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Some Classes of Potential and Semi-Potential Games

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Abstract

Continuing the development of a purely ordinal approach to potential games, we define a strong potential related to coalition improvements and strong equilibrium in the same way as the potential is related to individual improvements and Nash equilibrium. We describe games with perfect information giving rise to potential normal forms. A weaker concept of a semi-potential is introduced, providing an insight into better reply dynamics in models such as games with perfect information and voting by veto.

Key words: Potential game; Nash equilibrium; Strong equilibrium; Public and private objectives; Perfect information; Voting by veto.

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1. Introduction

This paper continues the development of a purely ordinal approach to potential games started in Kukushkin (1999). Since the latter paper has not yet appeared, its basic ideas are reproduced in the text below.

The main points are these. Generally following Monderer and Shapley (1996), we reject their (cardinalistic) view that a potential must be a numeric function: an order containing effective preferences of all players will do. Instead, we impose a topological restriction, monotone upper semi-continuity (MUSC), guaranteeing nice dynamics of better replies and the existence of a Nash equilibrium if every strategy set is compact in a metrics. Thus, in the finite case, the class of potential games as defined here coincides with the class of "generalized ordinal potential games" as defined by Monderer and Shapley. In the infinite case, both components of the symmetric difference of the two classes are not empty.

This paper pursues three main objectives. First, we define a strong potential, related to coalition improvements and strong equilibrium in the same way as the potential is related to individual improvements and Nash equilibrium, and provide examples of strong potential games. Second, we describe games with perfect information giving rise to potential normal forms; an important particular case are agent normal games, as was shown in Kukushkin (1999). Here the potential for such games is somewhat modified, providing an adequate expression for the backward induction principle. Finally, a weaker concept of a semi-potential is introduced, providing an insight into better reply dynamics in models such as games with perfect information and voting by veto.

In Section 2, basic definitions from Kukushkin (1999) are reproduced; Theorem 1 (cited from the same paper) provides a justification for the MUSC condition.

In Section 3, a definition of a (very) strong potential is given; two classes of strong potential games are described. The second one, "games with public and private objectives" is not devoid of serious economic meaning.

Section 4 contains the definition of a finite potential game form and a technical lemma useful for establishing the property. The important concept of a semi-potential is also defined there.

In Section 5, a necessary and sufficient condition for a finite game with perfect information to have a potential normal form is established. A potential for the agent normal form is introduced, a bit improving that constructed in Kukushkin (1999): A strategy profile is a maximizer for the potential if and only if it is a subgame perfect equilibrium.

Theorem 6 of Section 6 shows that every game with perfect information gives rise to a semi-potential game form.

Two classes of semi-potential game forms generated by the voting by veto principle are presented in the last Section 7.

2. Basic Concepts

A strategic game is defined by a set of players N and, for each $i \in N$, a set of strategies X_i and a preference relation U_i, a strict order on the set of strategy profiles $X = \Pi_{i \in N} X_i$. We usually assume that each U_i can be described with an ordinal utility function u_i, i.e. $y U_i x$ if and only if $u_i(y) > u_i(x)$. This assumption considerably simplifies the exposition even though I believe it is not necessary for the results as such. Anyway, we are only losing rather exotic preferences: incomplete or too complicated. Furthermore, we assume a metrics on each X_i such that each u_i is, at least, upper semi-continuous in own variable; normally, we assume each X_i compact in the metrics.

First, we define the effective preference relation P_i for each $i \in N$:

$$yP_i x \Leftrightarrow [\ u_i(y)>u_i(x)\ \&\ y_{-i}=x_{-i}\ ].$$

Obviously, x is a Nash equilibrium if and only if x is a maximizer for each P_i (or, the same, for their union). If preferences U_i are replaced with P_i, the set of Nash equilibria remains intact (although the new preferences are incomplete).

The most obvious property of a binary relation conducive to the existence of a maximizer is transitivity. Hence our preliminary definition (quite satisfactory for the finite case):

A potential P is a strict order (i.e. an anti-reflexive and transitive binary relation) on X containing each P_i (i.e. $yP_i x \Rightarrow yPx$).

For an infinite game, our preliminary definition does not guarantee either the existence of a Nash equilibrium, or good behaviour of "myopic" learning processes; this was shown in Kukushkin (1999), Example 2.

Let P be a strict order on a metric space X; we call P monotone upper semi-continuous (MUSC) if this condition holds:

$$[x^k \to x^\omega \ \&\ x^{k+1}Px^k \text{ for all } k=1,2,...] \Rightarrow x^\omega Px^k \text{ for all } k=1,2,...$$

Theorem 1. Every strict MUSC order on a compact metric space has a maximizer.

The idea behind the theorem is quite straightforward: suppose we try to build a P-increasing infinite sequence; if this proves impossible, we have a maximizer; otherwise, we pick a limit point and continue. The MUSC condition ensures that we never come back; therefore, "eventually" we will find a maximizer. A rigorous proof, based on Zorn's Lemma, is to be found in Kukushkin (1999).

Ultimately, we call P a potential if it is a strict, MUSC order containing every P_i. A game is called a potential game if it admits a potential.

Corollary. Every potential game with a compact set of strategy profiles has a Nash equilibrium.

Remark. There remains an interesting question of whether the MUSC condition on P can be derived from $P_i \subseteq P$ if P_i are defined with good enough utilities $u_i(x)$. In the above-mentioned Example 2 of Kukushkin (1999), the utilities were only upper semi-continuous; no similar example with continuous utilities is known.

Since every strict order on a finite set can be extended to one defined with a numeric function, in the finite case what we call "a potential game" is exactly what Monderer and Shapley call "a generalized ordinal potential game." In the infinite case, there are two differences: on the one hand, we impose the MUSC condition; on the other hand, we do not impose the condition that the potential should be represented by a numeric function. I must confess I am unable to understand motivation for the second restriction.

