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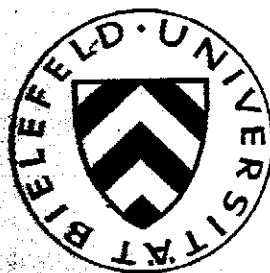
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## **Symmetries of Games with Public and Private Objectives**

by

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# Symmetries of Games with Public and Private Objectives<sup>†</sup>

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## Abstract

If a game is represented as a combination of simpler games, it seems natural to expect a connection between symmetries exhibited by the whole game and by its components. An exact analysis of such a situation is given for games with public and private objectives under additional assumption that the strategy sets are finite and all the players use the same aggregation function, which is strictly increasing in each variable. It turns out that each symmetry of a PP-game comes from symmetries of its components, but the converse need not be true. However, the group of motions of the whole game is determined by the groups of motions of its constituent components.

*Key words:* Strategic Games, Public and Private Objectives, Symmetries.

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## 0. Introduction

Although the idea of symmetry appears to have emerged in pre-historic times, its exact definitions in particular contexts are still being given. Quite recently, Peleg, Rosenmüller, and Sudhölter (1996) suggested a general definition of symmetry for normal-form and extensive games.

The purpose of this paper is to apply the new definition to a particular context of games with public and private objectives (henceforth, PP-games). This class of games characterized by a specific structure of utility functions emerged in Germeier and Vatel' (1974), Kukushkin (1992, 1994), and is remarkable for the existence of equilibria. In a sense, every PP-game is a combination of its two components, public and private, each of which is a strategic game with rather simple utilities.

Theorem 1 below shows that the group of motions of a PP-game is determined by the group of motions of its constituent components. For symmetry groups, the situation is a bit more complicated: each symmetry of a PP-game comes from symmetries of its components, but the converse need not be true.

The results obtained and the technique developed may be regarded as just a first step in the analysis of what happens to symmetries when several games are "combined" (in some sense) into a new one.

## 1. General Definitions

A general definition of a PP-game sounds as follows. There is a finite set of players  $N$ ; for each  $i \in N$ , there are a strategy set  $X_i$  and a (private) function  $f_i: X_i \rightarrow \mathbb{R}$ ; there is a (public) function  $f_0: X \rightarrow \mathbb{R}$ , where  $X = \prod_{i \in N} X_i$ ; for each  $i \in N$ , there is an aggregator, a function  $F_i: \mathbb{R}^2 \rightarrow \mathbb{R}$  increasing in each argument. Now the utility function of the game is defined as

$$u_i(x) = F_i(f_0(x), f_i(x_i)).$$

**Remark.** Without the monotonicity of  $F_i$ , every strategic game (with no

more than continuum of strategies) could be represented as a PP-game.

Such games will be considered here under additional assumptions, namely, each set  $X_i$  is finite, all the functions  $F_i$  are the same and it strictly increases in each variable. Actually, we will regard the aggregator as a kind of binary operation and denote  $\circ$ , so

$$u_i(x) = f_0(x) \circ f_i(x_i). \quad (1)$$

The assumed strict monotonicity implies cancellation properties:  $a \circ b = a \circ c$  implies  $b = c$  and  $a \circ b = c \circ b$  implies  $a = c$ .

With every PP-game  $G$ , two games,  $G^{\text{PB}}$  and  $G^{\text{PR}}$ , are associated. Each of them has the same set of players and the same strategies; in  $G^{\text{PB}}$ , we have  $u_i^{\text{PB}}(x) = f_0(x)$ ; in  $G^{\text{PR}}$ ,  $u_i^{\text{PR}}(x) = f_i(x_i)$ .

## 2. Motions

In the sequel, we will use definitions and notations from Peleg et al. (1996), in particular, denoting  $\mathcal{M}(G)$  the set of motions of  $G$  and  $\Sigma(G)$  the set of its symmetries. Obviously, the games  $G$ ,  $G^{\text{PB}}$  and  $G^{\text{PR}}$  have the same preform, so their motions are members of the same set.

**Theorem 1.**  $\mathcal{M}(G^{\text{PB}}) \cap \mathcal{M}(G^{\text{PR}}) = \mathcal{M}(G)$ .

The inclusion of the left-hand side into the right-hand side is quite obvious (it does not even need the strict monotonicity of  $\circ$ ). It also easily follows from the cancellation properties that  $\mathcal{M}(G^{\text{PR}}) \cap \mathcal{M}(G) \subseteq \mathcal{M}(G^{\text{PB}})$ . Therefore, we only need to prove  $\mathcal{M}(G) \subseteq \mathcal{M}(G^{\text{PR}})$ .

The following notion will be very useful. Let  $a = \langle a_1, \dots, a_m \rangle$  and  $b = \langle b_1, \dots, b_m \rangle$  be two corteges (vectors) of reals. We write  $a \approx b$  when  $a_i \leq a_j$  if and only if  $b_i \leq b_j$  for all  $i, j \in \{1, \dots, m\}$ ; this is an equivalence relation. If  $\lambda \in \mathbb{R} \setminus \{0\}$ , then  $\lambda a \approx \lambda b$  if and only if  $a \approx b$ .

