

03

# INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

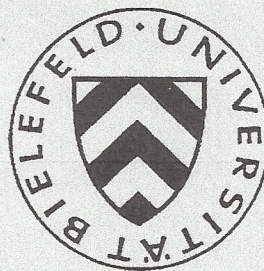
No. 295

## Adjusted Winner: An Algorithm for Implementing Bargaining Solutions in Multi-Issue Negotiations

by

Matthias G. Raith and Andreas Welzel

February 1998



University of Bielefeld

33501 Bielefeld, Germany

SE 050  
U5 B5I  
295

# Adjusted Winner: An Algorithm for Implementing Bargaining Solutions in Multi-Issue Negotiations

February 1998

Matthias G. Raith

Institute of  
Mathematical Economics  
University of Bielefeld  
P.O. Box 100131  
D-33501 Bielefeld  
Germany

MRaith@wiwi.uni-bielefeld.de

Andreas Welzel

Interdisciplinary Center for Research  
and Development in Higher Education  
University of Bielefeld  
P.O. Box 100131  
D-33501 Bielefeld  
Germany

Andreas.Welzel@post.uni-bielefeld.de

## Abstract

In this paper, we show that the procedure "Adjusted Winner," introduced by BRAMS AND TAYLOR (1996), implements the Kalai-Smorodinsky bargaining solution for a specific class of fair-division problems. By acknowledging this relationship, we generalize the algorithm in order to address a wider spectrum of bargaining problems. We are not only able to loosen the restriction on parties' preferences, but can also consider the effect of outside options on fair distributions. Moreover, we show that Adjusted Winner can easily be modified to implement alternative solutions, such as the Nash bargaining solution. Our approach combines formal reasoning with plausible argumentation, which is essential for the acceptance of theoretical solution concepts in real-life negotiations.

**Keywords:** Fair Division, Adjusted Winner, Kalai-Smorodinsky Solution

We are grateful to Steven Brams and Walter Trockel for critical comments and discussions.

## 1. Introduction

Research in negotiation analysis typically addresses practical questions of negotiation that mathematical bargaining theory has either deliberately excluded or has not been able to cope with. The main difficulty lies in the reconciliation of game theorists' formal modes of reasoning on the one side and negotiators' real-life problems on the other side. Major developments in the field of negotiation analysis over the past two decades are documented by the work of RAIFFA (1982, 1997).

Since bargaining always involves the distribution of benefits or costs, aspects of fair division are closely related, as is pointed out by YOUNG (1991, 1994). Although many methods of fair division have a very long tradition, a formal analysis of procedures is much more recent. BRAMS AND TAYLOR (1996) provide an assessment of fair division procedures, ranging from extremely simple practical approaches to quite sophisticated cake-cutting techniques with a more theoretical appeal. In their book, they introduce a new procedure labelled "Adjusted Winner" which has the remarkable property of offering two parties a division of multiple issues that is efficient, envy-free, and equitable (or egalitarian). Moreover, the procedure is practical since it requires only minor computational effort.

Adjusted Winner in its basic form is designed for a specific type of bargaining problem between two players over multiple issues. It assumes that players have linear, additively separable preferences over all issues. Additivity of preferences is a restrictive, but quite standard assumption that can be dealt with in various ways (cf. KEENEY AND RAIFFA (1991)). Linearity, however, is more difficult to justify, in particular when the issues in a negotiation involve several options. If preferences are not linear, efficiency, in general, will not hold under Adjusted Winner. In addition, the implied distribution rule for individual issues is based on a winner-take-all assumption, which suggests that the negotiated issues are viewed as individual goods: If a good is given to one party, the other party receives nothing.

Our objective in this paper is to generalize Adjusted Winner in order to address a wider spectrum of practically relevant bargaining problems. We not only loosen the restriction on parties' preferences, but also consider the effect of outside op-

tions on fair distributions. By explicitly formulating the underlying algorithm as a substitution process along the efficiency frontier, we reveal that the outcome of Adjusted Winner shares the same properties as the axiomatic solution of KALAI AND SMORODINSKY (1975), which was derived for more general bargaining problems.<sup>1</sup> We show, however, how Adjusted Winner can easily be modified to implement alternative axiomatic solutions for bilateral bargaining problems.

Despite all extensions, the computational steps involved are plausible and well manageable. Indeed, the individual steps contain all the components that are necessary for a cooperative negotiation process: First, there is the joint effort by both parties to attain efficiency without haggling over issues; second, there is a compensation between players, which is achieved at minimal cost in order to maintain efficiency; and third, there is the mutually accepted norm which legitimizes the final outcome. Instead of simply implementing a cooperative solution, Adjusted Winner thus encompasses the arguments which are crucial for the acceptance of a formal procedure in an actual negotiation process.

Our approach utilizes an additive scoring procedure for calculating utilities by weighting issues and valuing options, which is quite common in negotiation analysis. According to RAIFFA (1982), it was first introduced in the Panama Canal negotiations in 1974. For practical negotiators the issue-option characterization is of great relevance, since this is the form in which most complex negotiations are structured. The fact that this utility representation is often used for descriptive analyses of actual negotiations indicates that it can also be regarded as a legitimate approximation of parties' preferences for many multi-issue negotiations. Players are assumed to have piecewise linear, additively separable preferences. The scoring procedure is, thus, well suited for applying formal models of bargaining, which are typically based on more general assumptions concerning players' utilities.

We begin in Section 2 by characterizing Adjusted Winner as it was introduced

---

<sup>1</sup>In condensed form, MOULIN (1984) states that the Kalai-Smorodinsky solution amounts to normalizing players' utilities to a range between 0 for the worst and 100 for the best outcome, and then selecting an efficient allocation that equalizes the relative gains of cooperation.

by BRAMS AND TAYLOR (1996). We divide the underlying algorithm into three fundamental steps. Step 1 locates an initial outcome on the efficiency frontier. Step 2 then characterizes the substitution process along the efficiency frontier. Here we deviate from the characterization of BRAMS AND TAYLOR (1996) by decomposing issues into individual options in order to explicitly calculate substitution rates between options. Finally, step 3 imposes the equilibrium (equitability) condition for the final outcome. This stopping condition for the adjustment process is shown to correspond to the Kalai-Smorodinsky solution.

In Section 3, we consider a more general class of bargaining problems consisting of multiple issues with more than two efficient options, over which players have piecewise linear preferences. We modify step 2 of Adjusted Winner to take non-linear preferences of this type into account, so that the adjusted form of Adjusted Winner, again, implements the Kalai-Smorodinsky solution.

In Section 4, we consider the possibility that players may have valuable alternatives (i.e. non-zero outside options) to negotiation. This is considered to be a major source of bargaining power, since it shifts the status quo of negotiation away from the origin and, thus, affects the bargaining problem. By reformulating the equitability condition of Adjusted Winner (step 3), we show how outside options influence the structural bargaining power of players and, thereby, the equitable outcome. We modify Adjusted Winner in order to implement the Kalai-Smorodinsky solution, but we also offer a computationally simpler alternative that approximates this solution.

In Section 5, we demonstrate that the equilibrium condition for the adjustment process can also be adapted to the geometric properties of the NASH (1950) bargaining solution. Again, this requires a modification of step 3 only. Although debatable as a fair-division allocation, we find that the Nash bargaining solution does have practical advantages over the Kalai-Smorodinsky solution: As a stopping condition for the adjustment process along the efficiency curve, the Nash solution requires less computational effort when players have outside options. Moreover, it often induces a discrete outcome and does not require a convex combination of efficient options, which may be difficult to implement. For practitioners with time restrictions, these

computational and interpretational aspects are highly relevant, since the most intriguing theoretical solution concept loses its appeal if it is difficult to implement.

We conclude in Section 6 with some procedural implications of our extended version of Adjusted Winner.