At the end of this section, we introduce an abstract concept to be used in Sections 6 and 7. Let $v=\langle v_h \rangle_{h \in H}$ be a finite cortege of real numbers; we denote $\vartheta(v)=\langle \vartheta_1(v),...,\vartheta_m(v) \rangle$ a cortege of the same length obtained by arranging all v_h in the increasing order. Now, having two corteges of the same length m, v' and v'', we say that v' dominates v'', $v' \mathscr{D} v''$, if $\vartheta_k(v') \geq \vartheta_k(v'')$ for all $k=1,...,m$; we say that v' strictly dominates v'', $v' \mathscr{D}^o v''$, if $\vartheta(v')$ Pareto dominates $\vartheta(v'')$. For technical convenience we also allow empty corteges with $m=0$; for such corteges, $\mathscr{D}$ is always true and $\mathscr{D}^o$, never.

Remark. This relation is a partial case of the stochastic dominance relation. I have preferred an independent definition because there is no stochastics in our context. Actually, several equivalent definitions are possible.

3. Strong Potential Games

Let us turn to coalitions. Assuming I to be a non-empty subset of N, we define the effective coalition preference relation P_I in this way:

$$yP_I x \Leftrightarrow [\ (\forall i \in I\ u_i(y) > u_i(x))\ \&\ y_{-I} = x_{-I}\ ].$$

A strong potential is a strict, MUSC order on X containing every P_I.

We also define the effective weak coalition preference relation P_I:

$$yP_I^w x \Leftrightarrow [\ (\forall i \in I\ u_i(y) \geq u_i(x))\ \&\ (\exists i \in I\ u_i(y) > u_i(x))\ \&\ y_{-I} = x_{-I}\ ].$$

A very strong potential is a strict, MUSC order on X containing every P_I^w.

A (very) strong potential game is such that admits a (very) strong potential. Every (very) strong potential game with compact strategy sets has a strong equilibrium (there are two slightly different definitions for the latter in the literature).

Remark. When this paper was practically finished, I discovered the paper of Holzman and Law-Yone (1997), introducing a concept of a strong potential differing from ours in the same way as that of Monderer and Shapley's. Holzman and Law-Yone suggest different examples of strong potential games.

In the following, two classes of strong potential games are considered. Games of both classes may have arbitrary (finite) sets of players and arbitrary (compact) strategy sets X_i. It is the structure of utilities that ensures the existence of a strong potential in either case.

Suppose the set of players is ordered, say, $N=\{1,\ldots,n\}$, and each player's utility only depends on his own choice and the choices of the preceding players, i.e. $u_1(x)=u_1(x_1)$, $u_2(x)=u_2(x_1,x_2)$, etc. We define an order P on X as: zPx if and only if there exists $j \in N$ such that $y_i=x_i$ for all $i<j$ and $u_j(y)>u_j(x)$; its transitivity is checked easily.

Theorem 2. P is a strong potential if every utility has the structure indicated and is upper semi-continuous in own choice.

Let us prove the inclusion $P_I \subseteq P$ first. Suppose $yP_I x$ and denote $j=\min I$. Now $yP_I x$ implies $u_j(y)>u_j(x)$ while $y_i=x_i$ for all $i<j$ since $y_{-I}=x_{-I}$; therefore,

yPx.

Now assume $x^k \to x^\omega$ and $x^{k+1}Px^k$ for all $k=1,2,...$; we only need to prove $x^\omega Px^k$ for arbitrarily large k. For each k, we take $j(k)$ from the definition of $x^{k+1}Px^k$ and define $j=\lim\inf_{k\to\infty} j(k)$. For every $i<j$, the sequence of x_i^k stabilizes at some stage, so $x_i^\omega=x_i^k$ for all k large enough. Now we may restrict ourselves to those large k's; then at every step we have either $x_j^{k+1}=x_j^k$ or $u_j(x^{k+1})>u_j(x^k)$, with the latter inequality emerging for arbitrarily large k, so $u_j(x^\omega)>u_j(x^k)$ by the upper semi-continuity of u_j, hence $x^\omega Px^k$.

Remark. If every utility is upper semi-continuous in x, we can use a simpler potential, just lexicography in the utilities space. Although this potential also need not be very strong, it contains the Pareto dominance, so every maximizer is (strong) Pareto optimal. However, under milder topological assumptions such lexicography need not be MUSC (and indeed, there may be no Pareto optimal equilibrium).

This simple class of strong potential games demonstrates an inconvenience stemming from imposing the restriction that a potential must be numeric. Since the lexicography cannot generally be described by a numeric function, some games from the class would be classified as allowing no potential if such a restriction were imposed (see e.g. Example 4.1 of Voorneveld and Norde, 1996) while for others establishing the existence of a numeric strong potential could be a serious problem. On the other hand, game-theoretic implications of the existence of a potential are the same regardless of whether it is numeric or not.

Now let us consider a few modifications of a model due to Germeier and Vatel' (1974), see also Kukushkin (1992, 1994); a rather partial case of the model was, apparently independently, considered in Moulin (1995).

This time, the structure of utilities is as follows. There is a "public" characteristic of a situation described by a function $f_N(x)$ (e.g. minus logarithm of the concentration of a pollutant in the air) and a "private" characteristic described by a function $f_i(x_i)$ (e.g. monetary income) for each $i \in N$. Each player $i \in N$ has an aggregating function F_i: $\mathbb{R}^2 \to \mathbb{R}^2$, and

$$u_i(x) = F_i(f_N(x), f_i(x_i)\}.$$

Even assuming that all functions f_N, f_i are upper semi-continuous in their variables, we cannot hope for any good properties of such a game as long as "aggregators" F_i remain arbitrary.

If each F_i is the sum, i.e. the public and private characteristics are perfect substitutes for each other from every player's viewpoint, there is a cardinal potential

$$f_N(x) + \Sigma_{i \in N} f_i(x_i).$$

(The proof is straightforward.) Under milder (and ordinal) assumption that F_i is the sum up to monotonic transformations, $u_i(x)=\lambda_i(\mu_N(f_N(x))+\mu_i(f_i(x_i))$, the potential becomes only ordinal in the sense of Monderer and Shapley. Roughly speaking, this is the case for a Cournot model with identical linear costs.

More interesting is the situation where both characteristics are "absolutely complementary", i.e.

$$u_i(x) = \min \{f_N(x), f_i(x_i)\}. \quad (1)$$

Denote $\mathscr{L}$ the strict leximin order on $\mathbb{R}^N$ and define a strict order P on X as $yPx \Leftrightarrow u(y)\mathscr{L}u(x)$.

