\sigma^2}{d_{\tau\sigma}^1} > \frac{g_{\bar\sigma}^2}{d_{\tau\bar\sigma}^1}
$$

and

$$
(85) \qquad q_{\bar\sigma} > \frac{g_\tau^1}{d_{\tau\bar\sigma}^1}.
$$

This means that we can take a subblock of the shape

$$
\begin{aligned}
(86) \qquad & g_\tau^1 \cdots g_\tau^1 \\
& g_{\bar\sigma}^2 \cdots g_{\bar\sigma}^2
\end{aligned}
$$

(with number of weights on top $\frac{g_{\bar\sigma}^2}{d_{\tau\bar\sigma}^1}$, number of weights at bottom $\frac{g_\tau^1}{d_{\tau\bar\sigma}^1}$) out of the remainder of shape (79) , this block can be introduced into the partition since we have already enough blocks (76) introduced.

After having repeated this procedure as many times as necessary, we know that

$$
(87) \qquad p_\tau < \sum_{\sigma=2}^{T_2} \frac{g_\sigma^2}{d_{\tau\sigma}^1},
$$

and the analogue for q_σ is satisfied. Hence we can introduce the remaining block of shape (79) again in view of the number of blocks (76).

A similar consideraton can be offered if (79) does have a slightly different shape, e.g., some weights of shape 1 appear. **q.e.d.**

Remark 3.13 *(1) The condition that the smallest weight in each carrier equals 1 can be relaxed. g_1^ρ can be a divisor of all g_τ^ρ $(\tau \geq 2)$ which must, however, be the same in each carrier.*

(2) In a slightly changed setup, let λ^0 be an integer valued measure which is homogenous w.r.t some integer δ constituting a (nontrivial) simple homogeneous game 'without steps' (see [7],[12], or [9]). Let $\lambda^1, \cdots, \lambda^r$ be orthogonal copies of λ^0. Then with some milder conditions (e.g. $k_\tau^\rho \geq 2\frac{g_{\tau+1}^\rho}{g_\tau^\rho} - 1$) towards the number of players of the smaller weights the existence of a universally ordered partition and hence the stability of the core of $v = \bigwedge\{\lambda^1, \cdots, \lambda^r\}$ can be established.

(3) The numbers (74) cannot be called nice. The examples show that the introduction of larger weights can be managed by smaller weights other that 1; we cannot claim that we have found good lower bounds. This may be a combinatorial problem which is not quite straightforward.

(4) The numbers (74) are bounds for the numbers of players of smallest type, i.e., k_1^ρ which involve numbers of players of larger type, i.e., k_τ^ρ $(\tau = 2, \ldots, T_\rho)$. Therefore we have specified regions of distributions of players over the type (i.e. vectors $k_\bullet^\rho$) such that the core is stable. This is not uncommon in n.d. theory, see e.g. [11].

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