## 2. Adjusted Winner

We begin by describing the procedure “Adjusted Winner,” introduced by BRAMS AND TAYLOR (1996). Our exposition, however, is somewhat different. The algorithm is designed for a bargaining problem between two parties,  $a$  and  $b$ , over the division of  $n$  divisible goods.

Players’ preferences over the  $n$  required divisions are characterized by utility functions  $u^x : [0, 1]^n \rightarrow \mathbb{R}$ ,  $x = a, b$ , where  $u^x$  is assumed to be linear on the  $n$ -tuple of divisions. In particular, this implies that players’ utilities are additively separable across issues. Suppose that the standard of value for both players is the aggregate over all negotiated issues. Players’ valuation of individual issues can then be expressed in relation to the standard of value which we normalize to 100 (utility) points.

For illustrative purposes, we characterize Adjusted Winner for the case  $n = 3$ . Consider a bargaining problem composed of three goods or, more generally, three issues A, B, and C, with players’ utilities over all issues given by  $u^x = u_A^x + u_B^x + u_C^x$ ,  $x = a, b$ . Assume that Player  $a$  distributes his 100 utility points across all three issues, such that his maximum utility levels are given by  $u_A^a = 30$ ,  $u_B^a = 20$ , and  $u_C^a = 50$ . Player  $b$ ’s values for the same three issues are  $u_A^b = 60$ ,  $u_B^b = 20$ , and  $u_C^b = 20$ .

Underlying the algorithm of Adjusted Winner is a winner-take-all assumption. Hence, if Player  $a$  wins on an issue, then Player  $b$  will receive nothing, and vice versa. We make this explicit by introducing two distinct options for each issue. Option 1 makes Player  $a$  the winner of the issue, and option 2 Player  $b$ . Players’ assessments,  $u_X^a$  and  $u_X^b$  ( $X = A, B, C$ ) of the two discrete options for each issue are given in the three panels of Table 1.

		A	
		1	2
$u_A^a$		30	0
$u_A^b$		0	60

		B	
		1	2
$u_B^a$		20	0
$u_B^b$		0	20

		C	
		1	2
$u_C^a$		50	0
$u_C^b$		0	20

Table 1: Additively separable preferences over three issues

Since there are two distinct agreements to each issue, there are  $2^3 = 8$  possible discrete agreements over all three issues together. Their values to both players,  $u^x \equiv u_A^x + u_B^x + u_C^x$ ,  $x = a, b$ , are given in Table 2. In Figure 1, the 8 hollow points characterize the agreements of Table 2 in utility space.

$u^a$	100	50	80	30	70	20	50	0
$u^b$	0	20	20	40	60	80	80	100

Table 2: Players' valuations of  $2^3 = 8$  possible agreements

Due to the assumption that both players have linear preferences over the three divisible issues, their marginal utilities between individual options are constant. In Figure 1, the solid line then denotes the efficiency frontier of the bargaining set, which is necessarily convex.

We now apply "Adjusted Winner" in order to obtain an efficient, envy-free, and equitable solution to this bargaining problem. The algorithm consists of three fundamental steps. Step 1 is a distribution scheme, which locates an initial outcome on the efficiency frontier. Step 2 is a substitution procedure, which determines the movement along the efficiency frontier. And Step 3 imposes a stopping rule as an equilibrium condition, which characterizes the final efficient distribution.

The first step assigns each issue that is valued differently by the two players to the player who values it most. Consequently, issue A goes to player  $b$  and issue C to player  $a$ . Issue B is valued the same by both players. In this case, the algorithm

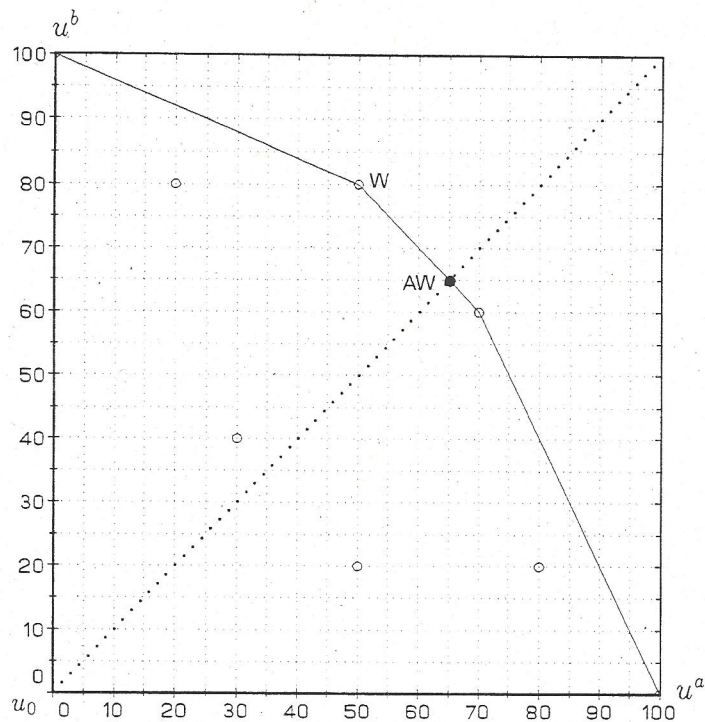


Figure 1: Adjusted Winner for of a three-issue negotiation

of Adjusted Winner prescribes to assign issue B to the player who already has the most points, in order to preserve his lead.<sup>2</sup> Thus, issue B goes to player  $b$ , since he has more points than player  $a$ . This results in the outcome A2, B2, and C1, denoted by point W in Figure 1. Since the assignment procedure ensures that there are no mutually beneficial trades, outcome W is guaranteed to be efficient. At this point, the temporary winner is Player  $b$  with 80 points versus Player  $a$ 's 50 points.<sup>3</sup>

Step 1 can be accomplished by having players submit their point allocations as sealed bids, which are opened by either a mediator or by both players together. We

---

<sup>2</sup>Technically, any division of issue B would do just as well. This does not apply, however, to more general bargaining problems, as we will see in the next section. If both players have the same number of points, then issue B can be allocated by simply tossing a coin.

<sup>3</sup>In our example, the outcome of step 1 is envy-free, since each player receives what he perceives to be at least 50% of the pie and neither player would, therefore, want to switch packages. Envy-freeness is, however, not necessarily implied by step 1.



assume that players' point allocations are truthful, but we discuss the possibility of strategic misrepresentation of preferences in Section 6.

The second step is the adjustment phase, where some of player  $b$ 's gain is shifted to player  $a$  until the division is equitable, i.e. until both players enjoy equal gains. The transfer is achieved by dividing one of the issues on which Player  $b$  wins. Since step 1 has already produced an efficient outcome, the adjustment process of step 2 is designed to preserve this feature.

With our modification in Table 1 that decomposes issues into options, the transfer is efficient, i.e. on the efficiency curve, if the rate of substitution between both players' gains is minimal. This minimizes player  $b$ 's cost of transferring gains to player  $a$ . For issue A, the rate of substitution is

$$RS_{A2,A1} := -\frac{u_{A2}^b - u_{A1}^b}{u_{A2}^a - u_{A1}^a} = 2,$$

while for issue B it is

$$RS_{B2,B1} := -\frac{u_{B2}^b - u_{B1}^b}{u_{B2}^a - u_{B1}^a} = 1,$$

where  $u_{Yi}^x$  denotes the points that player  $x = a, b$  receives for option  $i=1,2$  of issue  $Y=A,B,C$ . Since  $RS_{B2,B1} < RS_{A2,A1}$ , an efficient transfer is accomplished if issue B is divided before issue A. Passing issue B completely to player  $a$  implies a switch from option B2 to B1. This leaves Player  $b$  with only 60 points compared to Player  $a$ 's 70.

Step 3 imposes the stopping condition for the adjustment process of step 2. Under Adjusted Winner, the objective of transferring gains is to induce equitability between players. The adjustment is finished as soon as players receive equal gains.