Theorem 3. P just defined is a very strong potential for a game with utilities satisfying (1).

The MUSC property easily follows from the upper semi-continuity of u_i in x (somewhat similar to how it was done in the proof of Theorem 2); let us show $P_I^w \subseteq P$ for each $I \subseteq N$. Assuming that $y,x \in X$ are such that $y_{-I}=x_{-I}$, $u_i(y) \geq u_i(x)$ for all $i \in I$, and $u_i(y) > u_i(x)$ for some $i \in I$, we have to show $u(y) \mathcal{D} u(x)$.

Suppose $f_N(y) \geq f_N(x)$; then $y_j = x_j$ implies $u_j(y) \geq u_j(x)$ for all $j \notin I$. Therefore, $u(y)$ Pareto dominates $u(x)$.

Now let $f_N(y) < f_N(x)$; for each $i \in I$ we have $u_i(x) \leq u_i(y) \leq f_N(y)$ (and the first inequality is strict for some i). On the other hand, if $j \notin I$ and $u_j(x) < f_N(y)$, then $u_j(x) = f_j(x_j) = f_j(y_j) = u_j(y)$. Finally, if $j \notin I$ and $u_j(x) \geq f_N(y)$, then $f_j(y_j) = f_j(x_j) \geq u_j(x) \geq f_N(y)$, hence $u_j(y) = f_N(y)$. It follows immediately that $u(y) \mathcal{D} u(x)$.

If each F_i is the maximum, the exact analogue of Theorem 3 remains valid. One can also consider combinations of the minimum and maximum of a certain kind, see Kukushkin (1992); the latter paper did not use the concept of a potential explicitly, but the sufficiency part of its main theorem can be reformulated in these terms.

4. Potential and Semi-Potential Game Forms

A game form G is defined by a set of players N, a strategy set X_i for each $i \in N$, a set of outcomes A and a projection π: $X \to A$, where $X = \Pi_{i \in N} X_i$ is the set of strategy profiles. Once preferences of the players over the outcomes are specified (and we always assume this to be done with utilities u_i: $A \to \mathbb{R}$), a derivative game $G(u)$ (where u denotes a list $\langle u_i \rangle_{i \in N}$) emerges. A notational abuse is thus committed when we simultaneously have utilities $u_i(a)$ and $u_i(x) = u_i(\pi(x))$; however, as long as the abuse is recognized, it cannot have any disastrous consequences. Actually, we consider only finite sets X_i, so there is no need for any topological assumption.

Naturally enough, we call G a potential game form if every derivative game $G(u)$ admits a potential. Example 1 of Kukushkin (1999) shows that the agent normal form of any (finite) game with perfect information (see Selten, 1975) is a potential game form.

There are two principal ways to establish the existence of a potential for a finite strategic game: either to define a potential explicitly, or to show the impossibility of improvement cycles (for an infinite game, the second method is rarely workable, see, however, Nakamura, 1975). When it comes to game forms, we may use either method assuming arbitrary preferences; there also exists a "third way" based on the game form itself without any reference to utilities.

Let us introduce necessary definitions. A strategic path $\hat{x}$ is a sequence $x^0,x^1,\dots,x^m \in X$ together with $i(k) \in N$ for each $k=1,\dots m$ such that $x^{k-1}_{-i(k)}=x^k_{-i(k)}$; a strategic cycle is a strategic path such that $x^0=x^m$. With every path $\hat{x}$, we associate $N(\hat{x})=\{i(k) \mid k \in \{1,\dots m\} \}$ (the set of players involved) and, for each $i \in N(\hat{x})$, a binary relation $R_i(\hat{x})$ on A:

$$aR_i(\hat{x})b \Leftrightarrow \exists k \ [\ \pi(x^{k-1})=b \ \&\ \pi(x^k)=a \ \&\ i(k)=i].$$

A bad cycle is a strategic cycle $\hat{x}$ for which every $R_i(\hat{x})$ is acyclic (in particular, asymmetric).

Lemma 1. A finite game form is a potential one if and only if it allows no bad cycle.

If there is a bad cycle, then each $R_i(\hat{x})$ can be extended to a strict order relation, which can be taken for preferences of player i. Now the bad cycle becomes an improvement cycle (regardless of how the preferences for players not involved in the cycle are defined).

On the other hand, for whatever preferences, a strategic cycle generating $R_i(\hat{x})$ with a cycle cannot be an improvement cycle.

Examples. Let us consider two game forms with two players:

1.	a	a	a	**2.**	c	c	b
	b	c	c		c	a	a
	b	d	e		b	a	b

It can easily be seen that neither of them allows a bad cycle (actually, Example 1 belongs to Section 5 below and Example 2, to Section 7).

The idea of a potential may work (e.g. as a reason for equilibrium existence) even when there is no potential as defined so far.

Let G be a game form and $G(u)$ its derivative game. A semi-potential for $G(u)$ consists of a strict order S on X (semi-potential proper) and binary relations S_i (for all $i \in N$) such that

(i) $S_i \subseteq P_i$,

(ii) $S_i \subseteq S$,

(iii) $yP_ix \Rightarrow \exists z\ [\ zS_ix\ \&\ \pi(z)=\pi(y)\ ]$.

The relation yS_ix is pronounced as "y is an admissible improvement over x (by player i)". The existence of a semi-potential implies that no cycle can be formed of admissible improvements, while for any one-sided improvement there exists an equivalent admissible one. In particular, every maximizer of S is a Nash equilibrium. As to better replies dynamics, very much depends on the interpretation of the system of S_i's; the conclusion may range from "an improvement cycle is impossible unless the players deliberately create it" to "an improvement cycle cannot happen if the players are careful enough".

Quite similarly, a strong semi-potential for $G(u)$ consists of a strict order S on X and binary relations S_I (for all non-empty $I \subseteq N$) such that

(i) $S_I \subseteq P_I$,

(ii) $S_I \subseteq S$,

(iii) $yP_Ix \Rightarrow \exists z\ [\ zS_Ix\ \&\ \pi(z)=\pi(y)\ ]$.

The interpretation is largely the same. Every maximizer of S is a strong equilibrium. A very strong semi-potential could have been defined easily, but so far no example of such a thing is known.

