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Abstract

Several sufficient conditions for the existence of a pure-strategy Nash equilibrium in a strategic game with decreasing best replies are presented. The first presupposes restrictions on dependencies between the players, i.e. on "who may influence whom", described by a graph without odd cycles. The second, that each player is only affected by the maximal among choices of the relevant partners and this "relevance" is a symmetric relation. The third, that each player reacts to the sum of scalar characteristics of all the partners's strategies. A couple of fixed point theorems for lexicographically decreasing reactions, logically independent of the previous results, is also presented.

Key words: Nash equilibrium, order-reversing mappings, fixed points, maximum (minimum) aggregation, additive aggregation.

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0. Introduction

The purpose of this paper is to present a description of the current "Pareto border" in my search for conditions under which a strategic game with decreasing best replies has a pure-strategy Nash equilibrium. It may thus be regarded as a revised and updated version of Section 4 of Kukushkin (1995).

The search has been motivated by resentment against this unfair asymmetry: if a strategic game has increasing best replies, then, provided certain regularity of strategy sets, the existence of an equilibrium, its stability and nice comparative statics are ensured (Topkis, 1979; Vives, 1990; Milgrom and Roberts, 1990, 1994; Milgrom and Shannon, 1994); if the best replies are decreasing, almost nothing good about the game can be found in the literature. Meanwhile, both kinds of monotonicity emerge in economic models with more or less the same frequency (Fudenberg and Tirole, 1984; Bulow et al., 1985), and typical sufficient conditions for either of them only differ in the sign of an inequality.

To some extent, the asymmetry in the literature reflects that in reality and this cannot be helped. In particular, no attempt to address the stability or comparative statics of equilibria is made here. Still, if we concentrate on equilibrium existence problem, decreasing best replies appear to have a potential exceeding what was revealed in the early papers on the subject: Novshek (1985) on the Cournot model and Vives (1990) and Milgrom and Roberts (1990) on two-person games. The fact that the straightforward analogue of Tarski's (1955) fixed point theorem for decreasing mappings is not true makes the situation only more interesting.

Actually, we will work in a bit more abstract framework than strategic games, hence the "systems of decreasing reactions" in the title of this paper. From the game-theoretical viewpoint, each theorem below specifies conditions under which for any choice of decreasing single-valued selections from the best reply correspondences there exists a Nash equilibrium where each player uses this pre-specified selection; no assumption like upper hemi-continuity on the best reply correspondences is needed. This feature ensures a wider area of possible applications.

First, for the existence of the best reply correspondence we only need the upper semi-continuity of the utility function in own strategy, while for its upper hemi-continuity, we, generally speaking, need the continuity of the utility in the product topology. This is a big difference.

Second, this form of the results is useful for studying the set of all equilibria of a game: e.g. it may be important to know that there exists an equilibrium where each player chooses the greatest of his best replies. Apparently, some non-uniqueness theorems can be derived from them.

Third, our reactions need not be Nash best replies. For instance, when the existence of the best replies is not guaranteed, we may hope to find ε-optimal decreasing reactions and obtain the existence of an ε-equilibrium. (I am not prepared to formulate exact conditions for the existence of such reactions: the question seems rather complicated.)

The paper is organized as follows.

In Section 1, necessary definitions are introduced. The standard framework for a fixed point theorem, a mapping from a set into itself, is replaced with a bit more structured notion of a system of reactions. A mapping decreasing with respect to a preorder is defined, a crucial assumption in each theorem to follow being that reactions should be decreasing w.r.t. certain preorders.

In Section 2, we consider restrictions on "who may influence whom"; such restrictions can be represented by an oriented graph, the absence of an arc from i to j $(i \neq j)$ meaning that j cannot react to the choice made by i (in the strategic game interpretation, this means that the strategy x_i does not enter the utility function u_j). Theorem 1 shows that such a system of restrictions ensures, by itself, the existence of a fixed point for any collection of decreasing reactions if and only if the corresponding graph has no cycle with an odd number of arcs. The theorem includes the Vives-Milgrom-Roberts result on duopoly as a particular case and the proof is based on Milgrom and Roberts's reversing trick.

In Section 3, we add a restriction on the functional form of the reaction functions: each player is supposed to react only to the maximum of scalar characteristics of the relevant partners' choices. Theorem 2 shows that the mutuality condition - if i may influence j, then j may influence i - is sufficient for the existence of a fixed point under the restrictions. The condition is not necessary, but an example shows that it cannot simply be dropped.