It is possible to view the equitability condition of step 3 from a somewhat different perspective. Formally, the stopping condition requires the ratio of players' gains,

$$\gamma := \frac{u^b}{u^a} \equiv \frac{u_A^b + u_B^b + u_C^b}{u_A^a + u_B^a + u_C^a},$$

to be equal to unity, i.e.  $\gamma = 1$ . However, as BRAMS AND TAYLOR (1996) note "Adjusted Winner can be modified to reflect unequal shares to which the parties might be entitled."<sup>4</sup> The question then is: What determines players' entitlements?

<sup>4</sup>cf. BRAMS AND TAYLOR (1996), p. 70, footnote 8.

In their characterization of Adjusted Winner, BRAMS AND TAYLOR (1996) refer to players' *percentage* gains. Therefore, if equitability implies that players enjoy equal gains relative to their standards of value, i.e.  $u^a/100 = u^b/100$ , then the condition for stopping the adjustment is when players' gain ratio  $\gamma$  is equal to the ratio of their standards of value,

$$\epsilon := \frac{100}{100}.$$

Of course, the condition  $\gamma = \epsilon$  leads to the same allocation as the condition  $\gamma = 1$ , if players share the same standard of value. However, as we will see later, this need not be the case. For differing standards of value, the more general stopping condition,  $\gamma = \epsilon$ , then endogenously accounts for players' entitlements. We, therefore, refer to  $\epsilon$  as the 'entitlement ratio.'

In Figure 1, the equitable outcome under Adjusted Winner is indicated by point AW. It denotes the intersection of the Pareto frontier with the dotted 45°-line. Convexity of the efficiency frontier implies that the efficient and equitable outcome is envy-free as well.

In our example, equitability requires a convex combination of the alternative B1 and the (in step 1) temporarily chosen option B2; we denote their weights by  $\alpha$  and  $1 - \alpha$ , respectively. The equitable value of  $\alpha$  must then equate Player  $a$ 's share of the complete pie with that of Player  $b$ . This is determined by

$$\gamma = \frac{60 + [\alpha 0 + (1 - \alpha)20] + 0}{0 + [\alpha 20 + (1 - \alpha)0] + 50} = \frac{100}{100} = \epsilon,$$

which implies a value of

$$\alpha = \frac{3}{4}.$$

Adjusted Winner thus leads to an agreement consisting of A2, C1, and a compromise containing 75% of B1 and 25% of B2. This implies that players receive equitable shares of  $u^a = u^b = 65$ . A characteristic feature of the procedure is that it requires a division between the two options of one single issue only. This aspect becomes increasingly valuable as the number of issues rises.

It is important to note that the winner-take-all assumption, introduced above, implies that both players are not only supposed to value issues differently, but also

to have diametrically opposed interests. However, due to our specific approach, none of the three steps of Adjusted Winner makes use of this assumption. Our decomposition of issues into options and the introduction of substitution rates show that it is only the difference between a player's utilities and not the level of his utility which is relevant for the efficient transfer in step 2. So, in general, it is not necessary that players' less valued options are given 0 points. What matters are players' gains. Consequently, players do not need to have diametrically opposed interests, e.g. as in our example in Table 1. This generalization becomes crucial when players are negotiating over issues instead of goods. The options are then possible realizations over which players have differing, but not necessarily completely opposing views.

Since Adjusted Winner induces an efficient, equitable outcome for a convex bargaining set, where all issues together form the standard of value, the solution features the same characteristics as the axiomatic bargaining solution of KALAI AND SMORODINSKY (1975). Adjusted Winner, thus, provides an algorithmic implementation of the Kalai-Smorodinsky solution for bargaining problems based on issues, over which players have linear, additively separable preferences.

### 3. Adjusted Winner for Non-linear Preferences

We now extend our analysis to a class of bargaining situations, where players have *non-linear*, additively separable preferences over  $n$  divisible issues. This is characteristic for negotiations over issues (rather than goods) that consist of a variety of discrete options. With only a finite number of discrete options to each issue, each player's utility can be characterized by an additive scoring system: Issues are weighted by distributing 100 points, and then the options of each issue are valued by giving the worst option 0 points and the best option the number of points assigned to the issue. Intermediate options are valued accordingly. Formally, we assume that players' preferences over the  $n$  required divisions are, again, characterized by utility functions  $u^x : [0, 1]^n \rightarrow \mathbb{R}$ ,  $x = a, b$ , with  $u^x \equiv u_1^x + u_2^x + \dots + u_n^x$ , but where the subutility functions  $u_i^x$ ,  $i = 1, \dots, n$ , are now assumed to be piecewise linear and concave.

Adhering strictly to the basic algorithm of Adjusted Winner is now likely to lead to inefficiency. In order to illustrate this argument, consider again a negotiation over three issues A, B, and C. Both players assign to each issue the same share of 100 points as before, but their preferences over options are now characterized by the points in Table 3. The only difference to Table 1 is the modification of panel A.

A				B			C		
	1	2	3		1	2		1	2
$u_A^a$	30	20	0	$u_B^a$	20	0	$u_C^a$	50	0
$u_A^b$	0	45	60	$u_B^b$	0	20	$u_C^b$	0	20

Table 3: Additively separable preferences over three issues

With issue A consisting of 3 and issues B and C each still having 2 discrete options, there are now 12 possible (discrete) agreements over issues A, B, and C, together. Their values to both players are given in Table 4. In Figure 2, the 12 possible agreements of Table 4 are plotted (as hollow points) in utility space. The solid line, again, characterizes the efficiency frontier.

$u^a$	100	50	80	30	90	40	70	20	70	20	50	0
$u^b$	0	20	20	40	45	65	65	85	60	80	80	100

Table 4: Players' valuations of  $3 \times 2 \times 2 = 12$  possible agreements

If parties were to strictly apply Adjusted Winner to this negotiation problem, they would focus exclusively on the weights attached to the issues. In Table 3, the implicit winner-take-all assumption simply blends out option A2, thus reducing the whole negotiation problem to that of Table 1. The inefficiency of this procedure is shown in Figure 2, where the outcome is denoted by AW. Due to the assumption of linear preferences, the basic form of Adjusted Winner does not exploit the values of

efficient intermediate agreements. In our example, this is the intermediate option A2.

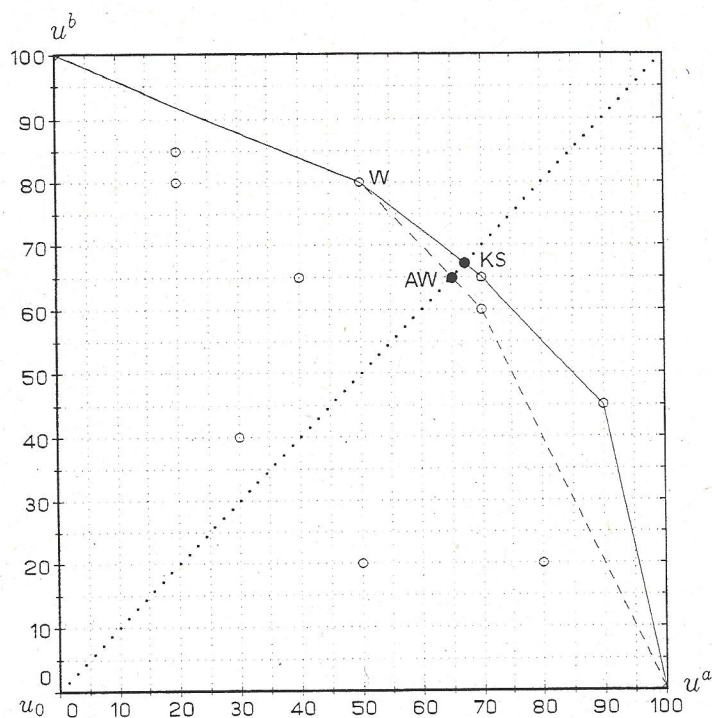


Figure 2: Adjusted Winner for a three-issue negotiation between players with nonlinear preferences

As Figure 2 shows, the efficient, equitable outcome is where the dotted 45°-line intersects the Pareto frontier. This point, denoted by KS, characterizes the Kalai-Smorodinsky solution. In order to implement this outcome with Adjusted Winner, one needs to modify the substitution process of step 2, which involves the following technical argument proved in the Appendix.