If every derivative game $G(u)$ admits a (strong) semi-potential, we call G itself a (strong) semi-potential game form. Examples of such forms are given in Sections 6 and 7 below.

5. Games with Perfect Information: Potentials

Since the concept itself should be familiar, I just fix notation. The game tree has the set $K=M\cup T$ of nodes, where T consists of terminal nodes and M of decision nodes. M is partitioned into M_i for $i\in N$ so that player i moves at nodes from M_i. The structure of the tree can be described by the tree order on K, $\alpha>\beta$ meaning α comes after β; in terms of this order, T is the set of maximizers and the origin α_0 is the least element of K. The set of moves available at each node $\alpha\in M$ may be identified with the set of immediate successors $X_\alpha\subseteq K$. For the necessity part of Theorem 4 below to be true, we demand $\#X_\alpha>1$ and $\alpha\in M_i$ $\Rightarrow$ [$X_\alpha\cap M_i$ is empty]; obviously, these restrictions inflict no loss in generality. For every $x\in X$ and $\alpha\in M$, $\tau(\alpha,x)\in T$ is uniquely defined as the result of playing x starting at α.

The standard normal form is a game form with $X_i=\Pi_{\alpha\in M_i}X_\alpha$ (so the set of strategy profiles is $X=\Pi_{i\in N}X_i=\Pi_{\alpha\in M}X_\alpha$), $A=T$, and $\pi(x)=\tau(\alpha_0,x)$. As is shown in Kukushkin (1999), a derivative game of a form of this kind need not allow a potential; however, a potential always exists if each M_i is a singleton. Here we provide a description of games with perfect information guaranteeing the existence of a potential regardless of preferences.

Just one definition more is needed. We say that a game with perfect information has well arranged moves if every M_i is a chain in the tree order (i.e. for every player there exists a play of the game containing all his decision nodes).

Remark. Although the property is quite easy to comprehend and to test for a particular game tree, it is not intuitively clear why it should play the key role in the problem of a potential.

Theorem 4. The normal form of a game with perfect information is a potential game form if and only if the game has well arranged moves.

The necessity proof essentially consists of two examples.

Example 3.

$$\begin{array}{ccccccccc} & & 2 & & 1 & & 2 & & \\ a & \leftarrow & \beta_1 & \leftarrow & \alpha_0 & \rightarrow & \beta_2 & \rightarrow & d \\ & & \downarrow & & & & \downarrow & & \\ & & b & & & & c & & \end{array}$$

We assume $M_1=\{\alpha_0\}$, $M_2=\{\beta_1,\beta_2\}$.

Now we note that the following strategic path is a bad cycle:

$$\langle\beta_1,ad\rangle,\ \langle\beta_2,ad\rangle,\ \langle\beta_2,bc\rangle,\ \langle\beta_1,bc\rangle,\ \langle\beta_1,ad\rangle;$$

indeed, the cycle generates these binary relations: dR_1a, bR_1c, cR_2d, aR_2b.

Example 4.

$$\begin{array}{ccccccccccccc} & & 1 & & 2 & & 1 & & 3 & & 1 & & \\ a & \leftarrow & \gamma_2 & \leftarrow & \beta_2 & \leftarrow & \alpha_0 & \rightarrow & \beta_3 & \rightarrow & \gamma_3 & \rightarrow & f \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ & & b & & c & & & & d & & e & & \end{array}$$

We assume $M_1=\{\alpha_0,\gamma_2,\gamma_3\}$, $M_2=\{\beta_2\}$, $M_3=\{\beta_3\}$.

This time, the bad cycle looks as follows:

$$\langle\beta_2 af,\gamma_2,\gamma_3\rangle,\ \langle\beta_2 af,c,\gamma_3\rangle,\ \langle\beta_3 ae,c,\gamma_3\rangle,\ \langle\beta_3 ae,c,d\rangle,\ \langle\beta_2 be,c,d\rangle,$$
$$\langle\beta_2 be,\gamma_2,d\rangle,\ \langle\beta_3 bf,\gamma_2,d\rangle,\ \langle\beta_3 bf,\gamma_2,\gamma_3\rangle,\ \langle\beta_2 af,\gamma_2,\gamma_3\rangle,$$

generating these binary relations: $eR_1cR_1dR_1b$, aR_1f, bR_2cR_2a, fR_3dR_3e.

It is important to note that the union of R_2 and R_3 is also acyclic. If we "consolidate" players 2 and 3 into a single player, the bad cycle will remain a bad cycle.

Let us now finish with the necessity proof. Suppose a player i has two decision nodes, α' and α'', incomparable in the tree order. Then we define $\alpha=\alpha'\cap\alpha''$ (K is a lower semi-lattice in the tree order) and denote j the player who moves at α. If $i\neq j$, we may use the bad cycle from Example 3 (fixing the decisions taken before α, or between α and α', etc. in an obvious way) with α as α_0, α' as β_1, and α'' as β_2. If $i=j$, we pick a decision node β_2 between α

and α' belonging to a different player and a decision node β_3 between α and α'' also belonging to a different player and use the bad cycle from Example 4 (as was noted there, it does not matter whether the same player moves at β_2 and β_3, or two different players).

The sufficiency proof goes by induction (e.g. in the number of nodes). A game with a single node obviously admits a potential.

Supposing the existence of a bad cycle $\hat{x}$ in a game with well arranged moves, we, first of all, note that subsequent changes of strategy by the same player can be combined together (if $i(k)=i(k+1)$, x^k can be deleted from the cycle), so we may assume $i(k)\neq i(k+1)$ for all $k=0,1,...m$. Then we define α_k for each $k=1,...m$ as the least node where a decision taken was changed in the transfer from x^{k-1} to x^k (it is well defined because each M_i is a chain). Finally, we pick a minimal α_k for $k=1,...m$ (actually, it must be the least among all α_k, but we do not need this statement). The following diagram may help a little.

$$\begin{array}{ccccccccccc} & & & & & \downarrow & & & & & \\ \leftarrow \alpha_{k-1} \leftarrow & \ldots & \leftarrow & \beta' & \leftarrow & \alpha_k & \rightarrow & \beta'' & \rightarrow \ldots \rightarrow & \alpha_{k+1} & \rightarrow \\ \downarrow & & & & & & & & & \downarrow & \end{array}$$