Now for every motion  $(\pi, \varphi) \in \mathcal{M}(G)$ , player  $i \in N$ , and strategy  $x_i^0 \in X_i$ , we have to show that

$$f_i(x_i^0) = f_{\pi(i)}(\varphi_i(x_i^0)). \quad (2)$$

For more convenient notations, we will usually assume  $i=1$ .

As is well known, every permutation of a finite set can be decomposed into cycles. We consider two basic cases.

Case 1. Let  $\pi=(1)(2)\dots$ , i.e. there are two players each matched to himself; the rest does not matter.

Pick  $x^0 \in X$  arbitrarily and define the sequence  $x^k$  by  $x^{k+1}=\varphi(x^k)$ . Then we define  $y^0$  by  $y_1^0=x_1^1(=\varphi_1(x_1^0))$ ,  $y_i^0=x_i^0$  for  $i \neq 1$  and the sequence  $y^k$  by  $y^{k+1}=\varphi(y^k)$ . We may assume  $x^0 \neq y^0$  because (2) for  $i=1$  is already true otherwise.

Since  $X$  is finite and  $\varphi$  is a bijection, there is an integer  $m$  for which  $x^m=x^0$  and  $y^m=y^0$ . In the following formulas we assume  $(m-1)+1=0$ .

For every  $k=0,1,\dots,m-1$ , we have  $u_i(x^k)=u_i(x^{k+1})$  and  $u_i(y^k)=u_i(y^{k+1})$  for  $i=1,2$ , which means

$$f_0(x^k) \circ f_1(x_1^k) = f_0(x^{k+1}) \circ f_1(x_1^{k+1}) \quad (3)$$

$$f_0(y^k) \circ f_1(x_1^{k+1}) = f_0(y^{k+1}) \circ f_1(x_1^{k+2}) \quad (4)$$

$$f_0(x^k) \circ f_2(x_2^k) = f_0(x^{k+1}) \circ f_2(x_2^{k+1}) \quad (5)$$

$$f_0(y^k) \circ f_2(x_2^k) = f_0(y^{k+1}) \circ f_2(x_2^{k+1}) \quad (6)$$

From (5), we have  $\langle f_0(x^k) \rangle \approx \langle f_2(x_2^k) \rangle$ ; from (6),  $\langle f_0(y^k) \rangle \approx \langle f_2(x_2^k) \rangle$ ; hence  $\langle f_0(x^k) \rangle \approx \langle f_0(y^k) \rangle$ . Similarly, from (3) and (4) we have  $\langle f_0(x^k) \rangle \approx \langle f_1(x_1^k) \rangle$  and  $\langle f_0(y^k) \rangle \approx \langle f_1(x_1^{k+1}) \rangle$ . Combining all that, we obtain  $\langle f_1(x_1^k) \rangle \approx \langle f_0(x^k) \rangle \approx \langle f_0(y^k) \rangle \approx \langle f_1(x_1^{k+1}) \rangle$ , i.e. a cyclical shift does not change the order relations on the set  $\langle f_1(x_1^k) \rangle$ . Clearly,  $f_1(x_1^k)$  does not depend on  $k$ . And this is exactly (2) for  $i=1$ .

Case 2. Let  $\pi=(1,2,\dots,s)\dots$  with  $s \geq 2$ , i.e. player 1 is involved in a non-trivial cycle.

Pick  $x_1^0$  and  $x_i^0$  for  $i \notin \{2,\dots,s\}$  arbitrarily and define  $x_i^k = \varphi_{i-1}(x_{i-1}^k)$  for  $i=2,\dots,s$ ,  $x_1^{k+1} = \varphi_s(x_s^k)$  for all  $k=0,1,\dots$ . Now we define  $y^0$  by  $y_i^0 = x_i^0$

for all  $i \in N$  and  $y^{k+1} = \varphi(y^k)$  for all  $k=0,1,\dots$ . Each step from  $y^k$  to  $y^{k+1}$  is accompanied with a single change, at most, in the coordinates  $y_i^k$  for  $i \in \{1, \dots, s\}$ ; more precisely, we have  $y_i^k = x_i^{(k+s-i) \text{ div } s}$  for all  $i=1, \dots, s$  and  $k=0,1,\dots$ . If it happens that  $x_1^1 = x_1^0$ , then  $y_i^k$  does not depend on  $k$  at all (for  $i=1, \dots, s$ ), but there is no necessity to treat this case differently.

Since  $X$  is finite and  $\varphi$  is a bijection, there is an integer  $m$  for which  $y^m = y^0$ . If  $x_1^1 \neq x_1^0$ ,  $m$  must contain  $s$ ; otherwise, we may repeat the cycle (determined by the other players)  $s$  times if needed and obtain  $m = r \cdot s$  in any case. In the following, we assume  $(m-1) \div s = 0$ .