**Theorem:** *Let A be an issue with  $n > 2$  efficient divisible options, over which players have preferences characterized by piecewise linear and concave subutility functions  $u_A^x$ ,  $x = a, b$ . Issue A can then be decomposed into  $n - 1$  subissues (with two options each), over which players have linear, additively separable preferences, such that there are  $n$  efficient agreements over the  $n - 1$  subissues, which yield the same utilities as the  $n$  efficient options of issue A.*

If issue A in our example is decomposed according to the Theorem, then Adjusted Winner will recognize all efficient options, because the negotiation problem over issues B, C, and the two subissues of A has the same structure as that of the previous section. However, the Theorem implies that the decomposition of issue A is not necessary. Since the adjustment process of Adjusted Winner considers only the substitution rates between efficient options, the substitution rates can be calculated directly from the efficient options of issue A.

We illustrate the implication of the Theorem by applying the modified algorithm to our example. At the end of this section, we then collect the individual steps of Adjusted Winner in the form of a general recipe.

Step 1 of Adjusted Winner selects options A3, B2, and C1, again allocating 50 points to player *a* and 80 points to player *b*. The outcome is point W in Figure 2. Since Player *b* is the temporary winner of issues A and B, only these are considered for division, leaving issue C completely to player *a*.

In step 2, issue B still allows the substitution from B2 to B1, with a substitution rate of

$$RS_{B2,B1} = -\frac{u_{B2}^b - u_{B1}^b}{u_{B2}^a - u_{B1}^a} = 1.$$

Since issue A now has three options, there are two alternatives to A3 to consider. Substitution to A1 again implies

$$RS_{A3,A1} = -\frac{u_{A3}^b - u_{A1}^b}{u_{A3}^a - u_{A1}^a} = 2,$$

but there is now also the substitution between A3 and A2, yielding

$$RS_{A3,A2} = -\frac{u_{A3}^b - u_{A2}^b}{u_{A3}^a - u_{A2}^a} = \frac{15}{20} = \frac{3}{4}.$$

Although the substitution rate over complete issues is still lower for issue B than for A, i.e.  $RS_{B2,B1} < RS_{A3,A1}$ , the substitution rate over options is the lowest between options A3 and A2. We, therefore, leave issue B completely for player *b* and concentrate on the division between A3 and A2.

In order to equalize (relative) gains between players in step 3, we must determine the convex combination between options A2 and A3, such that players' gain ratio  $\gamma$

is equal to the entitlement ratio  $\epsilon$ :

$$\frac{[\alpha 45 + (1 - \alpha)60] + 20 + 0}{[\alpha 20 + (1 - \alpha)0] + 0 + 50} = \frac{100}{100}$$

$$\Leftrightarrow \alpha = \frac{6}{7} = 0.86 .$$

The equitable share for both players is then  $u^a = u^b = 67.14$ . This is illustrated by point KS in Figure 2. Players thus reach an efficient and equitable allocation giving issue B to player  $b$  and issue C to player  $a$ , together with a compromise consisting of 86% of A2 and 14% of A3.

The following instructions completely characterize our generalization of the algorithm Adjusted Winner:

1. For each issue that is weighted differently by the two players, choose as the *temporary option* the one that is best for the player who values this issue most. The summation of points determines the *temporary winner* and the *temporary loser*. If both players have the same number of points, let a referee (or simply the toss of a coin) determine the **temporary winner**. For each issue that is weighted the same by both players, now choose as the temporary option the one that is best for the temporary winner.
2. Consider all issues for which the temporary option is not the best option for the temporary loser. Calculate the **substitution rates** with respect to all alternative options that benefit the temporary loser. Select as the *alternative option* the one which yields the lowest substitution rate. If, under the alternative option, the temporary winner still has more points than the temporary loser, then make the alternative option the new temporary option and repeat step 2; otherwise proceed with step 3.
3. Determine the convex combination between the temporary option and the alternative option that satisfies the **stopping condition**.

The individual steps embedded in the structure of the algorithm contain the necessary components of a cooperative negotiation process: The joint effort to attain

efficiency, the efficient compensation between players, and the mutually accepted allocative norm.

For negotiations with many issues that have several options, the most tedious part of the algorithm appears to be step 2. Note, however, that the number of issues to consider is already reduced in step 1. The more this step equalizes players' utilities the less adjustment is needed in step 2. In step 2, the substitution rates only have to be ranked. This requires less computational effort than a precise calculation. Moreover, with every iteration of step 2, there are only a few additional substitution rates to consider, since only one option is changed. Indeed, only step 3 requires a bit of algebra for the stopping condition.

In the following sections, we consider modifications of the stopping condition in step 3. The general algorithm, therefore, remains the same.

#### 4. Adjusted Winner for Outside Alternatives

Fairness of a distribution depends on the status quo of the parties involved. It is well-known in bargaining theory that parties' alternatives to negotiation can have a significant influence on the outcome of bargaining.<sup>5</sup> Indeed, the outside alternatives are considered to be a major determinant of structural bargaining power, i.e. the power which is determined by the bargaining problem and not the players' bargaining abilities. Consequently, any concept of fair division should take these structural aspects into account.

Assume, for example, that players negotiate over the three issues given in the previous section, but that player  $a$  now has an alternative to negotiating with player  $b$ , e.g. an opportunity provided by a third party. Compared with the standard of value, this outside option is worth a total of 50 points for player  $a$ . Player  $b$ , however, has no alternative and, therefore, can only achieve 0 points outside of this negotiation. The effect is that the players' status quo point shifts away from

---

<sup>5</sup>Depending on the specific literature, the *status quo* is referred to under various names, such as the *disagreement point*, the *outside option*, or the *best alternative to a negotiated agreement (BATNA)*. As a compromise, we often use the label *outside alternative*.



the origin. This is illustrated in Figure 3, where  $u_0 = (50, 0)$  denotes the new disagreement point.

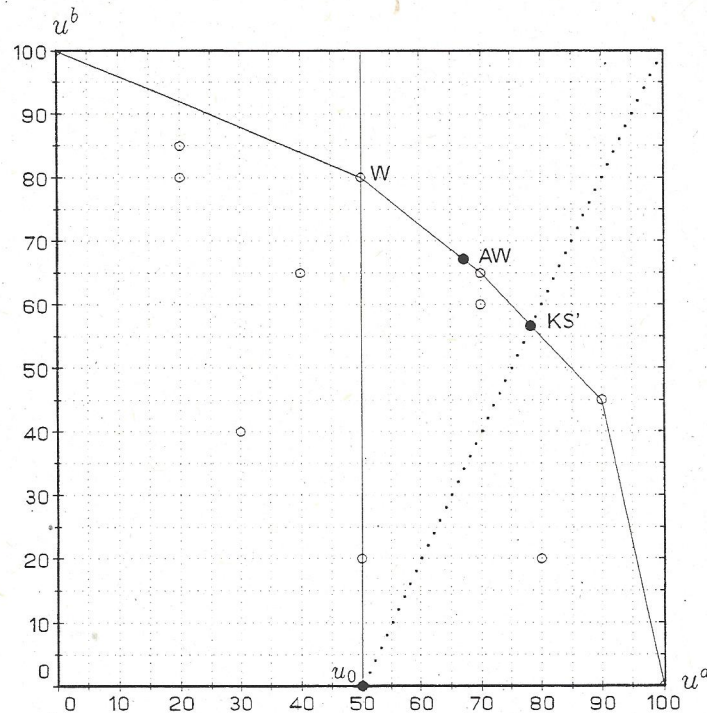


Figure 3: The Kalai-Smorodinsky Solution for a three-issue negotiation between players with outside alternatives

Of course, the outside alternatives will affect the bargaining problem, since the players' standards of value have changed: For player  $b$ , the pie to be divided is still worth 100 points, but for player  $a$  the size of the maximal achievable pie has shrunk to a total of 50 points.