In Section 4, the reactions are decreasing w.r.t. additive orderings; more precisely, each player reacts to the sum of scalar characteristics of all the partners' strategies (restrictions on dependencies are not allowed here). Theorem 3 establishes the existence of a fixed point under rather mild topological assumptions. An example shows that, generally speaking, the multi-dimensional addition would not do.

The story behind Theorem 3 goes back to the seminal paper of Novshek (1985), who discovered, in the context of the Cournot model, that decreasing best replies guarantee the existence of an equilibrium. In Kukushkin (1994), the result was reformulated as a fixed point theorem hinging on three essential assumptions: each player chooses a real number, each player reacts to the sum of the choices of the partners, and all reactions are decreasing. The theorem was given a short and rigorous proof, while Novshek's argument relied heavily on naive geometric intuition inapplicable to the truly general case (e.g. if one of the reactions jumps at every rational number, there cannot be any continuous branch at all). Unfortunately, a purely technical assumption - the upper hemi-continuity of all the reaction correspondences - had to be made. Now the assumption is, at last, dispensed with. Some could argue that the enhanced generality is not worth the price paid in the complexity of the proof, but, as stated above, I believe it important to have a result for single-valued reactions.

Theorems 4 and 5 of Section 5 just show that the previous results do not exhaust all the possibilities. Either of them can easily be extended, but it is not quite clear to what extent.

Section 6 contains a brief discussion of remaining open questions.

1. General Definitions

A mapping f from a partially ordered set to another will be called increasing if $x \geq y$ implies $f(x) \geq f(y)$ and decreasing if $x \geq y$ implies $f(y) \geq f(x)$.

A system of decreasing reactions Σ is given by a finite set N, and, for each $i \in N$, a partially ordered set X_i and a decreasing mapping r_i: $X_{-i} \to X_i$, where $X_{-i} = \Pi_{j \in N \setminus \{i\}} X_j$. A fixed point for such a system is a collection $x_i^o \in X_i$, $i \in N$, such that

$$x_i^o = r_i(x_{-i}^o), \qquad \text{for all } i \in N. \tag{1.1}$$

Obviously, the definition has been inspired by the concept of Nash equilibrium. However, it is relevant for other game-theoretic concepts (ε-equilibrium, for one) and looks nice enough by itself.

Remark. The model is meaningful only for $n = \#N \geq 2$; however, for induction processes to follow, it is convenient to consider $n=1$ admissible too, in which case the unique "reaction" is just an element of X_i so (1.1) is satisfied automatically.

Naturally, a system of r_i induces a mapping r: $X \to X$ (with $X = \Pi_{i \in N} X_i$), and $x^o \in X$ satisfies (1.1) if and only if $x^o = r(x^o)$, i.e. x^o is a fixed point of r. However, we cannot go this way because the straightforward analogue of

Tarski's (1955) fixed point theorem for decreasing mappings is not true. Moreover, essential additional assumptions are necessary.

Example 1. Let $N=\{1,2,3\}$, $X_i=\{0,1\}$ ($i\in N$) and $r_1(x_2,x_3)=1-x_2$, $r_2(x_1,x_3)=1-x_3$, $r_3(x_1,x_2)=1-x_1$. It is easy to see that no x^o satisfies (1.1): we should have $x_1^o=1-x_2^o=x_3^o=1-x_1^o$.

Remark. We could not provide so simple an example for $n=2$. When this paper was virtually finished, I learned, from Davey and Priestly (1990, Exercise 4.12), that Banach's proof of the Schröder-Bernstein theorem is based on the fixed point theorem for two decreasing reactions later rediscovered by Vives (1990). For Davey and Priestly, this is just an application of Tarski's theorem and they do not emphasize that it is actually applied to decreasing mappings.

A preorder ϑ is a reflexive and transitive binary relation; a complete preorder is called an ordering.

Let there be a mapping r: $X \to Y$, where X and Y are partially ordered sets, and a preorder ϑ on X. r is called decreasing w.r.t. ϑ if $x''\vartheta x'$ implies $r(x')\geq r(x'')$. We will only apply the definition in the case when ϑ is an extension of the order ($\geq$) on X; then a mapping decreasing w.r.t. ϑ is also decreasing in the sense of the previous definition.