By the symmetry conditions, we have

$$\begin{aligned} v &= f_0(y^0) \circ f_1(x_1^0) = f_0(y^1) \circ f_2(x_2^0) = \dots = f_0(y^{s-1}) \circ f_s(x_s^0) = f_0(y^s) \circ f_1(x_1^1) = \dots \\ w &= f_0(y^0) \circ f_2(x_2^0) = f_0(y^1) \circ f_3(x_3^0) = \dots = f_0(y^{s-1}) \circ f_1(x_1^1) = f_0(y^s) \circ f_2(x_2^1) = \dots \end{aligned}$$

Applying the cancellation property to expressions from different lines with identical terms  $f_i(x_i^k)$ , we obtain  $\langle v, w \rangle \approx \langle f_0(y^{k+1}), f_0(y^k) \rangle$  for all  $k=0,1,\dots,m-1$ . Therefore,  $f_0(y^k)$  does not depend on  $k$ ; then from the first equality we have  $f_1(x_1^0) = f_2(x_2^0) = f_{\pi(1)}(\varphi_1(x_1^0))$ .

If among the other players there is a player matched by  $\pi$  to himself, i.e.  $\pi = (1, 2, \dots, s) \dots (j) \dots$ ; we have  $f_0(y^0) \circ f_j(x_j^0) = f_0(y^1) \circ f_j(x_j^1)$ , hence  $f_j(x_j^0) = f_j(x_j^1) = f_{\pi(j)}(\varphi_j(x_j^0))$ .

To finish with the theorem, we only have to observe that the cases considered cover all possibilities. If player  $i$  is involved in a non-trivial cycle, then he is exactly in the same position as player 1 in Case 2. If player  $i$  is matched to himself and there is a non-trivial cycle among the other players, then player  $i$  is in the same position as player  $j$  in the previous paragraph. Finally, if every player is matched to himself, Case 1 applies. Theorem 1 is proved.

Without the cancellation properties, Theorem 1 would not be true.

**Example 1.** Let  $N=\{1,2\}$ ,  $X_i=\{1,2\}$ ,  $F_i(v_0, v_i)=\min\{v_0, v_i\}$  ( $i \in N$ ) and the functions  $f_0, f_i$  are given by the following matrices (player 1 chooses rows, player 2, columns):

$$f_0: \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad f_1: \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad f_2: [1 \quad 5].$$

Then we have the bi-matrix game (the first number is the utility of player 1, the second, of player 2):

$$\begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{bmatrix}$$

Clearly, the game is symmetric but its private component is not.

### 3. Symmetries

Before we go to symmetries, a terminological note is in order. Peleg et al. (1996) defined symmetries as a quotient group, which does not allow us to speak of  $\Sigma(G^{\text{PB}}) \cap \Sigma(G^{\text{PR}})$  even though both games have the same preform. However, we may notice that the projection to the first coordinate,  $p(\pi, \varphi) = \pi$ , maps  $\mathcal{M}(G)$  into  $\Sigma(N)$  and the kernel of this homomorphism is exactly the group of impersonal motions  $\mathcal{A}(G)$ ; therefore, its image,  $p(\mathcal{M}(G))$ , is isomorphic to  $\mathcal{M}(G)/\mathcal{A}(G) = \Sigma(G)$ . Now we may identify  $\Sigma(G)$  with this image, thus assuming  $\Sigma(G) \subseteq \Sigma(N)$ .

**Theorem 2.**  $\Sigma(G) \subseteq \Sigma(G^{\text{PB}}) \cap \Sigma(G^{\text{PR}})$ .

Easily follows from Theorem 1 and from the observation that the homomorphism  $p$  is defined by the preform rather than by the game itself.

The opposite inclusion, trivially true in the case of motions, is not at all true here. Consider the following example.

**Example 2.** Let  $N=\{1,2\}$ ,  $X_i=\{1,2\}$ ,  $F_i(v_0, v_i)=v_0+v_i$  ( $i \in N$ ) and the functions  $f_0, f_i$  are given by the following matrices (player 1 chooses rows,

player 2, columns):

$$f_0: \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad f_1: \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad f_2: [0 \quad 1].$$

Then we have the bi-matrix game (the first number is the utility of player 1, the second, of player 2):

$$\begin{bmatrix} (4,3) & (1,1) \\ (0,0) & (2,3) \end{bmatrix}.$$

Both components are symmetric, but the game itself is not.

### References

Germeier Yu.B., Vatel' I.A., 1974, On Games with a Hierarchical Vector of Interests, *Izvestiya Akademii Nauk SSSR. Tekhnicheskaya Kibernetika*, #3, 54-69 (in Russian; English translation in *Engineering Cybernetics*, V.12).

Kukushkin N.S., 1992, On Existence of Stable and Efficient Outcomes in Games with Public and Private Objectives, *International Journal of Game Theory* 20, 295-303.

Kukushkin N.S., 1994, A Condition for the Existence of a Nash Equilibrium in Games with Public and Private Objectives, *Games and Economic Behavior* 7, 177-192.

Peleg B., Rosenmüller J., Sudhölter P., 1996, The Canonical Extensive Form of a Game Form. Part I - Symmetries. University of Bielefeld, Institute of Mathematical Economics. Working Paper No. 253.