In the previous section, the application of Adjusted Winner in the modified form led to a gain of 67.14 points for each player. This is illustrated by point AW in Figure 3. Compared with the standards of value for the new bargaining problem, player  $a$  now receives  $17.14/50$  or approximately 34% of his standard of value, in contrast to player  $b$ , who still receives approximately 67% of his standard of value. Surely, it would be surprising if player  $a$  considered this to be a fair distribution.

The necessary modification of Adjusted Winner now requires an adjustment of

step 3, the stopping condition for the substitution process. The determination of the temporary winner in step 1 is unaffected, so that, as in our previous example, the initial allocation is at point W in Figure 3. The adjustment process (step 2) of moving along the efficiency frontier by determining the minimal substitution rates also remains the same. What must be modified in step 3 is not the criterion for stopping the adjustment,  $\gamma = \epsilon$ , but rather the definitions of  $\gamma$  and  $\epsilon$ .

In order to achieve equity, the solution must equalize the relative gains of both players,

$$\frac{u^a - u_0^a}{100 - u_0^a} = \frac{u^b - u_0^b}{100 - u_0^b},$$

or, equivalently, equalize the gain ratio of both players with the ratio of their standards of values, i.e. their entitlement ratio:

$$(1) \quad \gamma := \frac{u^b - u_0^b}{u^a - u_0^a} = \frac{100 - u_0^b}{100 - u_0^a} =: \epsilon.$$

With the left-hand side of equation (1) defined as  $\gamma$  and the right-hand side defined as  $\epsilon$ , the stopping condition for step 3 is still  $\gamma = \epsilon$ . Our modifications of the gain and entitlement ratios, however, now explicitly acknowledge the influence of outside alternatives, which affect the size of the pie to be divided. For the 'gain' ratio to deserve its label, it must adapt players' outcomes to their reservation values. And if parties differ in their status quo (i.e.  $u_0^a \neq u_0^b$ ), it is consistent to require that they are entitled to different shares of the negotiated pie.<sup>6</sup>

Consider, for example, a rise in  $u_0$ . This reduces the denominators on both sides of equation (1). With  $u^a - u_0^a < 100 - u_0^a$ , though, the increase in the gain ratio  $\gamma$  is greater than the increase in the entitlement ratio  $\epsilon$ . As a consequence,  $u^a$  must rise and  $u^b$  must fall in order to maintain equity along the efficiency frontier. This adjustment has nothing to do with  $a$ 's bargaining ability. It is only his structural bargaining power which has risen.

---

<sup>6</sup>Instead of equalizing *relative* gains, the equity condition could also be modified to equalize players' *absolute* gains. In equation (1), outside alternatives then only affect the gain ratio  $\gamma$ , while the entitlement ratio maintains its value of  $\epsilon = 1$ . This simplification, however, is not in the spirit of Adjusted Winner as it is described by BRAMS AND TAYLOR (1996).

With condition (1), the algorithm of Adjusted Winner remains as simple as before. In our example,  $u_0 = (50, 0)$  implies an entitlement ratio of  $\epsilon = 2$ , while at point AW, in Figure 3, the gain ratio is  $\gamma = 67.14/17.14 = 3.9$ . Consequently, the adjustment process described in the previous section must be continued.

Instead of considering only a convex combination between options A3 and A2, we switch to option A2 completely. Player  $b$  now has only 65 points compared with player  $a$ 's 70, so that the gain ratio now is

$$\gamma = \frac{65 - 0}{70 - 50} = \frac{65}{20} = 3.25 ,$$

which is still too large, since  $3.25 > 2 = \epsilon$ . Comparing the substitution rates between A2 and A1 ( $RS_{A2,A1} = 4.5$ ) and between B2 and B1 ( $RS_{B2,B1} = 1$ ), we find the latter to be smaller. By moving to option B1, player  $b$  is then down to 45 points and player  $a$  now has 90, which yields a gain ratio of

$$\gamma = \frac{45 - 0}{90 - 50} = \frac{45}{40} = 1.125 .$$

Since  $1.125 < 2 = \epsilon$ , an adjustment to this point would be too strong. Equity, therefore, requires a convex combination between options B1 and B2.

We denote again by  $\alpha$  the weight given to the chosen alternative, B1. The equitable allocation is then determined by equation (1), i.e.  $\gamma = \epsilon$ , which implies

$$\frac{\{45 + [\alpha 0 + (1 - \alpha)20] + 0\} - 0}{\{20 + [\alpha 20 + (1 - \alpha)0] + 50\} - 50} = 2 .$$

This yields an equitable weight of

$$\alpha = \frac{5}{12} = 0.42 .$$

Hence players agree on options A2, C1, and a compromise consisting of approximately 42% of B1 and 58% of B2. Player  $a$  receives a total of  $u^a = 78.33$  and Player  $b$  a total of  $u^b = 56.67$  points. In Figure 3, this agreement is denoted by point KS'. The prime indicates that the solution is only an approximate implementation of Kalai-Smorodinsky. An exact implementation requires the following, more detailed analysis of how outside-options affect the entitlement ratio.

According to equation (1), a player's standard of value is given by the difference between his outside option and the value of the complete pie, which is worth 100 points. For the previous example, Figure 3 illustrates that this appears to be a reasonable approach for player  $a$ , but it is not necessarily plausible for player  $b$ . This is because player  $b$  has no chance of achieving 100 points if player  $a$  is committed to realizing his outside option should the negotiation yield a worse outcome. Indeed, under the restriction  $u^a \geq u_0^a = 50$ , the highest utility that player  $b$  can achieve is 80. Since this value depends on the outside option of player  $a$ , we denote the maximum feasible outcome of player  $b$  by  $\hat{u}^b(u_0^a)$ , where  $\hat{u}^b$  is monotonically non-increasing in  $u_0^a$ . Accordingly, for player  $a$  we define the maximum achievable outcome by  $\hat{u}^a(u_0^b)$ , with  $\hat{u}^a$  monotonically non-increasing in  $u_0^b$ . In our specific example  $\hat{u}^a(0) = 100$ .<sup>7</sup>

A player's standard of value is then given by the difference between his maximum feasible outcome and the value of his outside option. This only affects the definition of  $\epsilon$ , the equity ratio between players. The equity condition (1) must, therefore, be modified to

$$(2) \quad \gamma := \frac{u^b - u_0^b}{u^a - u_0^a} = \frac{\hat{u}^b(u_0^a) - u_0^b}{\hat{u}^a(u_0^b) - u_0^a} =: \epsilon.$$

Equation (2) reveals an additional effect of structural power: As player  $a$ 's outside option improves, this leads to a decrease in player  $b$ 's standard of value by lowering his aspiration level and, thus, his range of possible outcomes. As a result, player  $b$ 's outcome must fall relative to  $a$ 's in order to maintain equity.