When the preorder ϑ is defined by a real-valued function F, i.e. $x''\vartheta x'$ iff $F(x'')\geq F(x')$ (in which case it is an ordering), this property is equivalent to the existence of a representation $r=q\circ F$ with a decreasing mapping q: $F(X) \to Y$; such mappings r are also called decreasing w.r.t. F. For the consistency, it is natural to restrict ourselves to increasing functions F.

In Sections 2 and 3, an important part is played by "partial product" preorders. Suppose, for each $i\in N$, a subset $I(i) \subseteq N\setminus\{i\}$ is given. Then we say that a system Σ satisfies restrictions on dependencies $\langle I(i)\rangle_{i\in N}$ if each r_i is decreasing w.r.t. preorder ϑ_i: $x\vartheta_i y$ iff $x_j\geq y_j$ for all $j\in I(i)$. When $I(i)$ is not empty, we may regard r_i as a decreasing mapping $\Pi_{j\in I(i)}X_j \to X_i$;

otherwise (the exclusion of which case would be technically inconvenient), r_i is just a constant.

Restrictions on dependencies, $<I(i)>_{i \in N}$, can be described by an oriented graph. More formally, we say that an oriented graph G describes the system $<I(i)>_{i \in N}$ if its set of vertices is N and $j \in I(i)$ is equivalent to the existence of an arc from j to i in G (for all $i,j \in N$).

2. Restrictions on Dependencies

We call a graph G stable if every Σ having complete lattices as X_i, $i \in N$, and satisfying the restrictions on dependencies described by G has a fixed point in the sense of (1.1).

Remark. The restriction that each X_i should be a complete lattice is, naturally, motivated by the similar assumption in Tarski's theorem. In principle, other fixed point theorems for increasing mappings can also do. For instance, Theorem 1 remains true if, in the definition of a stable graph, we demand that each X_i be a partially ordered, finite set having the fixed point property (Roddy, 1994; I thank Sergei Tarasov, who brought this paper to my attention).

Theorem 1. An oriented graph G is stable if and only if every cycle in G includes an even number of arcs.

For the simplicity of notations, we assume that each r_i is a decreasing mapping $\Pi_{j \in I(i)} X_j \to X_i$ (remembering the reservation about the case of empty $I(i)$). (1.1) then transforms into

$$z_i = r_i(z_{I(i)}), \tag{2.1}$$

for all $i \in N$, where $z_{I(i)}$ denotes the vector of z_j for $j \in I(i)$.

1. Necessity. Let G have an odd cycle i_0, i_1, ..., i_{2m} ($i_k \in N$, there is an arc from i_k to i_{k+1} as well as from i_{2m} to i_0). Without restricting generality, we may assume $i_j \neq i_k$ for $j \neq k$. Now we can define a system Σ without a fixed point: $X_i = \{0,1\}$ for $i \in \{i_0, i_1, \ldots, i_{2m}\}$, $X_j = \{0\}$ for all other $j \in N$;

$r_{i_{k+1}}(x_{I(i_{k+1})})=1-x_{i_k}$ for $k=0,1,...,2m-1$, $r_0(x_{I(i_0)})=1-x_{i_{2m}}$, $r_j(x_{I(j)})=0$ for all other $j\in N$. Supposing the existence of a fixed point z_N, denote $s=\sum_{k=0}^{2m} z_{i_k}$. Summing up (2.1) for $i=i_0, ..., i_{2m}$, we obtain $2s=2m+1$; on the other hand, s must be an integer.

2. Sufficiency. The proof goes by induction in the cardinality of N. For $\#N=1$, the theorem is trivially true.

Let us consider a graph G without any odd cycle and a system of reactions Σ with complete lattices as X_i and satisfying the restrictions on dependencies described by G. Introduce a relation R on N: iRj if and only if $i=j$ or there exists a path in G from i to j, i.e. $i_0,...,i_m\in N$ such that $i_0=i$, $i_m=j$ and there exists an arc from i_k to i_{k+1}, $k=0,1,...,m-1$. R is reflexive and transitive; roughly speaking, it is the transitive closure of the basic relation "be connected with an arc in G". We call i and j equivalent if iRj and jRi; thus N is partitioned into equivalence classes and R defines a partial order on the set of the classes. Now let us take a maximal, w.r.t. R, equivalence class. In other words, we take a subset $N^o\subseteq N$ such that iRj for all $i,j\in N^o$ and iRj for no $i\in N\setminus N^o$, $j\in N^o$ (i.e. there is no arc leading from a vertex outside N^o to a vertex in N^o). The following procedure defines z_i satisfying (2.1) for all $i\in N^o$.