By the choice of α_k, no change is ever made at the nodes between α_0 and α_k, so we may just forget about them. Since $M_{i(k)}$ is a chain, at least one of the subtrees starting at β' or β'' contains no node from $M_{i(k)}$; we may assume it to be that starting at β'. Denote $L=\{\gamma\in K \mid \gamma\geq\beta'\}$ and $N(L)=\{i\in N(\hat{x}) \mid M_i\cap L$ is not empty$\}$; $N(L)$ itself is not empty because $i(k\text{-}1)\in N(L)$. With every strategy profile x of the initial game, we associate its projection x_L to the subgame L; in particular, every x^h is transformed into $x_L^{\ h}$ (for $h=0,1,...m$). Let us show that $\hat{x}_L$ turns, after deleting repetitions, into a bad cycle for the subgame. This will contradict the induction hypothesis.

Indeed, if $i(h)\notin N(L)$, then player $i(h)$ has no decision node in L and $x_L^{\ h}=x_L^{\ h-1}$; when considering $\hat{x}_L$, we simply delete such steps. If $i(h)\in N(L)$, then all the decision nodes player $i(h)$ has outside L precede $\alpha_{i(k)}$, hence every actual change is done inside L; therefore $N(\hat{x}_L)=N(L)$ and each time when $i(h)\in N(L)$, we have $\pi(x^{h-1})=\tau(\alpha_k,x_L^{h-1})$ and $\pi(x^h)=\tau(\alpha_k,x_L^{\hat{h}})$. It follows easily that $R_i(\hat{x}_L)=R_i(\hat{x})$ for every $i\in N(L)$. The theorem is proved.

It should be noted that we thus only have an indirect proof of the existence of a potential. For an important particular case of a game with singleton M_i's, an explicit potential was defined in Kukushkin (1999), every maximizer of which is a subgame perfect equilibrium. The importance of the case stems from the fact that we may, in any game, replace every player with a team of agents having the same preferences so that each agent has just one decision node (Selten, 1975); the sets of subgame perfect equilibria of both games coincide. Now I present a modification of that potential.

Suppose $\#M_i=1$ for every $i\in N$; we may then identify M with N and consider the tree order as an order on the set of players. First we define auxiliary, "subgame perfect" utilities: $w_i(x)=u_i(\tau(i,x))$, i.e. each player is only interested in the result of an "imaginary" play starting at his decision node; we may use notation $G(w)$ for the game with strategy sets X_i and utilities w_i, but it should be stressed that $G(w)$ is *not* a derivative game of the same game with perfect information G. The set of Nash equilibria of $G(w)$ coincides with the set of subgame perfect equilibria of $G(u)$.

Now let us define a strict order P on X:

$$yPx \Leftrightarrow [\ \exists i\in N\ w_i(y)>w_i(x)\ \&\ \forall j\in N\ (\ y_j\neq x_j\ \Rightarrow\ \exists i\geq j\ w_i(y)>w_i(x)\ )].$$

(Checking the transitivity of P is a bit more sophisticated than in Theorem 2, but still quite straightforward.)

Theorem 5. If every player has just one decision node in G, then P just defined is a potential for $G(u)$, and a profile $x\in X$ is a maximizer for P if and only if x is a subgame perfect equilibrium of $G(u)$.

Let $yP_i x$; then $\tau(i,x)=\pi(x)$ and $\tau(i,y)=\pi(y)$, so $w_i(y)=u_i(y)>u_i(x)=w_i(x)$. Since $y_j=x_j$ for all $j\neq i$, we have yPx.

If $x\in X$ is not a subgame perfect equilibrium, it cannot be a maximizer of P - just pick any profitable individual deviation in a subgame. If $x\in X$ is not a maximizer of P, then yPx, hence $\{i\in N \mid y_i\neq x_i\}$ is not empty, so we may pick a maximal element of it, i.e. $i\in N$ such that $y_i\neq x_i$ and $y_j=x_j$ for all $j>i$. Since yPx, there must be $w_i(y)>w_i(x)$ but $w_i(y)=w_i(x_{-i},y_i)$; therefore the equilibrium condition for i at x in the subgame starting at i is violated.

Remark. It is funny to note that P is a strong potential for $G(w)$, cf. Theorem 2. This property is lost when we return to utilities u_i.

6. Games with Perfect Information: Semi-Potentials

Now the concept of a semi-potential comes to the scene. Both bad cycles of Examples 3 and 4 contained changes of decisions outside actual play, which cannot serve any useful purpose. A hypothesis naturally emerges that cycling would become impossible if we prohibited such unnatural behaviour. The hypothesis proves correct.

From now on, we consider the normal form of an arbitrary game with perfect information with fixed (arbitrary) preferences. Let $x\in X$ and $i\in N$; we call $y\in X$ an admissible improvement over x by player i, $yS_i x$, if $yP_i x$ and for each $\alpha\in M$ [$x_\alpha\neq y_\alpha \Rightarrow y_\alpha\leq\pi(y)$]. To define a semi-potential proper, we need further notation. Let $x\in X$, $a\in T$; we say that a is blocked by player i at x if there exist $\alpha\in M_i$ and $\beta\in X_\alpha$ such that $\tau(\beta,x)=a$ and $x_\alpha\neq\beta$. It is easily checked that every $a\in T\setminus\{\pi(x)\}$ is blocked by a unique player i; we denote this fact by $a\in T(i,x)$, obtaining a partitioning of $T\setminus\{\pi(x)\}$. It is important to note that $m_i=\#T(i,x)$ does not depend on x; actually, $m_i=[\Sigma_{\alpha\in M_i}\#X_\alpha]-\#M_i$. With every profile $x\in X$, we associate n corteges $v^i(x)=\langle u_i(a)\rangle_{a\in T(i,x)}$ of the lengths m_i.

Now we define our semi-potential as

$$ySx \Leftrightarrow [\ \forall i{\in}N\ v^i(x)\mathscr{D}v^i(y)\ \&\ \exists i{\in}N\ v^i(x)\mathscr{D}^o v^i(y)\ ],$$

where the relations $\mathscr{D}$ and $\mathscr{D}^o$ were defined at the end of Section 2.