In our example,  $\epsilon = (\hat{u}^b(u_0^a) - u_0^b) / (\hat{u}^a(u_0^b) - u_0^a) = 80/50 = 1.6$ . For an equitable allocation, i.e. where  $\gamma = \epsilon$ , this implies

$$\frac{\{45 + [\alpha 0 + (1 - \alpha)20] + 0\} - 0}{\{20 + [\alpha 20 + (1 - \alpha)0] + 50\} - 50} = 1.6,$$

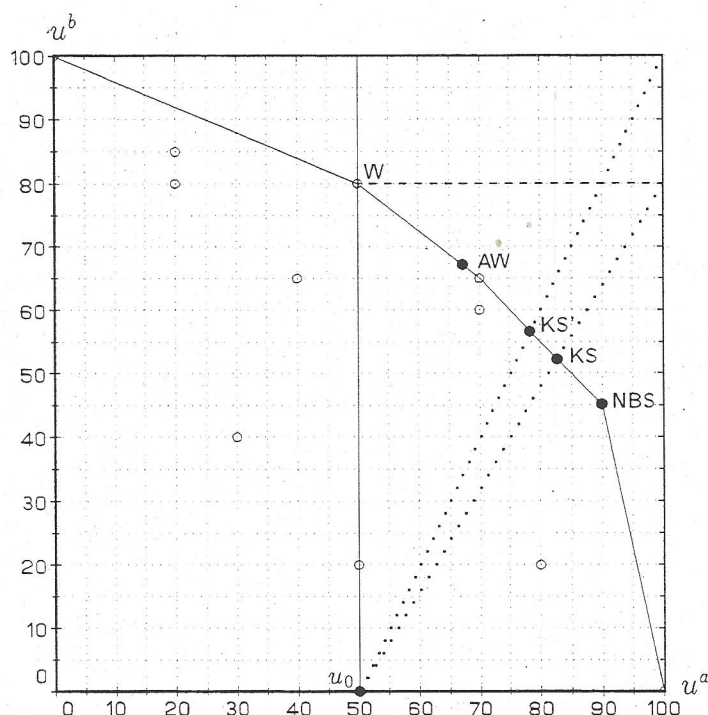
---

<sup>7</sup>It is a fortunate feature of our example that the maximum feasible outcomes for both players happen to be actual options of the negotiation, i.e.  $(\hat{u}^a, u_0^b) = (100, 0)$  and  $(u_0^a, \hat{u}^b) = (50, 80)$  are both entries in Table 4. When this is not the case, one can alternatively define the maximum outcome as the highest feasible allocation, or the lowest non-feasible allocation among actual options, or a convex combination of both.

which yields an equitable weight of

$$\alpha = \frac{33}{52} = 0.64 .$$

This leaves player  $a$  with a total of 82.7 points and player  $b$  with 52.3 points. In Figure 4, we denote this outcome, which characterizes the Kalai-Smorodinsky solution, by  $KS$ , in contrast to point  $KS'$  from before.



**Figure 4:** Bargaining solutions for a three-issue negotiation between players with outside alternatives

The importance of letting Adjusted Winner take outside alternatives into account is evident from the effect that it has on the outcome. In our example, illustrated in Figure 4, the acknowledgement of player  $a$ 's outside alternative improves his outcome by approximately 23% (point  $KS$  as compared to  $AW$ ). Clearly, this is not an aspect that can simply be neglected if an allocation is to be considered as fair by both players.

One must not forget that, without graphical support to determine  $\hat{u}^a$  and  $\hat{u}^b$ ,

the algorithm of Adjusted Winner involves additional computational effort, as one must trace out the efficiency frontier with the help of the substitution process. It is, however, not necessary to begin with point  $W$ ; any other efficient outcome can serve as a starting point as well. It may, in fact, be more convenient to begin the substitution process at a point which is closer in value to one of the outside alternatives  $u_0^a$  or  $u_0^b$ . The required modification of step 1 of Adjusted Winner is easily accomplished, e.g. by assigning all issues to Player  $a$  or to Player  $b$ . Of course, as a practical alternative, one may simply consider equation (1) instead of (2). This implements a solution that one might refer to as a satisficing version of Kalai-Smorodinsky.

## 5. Implementing the Nash Bargaining Solution

The criterion of fairness enters Adjusted Winner via the equitability condition (2) as a stopping rule for the adjustment process along the efficiency frontier. Hence, it is only this condition which needs to be changed if one is interested in using the algorithm to implement alternative bargaining solutions.

The criterion of fairness behind the Kalai-Smorodinsky Solution is *egalitarian*, since both players receive the same fraction of their standard of value. As an alternative, YOUNG (1991) also considers the problem of fair division using a *utilitarian* criterion, where the objective is first to find a distribution that maximizes joint utility. The distribution of gains may then be considered as fair if there is no redistribution that would cause one player's utility to increase by a larger percentage than the other player's utility decreases. This type of parity between players, as measured by their percentage gains in utility, is induced by the Nash bargaining solution. As NASH (1950) proved, there exists a unique efficient outcome with this property.

Consider a negotiation over  $n$  issues  $X_1, \dots, X_n$ , each with its own given number of options. For each issue  $X_l$  ( $l = 1, \dots, n$ ), let  $u_{X_l}^c$  and  $u_{X_l}^b$  denote the achieved utility points of player  $a$  and  $b$ , respectively, and let  $u_{X_{li}}^a$  and  $u_{X_{li}}^b$  denote their utilities from a specific option  $i$  of issue  $X_l$ . As we have seen in the preceding sections, the

substitution process along the efficiency frontier requires the convex combination of at most two options of a single issue. Consider, therefore, a combination of two options  $X_{li}$  and  $X_{lj}$ . With the utilities of all other issues  $X_k$ ,  $k \neq l$ , given by  $u_{X_k}^x$  ( $x = a, b$ ), the utility of player  $x$  can be written as

$$u^x = \sum_{k \neq l} u_{X_k}^x + [\alpha u_{X_{lj}}^x + (1 - \alpha) u_{X_{li}}^x], \quad \alpha \in [0, 1], \quad x = a, b.$$

Consequently, the percentage change in utility from moving to another convex combination of the same two options is

$$\frac{\Delta u^x}{u^x - u_0^x} = \frac{[\beta u_{X_{lj}}^x + (1 - \beta) u_{X_{li}}^x] - [\alpha u_{X_{lj}}^x + (1 - \alpha) u_{X_{li}}^x]}{\sum_{k \neq l} u_{X_k}^x + [\alpha u_{X_{lj}}^x + (1 - \alpha) u_{X_{li}}^x] - u_0^x}, \quad \alpha, \beta \in [0, 1], \quad x = a, b.$$

In order for parity to hold at this point, the percentage increase in one player's utility must be equal to the percentage decrease in the other's:

$$\frac{\Delta u^a}{u^a - u_0^a} = -\frac{\Delta u^b}{u^b - u_0^b}.$$

By rearranging terms, one then obtains

$$(3) \quad \frac{u^b - u_0^b}{u^a - u_0^a} = -\frac{\Delta u^b}{\Delta u^a},$$

which is a common characterization of the Nash bargaining solution. The left-hand side of equation (3) is the ratio of players' utility gains,

$$\gamma := \frac{u^b - u_0^b}{u^a - u_0^a} \equiv \frac{[\sum_{k \neq l} u_{X_k}^b + \alpha u_{X_{lj}}^b + (1 - \alpha) u_{X_{li}}^b] - u_0^b}{[\sum_{k \neq l} u_{X_k}^a + \alpha u_{X_{lj}}^a + (1 - \alpha) u_{X_{li}}^a] - u_0^a},$$

while the right-hand side of (3) denotes the slope of the efficiency curve at this point or the rate of substitution between options  $X_{li}$  and  $X_{lj}$ :

$$RS_{X_{lj}, X_{li}} := -\frac{\Delta u^b}{\Delta u^a} \equiv -\frac{u_{X_{lj}}^b - u_{X_{li}}^b}{u_{X_{lj}}^a - u_{X_{li}}^a}.$$

Hence, parity in the above form implies that the gain ratio between players is equal to the rate of substitution, i.e.  $\gamma = RS$ .

If the efficiency curve is piecewise linear, as in our example above, then there may not be an outcome for which condition (3) holds. On the one hand,  $\gamma > RS$  implies

that parity can be achieved only by redistributing gains from player  $b$  to player  $a$ . On the other hand,  $\gamma < RS$  indicates that parity requires a redistribution from  $a$  to  $b$ . With a finite number  $n$  of substitution rates (given by the flat segments of the efficiency curve), there may be a unique outcome at a kink in the efficiency curve, i.e. a unique value of  $\gamma$  between two substitution rates, where both inequalities are valid. In this case, there is no further redistribution that can increase parity. With the  $n$  substitution rates ranked in ascending order, the stopping condition, thus, requires that  $\gamma$  lies in the closed interval given by two neighboring substitution rates:

$$(4) \quad \gamma \in [RS_i, RS_{i+1}] , \quad i = 1, \dots, n - 1 .$$

Note that condition (3) is a special case of (4), where a unique value of  $\gamma$  simultaneously satisfies both  $\gamma \in [RS_{i-1}, RS_i]$  and  $\gamma \in [RS_i, RS_{i+1}]$ .

In order to implement the Nash bargaining solution, we follow step 1 of Adjusted Winner in order to obtain an efficient initial distribution. In step 2, we calculate substitution rates for the adjustment process along the efficiency frontier. For the modification of step 3, we note that the Nash bargaining solution induces an efficient distribution which satisfies condition (3) or, more generally, condition (4).

Consider again our example from the previous section. The Kalai-Smorodinsky solution, denoted by point KS in Figure 4, induces an outcome based on a compromise between options B1 and B2 which feature a substitution rate of  $RS_{B2,B1} = 1$ . The gain ratio  $\gamma$  at this outcome, however, is equal to the entitlement ratio  $\epsilon$ . Since  $\gamma = 1.6 > 1 = RS$ , further adjustment from  $b$  to  $a$  is necessary in order to implement the Nash solution. By substituting all the way to option B1, we found the gain ratio to be  $\gamma = 1.125 > 1$ , which is still too large. The next highest substitution rate is then between options A2 and A1, where  $RS_{A2,A1} = 4.5$ . Since at A2,  $\gamma = 1.125 < 4.5$ , further substitution now should be from  $a$  to  $b$ . The two inequalities  $1 < \gamma < 4.5$  at a single point imply that there is no outcome that satisfies condition (3); the Nash outcome is, therefore, at the kink in the efficiency curve where  $\gamma = 1.125 \in [1, 4.5]$ . In Figure 4, this is denoted by point NBS. The Nash agreement consists of options A2, B1, and C1, giving player  $a$  a total of 90 points and player  $b$  45 points.



Although the criterion of fairness behind the Nash bargaining solution is debatable, it has some practical advantages over the Kalai-Smorodinsky solution as a stopping condition for the substitution process of Adjusted Winner. First, the Nash solution is generally easier to calculate, since the equilibrium condition, (3) or (4), is simpler than (2). Analytically, the Nash bargaining solution merely requires maximizing the product of players' gains. This is easier than determining the intersection of a non-linear efficiency curve and an equity line, the slope of which is given by the standards of value, which, again, are determined by the efficiency curve. This feature has surely contributed to the popularity of the Nash bargaining solution in theoretical research. Second, with piecewise linear preferences, the Nash solution is often easier to implement than the Kalai-Smorodinsky solution. With only a finite number of substitution rates, there are only finitely many values of  $\gamma$  that can satisfy the Nash criterion (3). Hence, with an arbitrary disagreement point, chances are high that only condition (4) applies. This is quite fortunate, however, since an allocation that is located at a kink in the efficiency curve does not require a convex combination of individual options, which may be difficult to implement.

## 6. Generalized Adjusted Winner: Procedural Implications

A characteristic feature of an axiomatic approach to bargaining is that it is based on a list of desirable properties that one might expect of a reasonable solution to a bargaining problem. However, even if one focuses on the criterion of fairness, the appropriate solution depends on the precise definition of what is to be considered as fair. From an egalitarian perspective, the Kalai-Smorodinsky solution appears to offer an appropriate distribution of gains. With a variety of alternatives at hand, though, it is very well likely that a negotiating party may prefer, for example, to "play Nash" instead.

As we have shown, the standard algorithm underlying Adjusted Winner can be combined with the general equitability condition of the Kalai-Smorodinsky solution, given by (2), as well as with the parity condition of the Nash bargaining solution, given by (3). The implementation of other bargaining solutions only requires a

modification of the stopping condition for the adjustment process.

The main drawback of Adjusted Winner is its strategic manipulability. In our example, if player *b* knows player *a*'s preferences, then he can strategically rearrange his own points in order to improve his payoff. Indeed, if player *b* allocates his points such that they are only slightly above or just below player *a*'s, then he can still be the temporary winner in the first step of Adjusted Winner. In the adjustment phase, however, player *b* does not have to offer as much compensation as before, since his (false) preferences indicate that he has only a minor advantage over player *a*.

BRAMS AND TAYLOR (1996) argue that, in practice, such strategic behavior is quite likely to backfire as soon as players are only slightly uncertain about each others preferences. This is because the procedure underlying Adjusted Winner requires a distribution of 100 points across all issues. Increasing the points on one issue is thus only possible if the points on other issues are decreased. In order to take advantage of the other player one must, therefore, consider several issues simultaneously. Consequently, Adjusted Winner is de facto more robust against manipulation than similar procedures such as the Knaster/Steinhaus procedure of sealed bids, where players can bid strategically for single issues.<sup>8</sup>

With the inclusion of outside alternatives, however, misrepresentation becomes a serious problem, because, as equations (2) and (3) both reveal, an increase of a player's disagreement point has an unambiguously positive effect on his own outcome. Our analysis and, in particular, the allocations in Figure 4 illustrate that it is not legitimate to simply neglect outside options, since the fair outcome depends on the structure of the bargaining problem, which includes the status quo before negotiation begins. Assessing the alternatives to negotiation thus becomes a major aspect of the negotiation process. When negotiations are of longer duration or part of an ongoing relationship, bluffing with respect to outside options becomes more difficult. But if outside alternatives are difficult to verify, then parties may require the help of a mediator, who has at least some chance of assessing players' reservation

---

<sup>8</sup>A description of the Knaster/Steinhaus procedure as well as a comparison with Adjusted Winner is given by BRAMS AND TAYLOR (1996).

values.

Nevertheless, a fair-division procedure works best when strategic misrepresentation of preferences or alternatives is not a problem. This is the case in conflicts where parties have an interest to play cooperatively, but simply do not trust each other. A formal procedure then serves as a commitment device. Of course, binding oneself to a mechanism that induces a cooperative outcome is equivalent to agreeing directly to a cooperative solution. A comprehensible procedure consisting of plausible steps, however, has a high acceptability, since it reproduces and manages a cooperative negotiation process, which both parties desire but are not capable of without support.

Adjusted Winner can be applied to a wide spectrum of complex negotiations over multiple issues with several options. Despite all extensions, the procedure remains a back-of-the-envelope exercise. Moreover, all of our modifications showed that a fair division involves at most one convex combination between two options of only one single issue. And if a linear combination of options is difficult to implement, Adjusted Winner even offers creative support by showing parties precisely what agreement they must improve on through negotiation. Creativity is undeniably an important characteristic of successful negotiators, but if innovations are not directed towards a specific goal, they are not of much use. Combined with the appropriate bargaining solution, Adjusted Winner can lead negotiating parties in the direction of joint problem solving.