Theorem 6. The relations S and S_i $(i{\in}N)$ just defined form a semi-potential for the normal form of any finite game with perfect information.

The conditions (i) and (iii) from the definition are checked easily; only condition (ii) may deserve a few words. Let yS_ix; we have to show ySx.

We denote $a{=}\pi(x)$ and $b{=}\pi(y)$; let us show $T(j,x){\subseteq}T(j,y)$ for each $j{\neq}i$. Assuming $c{\in}T(j,x)$, we, by the definition of blocking, have $\alpha{\in}M_j$ and $\beta{\in}X_\alpha$ such that $\tau(\beta,x){=}c$ and $x_\alpha{\neq}\beta$. Since $j{\neq}i$, $y_\alpha{=}x_\alpha{\neq}\beta$, so $c{\notin}T(j,y)$ could only be possible if $\tau(\beta,y){\neq}c$. However, $\gamma{\geq}\beta$ is incompatible with $\gamma{\leq}b$ because β is incomparable with b; therefore, by yS_ix, $y_\gamma{=}x_\gamma$ for all $\gamma{\geq}\beta$, hence $\tau(\beta,y){=}\tau(\beta,x){=}c$. Now $T(j,x){\subseteq}T(j,y)$ for all $j{\neq}i$ implies $T(j,x){=}T(j,y)$ for all $j{\neq}i$, hence $T(i,y){=}(T(i,x)\setminus\{b\}){\cup}\{a\}$. Since $u_i(b){>}u_i(a)$, $v^i(x)\mathscr{D}^o v^i(y)$; since $v^j(x){=}v^j(y)$ for $j{\neq}i$, we have ySx. The theorem is proved.

Example 5.

$$\begin{array}{ccccc}
1 & & 2 & & \\
\alpha_0 & \rightarrow & \beta & \rightarrow & c \\
\downarrow & & \downarrow & & \\
a & & b & &
\end{array}$$

We assume $M_1{=}\{\alpha_0\}$, $M_2{=}\{\beta\}$; $u(a){=}{<}1,2{>}$, $u(b){=}{<}2,1{>}$, $u(c){=}{<}0,0{>}$.

There are just two equilibria here: $<\beta,b>$ and $<a,c>$, only the first one being subgame perfect. According to Theorem 5, $<\beta,b>$ is the unique maximizer of the potential defined there. Let us look at the semi-potential from Theorem 6. At $<\beta,b>$, we have $T(1,\beta,b){=}\{a\}$, $T(2,\beta,b){=}\{c\}$, so $v^1(\beta,b){=}{<}1{>}$, $v^2(\beta,b){=}{<}0{>}$; at $<a,c>$, $T(1,a,c){=}\{c\}$, $T(2,a,c){=}\{b\}$, so $v^1(a,c){=}{<}0{>}$, $v^2(a,c){=}{<}1{>}$. Both equilibria are maximizers for the semi-potential (which, incidentally, is a potential in this example).

I must confess I do not believe that Example 5 points out any deficiency of our semi-potential. The very idea of a potential is naturally associated with gradual, "myopic" adaptation of the players to a situation they may not perceive as a whole. The subgame perfectness concept, on the contrary, assumes the whole tree under consideration at once. If the players start at the profile $<\beta,c>$ and player 1 adapts the first, player 2 will never have an opportunity to reconsider his choice. After all, the choice between the two equilibria here is a question of leadership, of who is to adapt the first, and the semi-potential catches this aspect perfectly. Certainly, evolutionists would argue that myopic adaptation implies subgame perfectness if small mistakes are allowed for. However, there is no place for "small mistakes" in the strictly ordinal world considered here.

7. Voting by Veto

The principle of voting by veto appears to be due to Peleg (1978) and Mueller (1978). Here we are not concerned with the importance of the idea for social choice theory; I will consider strategic games based on the principle just as examples of game forms ensuring the existence of strong equilibria for every preferences of the players. And the main objective of this section is to establish the presence of a strong semi-potential in such games.

There is the set of players N and the set of outcomes, or alternatives, A. The players may have arbitrary preferences over outcomes described by (ordinal) utilities u_i: $A \to \mathbb{R}$. A voting by veto procedure specifies positive integer numbers $\lambda(a)$ and μ_i for each $a \in A$ and $i \in N$ such that

$$\Sigma_{a\in A}\ \lambda(a) = \Sigma_{i\in N}\ \mu_i + 1. \qquad (2)$$

μ_i is the number of black balls given to player i; $\lambda(a)$ is "veto-resistance" of outcome a. We assume a finite set T_i with $\#T_i = \mu_i$ for each $i \in N$ such that all T_i's are disjoint; we denote $T = \cup T_i$. A strategy of player i is a mapping x_i: $T_i \to A$. Having a strategy profile x, which can be understood as a mapping x: $T \to$

A, we denote $\kappa(a,x)=\#x^{-1}(a)$. An outcome a is vetoed at x if $\kappa(a,x)\geq\lambda(a)$ and over-vetoed if the inequality is strict. Equality (2) ensures the existence of non-vetoed outcomes at every strategy profile; if no outcome is over-vetoed, there is just one non-vetoed outcome. A voting procedure is ultimately described by a mapping π: $X \to A$ such that $\pi(x)$ is not vetoed at x for all $x \in X$.

Peleg (1978) effectively showed that every such game has a strong equilibrium, at which no outcome is over-vetoed; more good properties are described in Moulin (1983). Unfortunately, I cannot say anything about potentials or semi-potentials for all games from this wide class; we have to specify π. Actually, we consider two such mechanisms. The first of them uses a more or less standard procedure for tie-breaking. The second was introduced by Gol'berg and Gurvich (1986); its only justification lies in good formal properties of the games generated (Gol'berg and Gurvich showed that the mechanism ensures dominance-solvability in most cases; here it turns out that it also allows a rather simple semi-potential). I know of no convincing interpretation for the mechanism in any voting context.

The first mechanism, π_P, presupposes a complete order on A. If more than one outcome is not vetoed, the highest of them is selected.

The second mechanism, π_G, assumes that A is arranged into an oriented circle and superfluous black balls are shifted clockwise till all the empty places but one are filled; the only outcome that remains not vetoed after this procedure is declared winner (formal definitions are given below).