## References

- BRAMS, S.J., A.D. TAYLOR (1996): *Fair Division*, Cambridge, Mass.: Cambridge University Press
- KALAI, E., M. SMORODINSKY (1975): "Other Solutions to Nash's Bargaining Problem," *Econometrica*, 43, 513-518
- KEENEY, R.L. AND H. RAIFFA (1991): "Structuring and Analyzing Values for Multiple-Issue Negotiations," in H.P. Young [ed.]: *Negotiation Analysis*, Ann Arbor: University of Michigan Press
- MOULIN, H. (1984): "Implementing the Kalai-Smorodinsky Bargaining Solution," *Journal of Economic Theory*, 33, 32-45
- NASH, J.F. (1950): "The Bargaining Problem," *Econometrica*, 18, 155-162
- RAIFFA, H. (1982): *The Art and Science of Negotiation*, Cambridge, MA: Harvard University Press
- RAIFFA, H. (1997): *Lectures on Negotiation Analysis*, Cambridge, MA: The Program on Negotiation
- YOUNG, H.P. (1991): "Fair Division," in H.P. Young [ed.]: *Negotiation Analysis*, Ann Arbor: University of Michigan Press
- YOUNG, H.P. (1994): *Equity in Theory and Practice*, Princeton: Princeton University Press

## Appendix

### Proof of the Theorem:

Consider an issue A with  $n$  distinct efficient options. Players' utilities from these  $n$  options are given in Table A1. The options in Table A1 are ordered such that  $A1^a > A2^a > \dots > An^a \geq 0$  and  $0 \leq A1^b < A2^b < \dots < An^b$ . Moreover, the concavity of utility functions implies that  $RS_{Ai+1,Ai} < RS_{Ai,Ai-1}$ ,  $i = 2, \dots, n-1$ .

		A							
		1	2	...	$i$	$i+1$	...	$n-1$	$n$
$u_A^a$		$A1^a$	$A2^a$	...	$Ai^a$	$Ai+1^a$	...	$An-1^a$	$An^a$
$u_A^b$		$A1^b$	$A2^b$	...	$Ai^b$	$Ai+1^b$	...	$An-1^b$	$An^b$

Table A1: Non-linear preferences over  $n$  options

Consider now  $n-1$  issues  $X_1, \dots, X_{n-1}$ , over which players have linear, additively separable preferences. Each issue has two options. Players' utilities over the options are given in the three panels of Table A2, where the second panel is only relevant for  $n > 3$ .

		$X_1$		$X_{i=2, \dots, n-2}$		$X_{n-1}$	
		1	2	1	2	1	2
$u_{X_1}^a$		$A1^a - A2^a$	0	$Ai^a - Ai+1^a$	0	$An-1^a$	$An^a$
$u_{X_1}^b$		$A1^b$	$A2^b$	0	$Ai+1^b - Ai^b$	0	$An^b - An-1^b$

Table A2: Additively separable preferences over  $n-1$  issues

The issues of Table A2 can be viewed as a decomposition of issue A in Table A1. First, the options of Table 1 can be reproduced from those of Table A2 according to

$$(A1) \quad Ai = \sum_{j=1}^{i-1} X_j 2 + \sum_{j=i}^{n-1} X_j 1, \quad i = 1, \dots, n,$$

where  $X_{j1}$  and  $X_{j2}$  denote the first and second options of issue  $X_j$ , respectively. Second, the substitution rates over individual issues  $X_i$  are equal to the  $n - 1$  substitution rates between neighboring options of issue A:

$$RS_{X_{i2}, X_{i1}} = -\frac{X_{i2}^b - X_{i1}^b}{X_{i2}^a - X_{i1}^a} = -\frac{A_{i+1}^b - A_i^b}{A_{i+1}^a - A_i^a} = RS_{A_{i+1}, A_i}, \quad i = 1, \dots, n-1.$$

And third, although the negotiation over the  $n - 1$  issues of Table A2 has  $2^{n-1}$  possible agreements, only the  $n$  agreements given by equation (A1) are efficient, where  $n < 2^{n-1}$ , for  $n \geq 3$ .

In order to see the third aspect, consider the four options of two joint issues  $X_i$  and  $X_{i+1}$ . For  $i = 2, \dots, n - 3$ , the utilities of the four options are given in Table A3; the analysis for  $i = 1$  and  $i = n - 2$  is analogous.

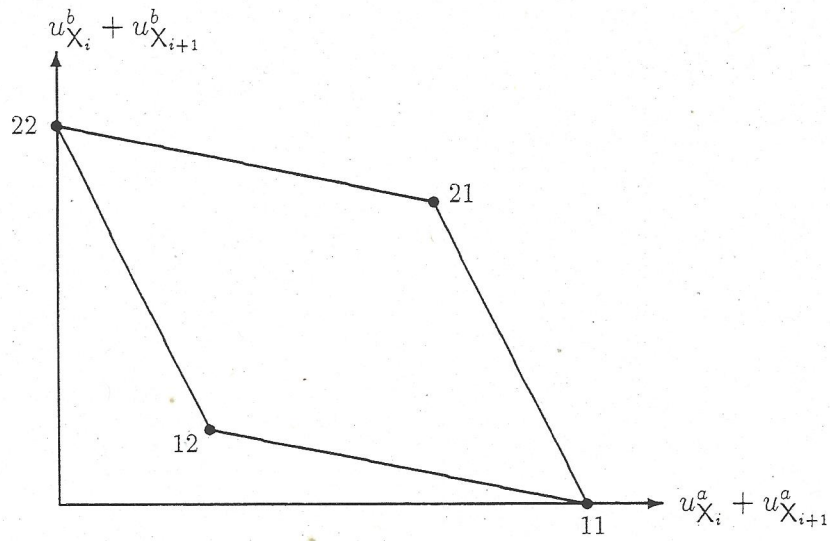
		$X_i, X_{i+1}, \quad i = 2, \dots, n - 3$			
		11	12	21	22
$u_{X_i}^a + u_{X_{i+1}}^a$		$A_i^a - A_{i+2}^a$	$A_i^a - A_{i+1}^a$	$A_{i+1}^a - A_{i+2}^a$	0
$u_{X_i}^b + u_{X_{i+1}}^b$		0	$A_{i+2}^b - A_{i+1}^b$	$A_{i+1}^b - A_i^b$	$A_{i+2}^b - A_i^b$

Table A3: Additive preferences over issues  $X_i$  and  $X_{i+1}$ .

For the given utilities of options  $A_i$  in Table A1, Table A3 implies that the substitution rate between options 22 and 21 is the same as between options 12 and 11, and the substitution rate between 22 and 12 is the same as between options 21 and 11. Since the substitution rate between 21 and 11 is greater than between 22 and 21, option 12 must be inefficient. This is illustrated in Figure A1.<sup>9</sup>

We characterize an agreement over all  $n - 1$  issues by the vector of options  $\mathbf{o} := (\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_{n-1})$ , where  $\mathbf{o}_i \in \{1, 2\}$  denotes the chosen option of issue  $X_i$ ,  $i = 1, \dots, n - 1$ . We denote by  $\mathbf{o}_{i,k,l} := (\mathbf{o}_1, \dots, \mathbf{o}_{i-1}, k, l, \mathbf{o}_{i+2}, \dots, \mathbf{o}_{n-1})$  a given vector  $\mathbf{o}$ , for which  $k, l \in \{1, 2\}$  are the chosen options of issues  $X_i$  and  $X_{i+1}$  ( $i = 1, \dots, n - 2$ ),

<sup>9</sup>With  $A_1^b > 0$  ( $A_n^a > 0$ ) the illustration of Figure A1 would be shifted upwards (rightwards) for  $i = 1$  ( $i = n - 2$ ).



**Figure A1:** Utilities of the four options of issues  $X_i$  and  $X_{i+1}$ .

respectively. From the analysis of Table A3 and Figure A1 we know that any agreement  $o_{i,1,2}$  is inefficient, because it is dominated by some convex combination of  $o_{i,2,2}$  and  $o_{i,2,1}$  or some convex combination of  $o_{i,2,1}$  and  $o_{i,1,1}$ . With an efficient agreement,  $o_{i,2,2}$  implies  $o_{i+1,2,2}$  or  $o_{i+1,2,1}$ , and  $o_{i,2,1}$  or  $o_{i,1,1}$  implies  $o_{i+1,1,1}$ . Hence, an agreement  $o$  is efficient if, and only if, it satisfies equation (A1).

□