Theorem 7. The mechanism π_P always gives rise to a semi-potential game form.

Let us construct a strong semi-potential for any game from the class. As a preliminary explanation of the concept of admissible changes adopted, I can put forward two general ideas. First, the players should avoid unnecessary changes (e.g. a coalition might order two members to exchange some of their ballots cast to different outcomes; such a change cannot promote any change in the outcome selected, but may well upset a forming equilibrium); the admissibility concept in the previous section was based on this idea alone. Second, the players should not rely too heavily on peculiarities of the procedure as contrasted to its general spirit (if you are not satisfied with the currently selected outcome, add more black balls to it rather than inventing indirect ways to have another one selected). In particular, we assume that the players should never deliberately create over-vetoing.

The exact definition of an admissible improvement, $yS_I x$, naturally, includes the requirement that $yP_I x$, i.e. y must be an improvement over x for coalition I, and the following four "rules" (which seem easier to perceive than a single logical formula):

1) if $\kappa(a,y)>\lambda(a)$ and $y(t)=a$, then $x(t)=a$ (for any $a \in A$ and $t \in T$);

2) if $\kappa(a,x)>\lambda(a)$ and $\kappa(a,y)<\kappa(a,x)$ for some $a \in A$, then no further rule applies;

3) if $b=\pi(y)>\pi(x)=a$, then there exists $t^{\circ} \in T$ such that $x(t^{\circ})=b$, $y(t^{\circ})=a$, and $y(t)=x(t)$ for all $t \neq t^{\circ}$;

4) if $b=\pi(y)<\pi(x)=a$, then $[y(t)=c \Rightarrow x(t)=c]$ and $[x(t)=d \Rightarrow y(t)=d]$ for all $c<b<d$ and $t \in T$.

Rule 1) says that the players should not create new over-vetoing or add to existing one; rule 2), that if over-vetoing is actually diminished then nothing else matters. If the change does not touch over-vetoed outcomes, rules 3) and 4) prescribe economy of effort. To shift the outcome selected upwards, it is sufficient to remove just one ball from the outcome desired, so rule 3) demands that exactly one ball should be removed and put to the previously selected outcome. To shift the outcome downwards, one has to veto everything between the former and the new outcomes; however, there cannot be any need to put new balls below the desired outcome or remove balls from above it, so rule

4) prohibits such superfluous activity.

The explanation of the rules simultaneously proves condition (iii) from the definition at the end of Section 4 (it may be worthwhile to add that equality (2) implies that a ball removed from an over-vetoed outcome can always be put somewhere without creating new over-vetoing); condition (i) is included into the definition of S_i. Let us define the semi-potential proper.

I again avoid complicated logical formulas; S is lexicography over the three following orders. Comparing two outcomes, y and x, we look at their "total over-vetoing" first: if $\Sigma_{a\in A}$ max[$\kappa(a,x)$-$\lambda(a)$,0] > $\Sigma_{a\in A}$ max[$\kappa(a,y)$-$\lambda(a)$, 0], then ySx. In the case of equality, we invoke the second criterion. For every profile $x\in X$, we introduce notation $s(x)$=max{$a\in A$ | $\Sigma_{b\geq a}$ max[$\lambda(b)$-$\kappa(b,x)$,0] $\geq$ 2}, (assuming the maximum of an empty set to be min$A$), $B(x)$ = {$a\in A$ | $\lambda(a)\geq\kappa(a,x)$ & $a\geq s(x)$}, (not over-vetoed outcomes not below "the second blank"), $T(i,x)$ = $x_i^{-1}(B(x))$, $m_i(x)$ = $\#T(i,x)$, $m(x)$ = $<m_i(x)>_{i\in N}$, $v^i(x)$ = $<u_i(x(t)>_{t\in T(i,x)}$. Now, if $m(y)$ Pareto dominates $m(x)$, we set ySx. Finally, if $m(y)$=$m(x)$, then ySx if and only if $v^i(x)\mathcal{D}v^i(y)$ for all $i\in N$ and $v^i(x)\mathcal{D}^o v^i(y)$ for some $i\in N$ (where the relations $\mathcal{D}$ and $\mathcal{D}^o$ were defined at the end of Section 2).

Suppose yS_ix; we have to show ySx. If rule 2) is applicable, then the first criterion works; so we may assume that no over-vetoed outcome was touched. Let us denote a=$\pi(x)$, b=$\pi(y)$.

Suppose $b>a$ first; then rule 3) must be applicable. Everything above a is vetoed at x, so $s(x)\leq a$. To be more precise, $s(x)=a$ if $\lambda(a)>\kappa(a,x)+1$, and $s(x)<a$ if $\lambda(a)=\kappa(a,x)+1$; in either case, $s(y)=s(x)$, hence $B(y)=B(x)$ and $m(y)=m(x)$. Moreover, for all $j\neq i$ (where $t^o\in T_i$ for t^o from rule 3)) we have $v^j(y)=v^j(x)$, while $v^i(x)\mathcal{D}^o v^i(y)$ because $i\in I$, hence $u_i(b)>u_i(a)$. Therefore, ySx.

Suppose $b<a$; then rule 4) must be applicable. Since everything above b is

vetoed at y, $s(y)\leq b$. Consider four alternatives. If $\lambda(a)>\kappa(a,x)+1$, then $s(x)=a>s(y)$, so $B(x)\subset B(y)$, $m_i(y)\geq m_i(x)$ for all $i\in N$, and $m_i(y)>m_i(x)$ e.g. for every player i with $x(t)\neq y(t)=a$ for $t\in T_i$. If $\lambda(c)>\kappa(c,x)$ for $b<c<a$, then $s(x)\geq c>s(y)$, so $B(x)\subset B(y)$, $m_i(y)\geq m_i(x)$ for all $i\in N$, and $m_i(y)>m_i(x)$ e.g. for every player i with $x(t)\neq y(t)=c$ for $t\in T_i$. If both inequalities considered are wrong, a unique ball $t^o\in T_i$ $(i\in I)$ was shifted from somewhere $\leq b$ to a. If $x(t^o)<b$, then $\lambda(b)>\kappa(b,x)$, so $s(x)=b\geq s(y)$, hence $x(t^o)\notin B(x)$, hence m_i strictly increased. Finally, if $x(t^o)=b$, then $s(y)=s(x)$, hence $B(y)=B(x)$, hence $m(y)=m(x)$, but $v^j(y)=v^j(x)$ for $j\neq i$ while $v^i(x)\mathcal{D}^o v^i(y)$ because $u_i(b)>u_i(a)$.

The theorem is proved.

Example 6. Let $A=\{a,b,c,d,e\}$ arranged in the alphabetic order, $N=\{1,2,3\}$, all μ's and λ's be equal to 1; certainly, equality (2) is thus violated, but we may assume the presence of other, over-vetoed outcomes not touched in the following. Assume $u_1(c)>u_1(a)>u_1(b)$, $u_2(b)>u_2(a)>u_2(d)$, $u_3(d)>u_3(b)>u_3(c)$, and consider the following cycle: $x^o=<a,d,e>$, $x^1=<c,d,e>$, $x^2=<c,a,e>$, $x^3=<c,a,b>$, $x^4=<c,d,b>$, $x^5=<a,d,b>$, $x^6=x^o$; it is easily checked that x^k form an improvement cycle. On the steps from x^o to x^1 and from x^5 to x^o, rule 3) is not complied with - indirect vetoing is used instead.

Turning to our second mechanism, π_G, we start with formal definitions. A one-to-one mapping σ: $A\rightarrow A$ with a single orbit is fixed. For each $a\in A$, $0\leq k\leq \#A$-1, we define $\upsilon_k(a,x)=\Sigma_{s=0}^{k}[\lambda(\sigma^s(a))-\kappa(\sigma^s(a),x)]$ and $\upsilon(a,x)=\min\{\upsilon_k(a,x)\}$. It can be derived from (2), see Gol'berg and Gurvich (1986), that $\upsilon(a,x)>0$ (in fact, $\upsilon(a,x)=1$) for a unique $a\in A$ at each $x\in X$; $\pi_G(x)$ is this outcome.

An interpretation with the rotating of A was given above; note that whenever $\upsilon(a,x)\leq 0$, $-\upsilon(a,x)$ is the number of black balls that passes between a and its "leeward" neighbour, $\sigma^{-1}(a)$, during the rotation of balls. When it comes to improvement paths, it should be understood clearly that the rotation is only performed in imagination or simulated in a computer: the real balls remain where the players put them until the players themselves shift them elsewhere.

Theorem 8. The mechanism π_G always gives rise to a semi-potential game form.

As was noted above, the mechanism lacks a serious interpretation, which drastically diminishes the importance of the theorem. I restrict myself to an explicit formulation of a semi-potential and an informal sketch of the proof.

Let us start with the definition of a strict order S on X, the semi-potential proper. Denote $B(x)=\{a\in A \mid \upsilon(a,x)\geq 0\}$ ("outcomes over which no balls pass further leeward"), $T(i,x)=x_i^{-1}(B(x))$, $m_i(x)=\#T(i,x)$, $m(x)=<m_i(x)>_{i\in N}$, $v^i(x)=<u_i(x(t)>_{t\in T(i,x)}$. We set ySx if and only if $m(y)$ Pareto dominates $m(x)$, or if $m(y)=m(x)$, $v^i(x)\mathscr{D}v^i(y)$ for all $i\in N$ and $v^i(x)\mathscr{D}^o v^i(y)$ for some $i\in N$ (where the relations $\mathscr{D}$ and $\mathscr{D}^o$ were defined at the end of Section 2).

Now we define admissible improvements. Besides yP_Ix, two more conditions are imposed. Denote $a=\pi(x)$, $b=\pi(y)$, $T^*=\{t\in T \mid y(t)\neq x(t)\}$, $m^*=\#T^*$. The first condition is: $m^*\leq 1-\upsilon(b,x)$; the second: $a\in y(T^*)$. Interpretation: m^* is the total number of balls shifted; the first condition demands that the number of balls shifted should not exceed that of balls vetoing b (directly or indirectly); the second, that at least one ball should be shifted to a.

Let us show that condition (iii) from the definition of a semi-potential is satisfied. Let yP_Ix, $a=\pi(x)$ and $b=\pi(y)$; since $\upsilon(b,x)\leq 0$ and $\upsilon(b,y)=1$, coalition I was able to remove $1-\upsilon(b,x)$ balls from "windward" of b. At the equivalent admissible improvement, it shifts only these balls, fulfilling the first condition as an equality. Further, the $-\upsilon(b,x)$ balls that passed from b "leeward" during the rotating procedure at x, found empty places for themselves no later than at a, and just one empty place at a was still left - otherwise, $\upsilon(a,x)=1$ would be impossible. Now we put one ball to a and $-\upsilon(b,x)$

others to those empty places, denoting the new profile z; clearly, $\upsilon(b,z)=1$, so $\pi(z)=b=\pi(y)$.

Finally, suppose $yS_I x$ and show ySx. As is easily understood, the shift from x to y must be organized exactly as described in the previous paragraph. Consider two alternatives.

Suppose $\upsilon(b,x)<0$ first. Then $b\notin B(x)$, and every ball of $1\text{-}\upsilon(b,x)$ that were shifted was removed from $c\notin B(x)$ (otherwise, the ball would have been shifted without necessity and then, since the total number of balls shifted is just enough, $\upsilon(b,y)>0$ would be impossible); as to their new positions, at least one of them went to $a\in B(y)$, while the others cannot "spoil" their new positions because they occupy places that are filled during the rotating procedure at x anyway (to be more formal, $[\kappa(c,y)>\kappa(c,x)\ \&\ c\in B(x)] \Rightarrow c\in B(y)$). Thus, ballots cast to outcomes from $B(x)$ remain in $B(y)$, while at least one ball shifts from outside $B(x)$ to inside $B(y)$; therefore, $m(y)$ Pareto dominates $m(x)$.

If $\upsilon(b,x)=0$, just one ball is shifted from b to a. Since $B(x)=B(y)$, we have $m(x)=m(y)$; now the last term in the definition of S works because $u_i(b)>u_i(a)$ for all $i\in I$, in particular, for the player who actually shifted the ball.

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