If $N^o=\{i\}$, then r_i must be a constant; we take it as z_i. Supposing $\#N^o>1$, we fix an $i\in N^o$; for any $j\in N^o$, there exists a path from i to j and a path from j to i. Since G has no odd cycle, there cannot be a path from i to j with an even number of arcs and another path with an odd number of arcs. Therefore, we have a partitioning $N^o=E\cup O$, where $j\in E$ if there exists an even path from i to j (so $i\in E$), $j\in O$ if there exists an odd path from i to j, and $E\cap O$ is empty. Obviously, no arc can connect vertices belonging to the same element of the partitioning.

Now we retain the existing order on X_i for $i\in E$, while reversing it on

X_i for $i\in O$ (similarly to Milgrom and Roberts, 1990); all the mappings r_i, $i\in N^o$, become increasing and Tarski's theorem (applied to the Cartesian product of X_i, $i\in N^o$) implies the existence of a "partial" fixed point $<z_i>_{i\in N^o}$ satisfying (2.1).

If, by chance, $N^o=N$, the theorem is proved. Otherwise, we define a new system Σ' with $N'=N\setminus N^o$, $I'(i)=I(i)\setminus N^o$, $X_i'=X_i$ $(i\in N')$,

$$r_i'(x_{I'(i)}) = r_i(x_{I'(i)}, z_{I(i)\cap N^o}). \tag{2.2}$$

We also define a new graph G' with N' as the set of vertices and the old arcs between $i,j\in N'$. Obviously, G' describes $<I'(i)>_{i\in N'}$ and still has no odd cycle; by the induction hypothesis, there exists a fixed point $<z_i>_{i\in N'}$. Combining z_i for $i\in N^o$ and $i\in N'$, we obtain the fixed point needed as (2.1) for r_i' and (2.2) imply (2.1) for r_i for all $i\in N'$.

As a kind of application of Theorem 1, let us consider a game where the players are arranged in a circle and each player only interacts with his neighbours. Assume that the strategy sets are nice enough and the best replies are decreasing. Can we be sure of the existence of an equilibrium? Theorem 1 gives a positive answer for an even number of players.

3. Maximum (Minimum) Aggregation

A system of restrictions $<I(i)>_{i\in N}$ is called mutual if $j\in I(i)$ implies $i\in I(j)$ for all $i,j\in N$. Under this condition, possible dependencies can be described by an unoriented graph.

Theorem 2. Suppose we have a system of decreasing reactions with mutual restrictions $I(i)\subseteq N\setminus\{i\}$, and, for each $i\in N$, an increasing function f_i: $X_i \to \mathbb{R}$ such that $f_i(X_i)$ is compact in its intrinsic topology, see Birkhoff (1967), p.241-242. Effectively, this means that every subset has the least upper bound in $f_i(X_i)$. Suppose also that each r_i is decreasing w.r.t. $F_i(x_{-i})=\max_{j\in I(i)}f_j(x_j)$ (if $I(i)$ is empty, F_i is a constant). Then there exists a fixed point x^o satisfying (1.1).

The key role is played by the following particular case.

Fundamental Lemma on Maximum Aggregation. Consider a system Σ defined by a finite set N, a closed interval $[a,b]\subseteq\mathbb{R}$, and, for each $i\in N$, a subset $I(i)\subseteq N\setminus\{i\}$ and a decreasing function r_i: $[a,b] \to [a,b]$. Suppose also that the system of $<I(i)>_{i\in N}$ is mutual. Then there exists a vector $z\in[a,b]^N$ satisfying

$$z_i = r_i(\max_{j\in I(i)} z_j), \tag{3.1}$$

for all $i\in N$ (here and in the proof, we, quite naturally, assume that the maximum of an empty set is a).

Proof of the Fundamental Lemma

For each $i\in N$, since r_i is decreasing, there exists $x_i^*\in[a,b]$ such that

$$r_i(x)\geq x_i^* \text{ for } x<x_i^* \text{ and } r_i(x)\leq x_i^* \text{ for } x>x_i^*. \tag{3.2}$$

Denote $r_{-i}(x)=\max_{j\in I(i)}r_j(x)$, $v_i(x)=r_i\circ r_{-i}(x)$, $L_i=\{x\in[a,b] \mid r_i(x)\leq x\}$, $L=\bigcap_{i\in N}L_i$; note that v_i is increasing and $L_i\supseteq]x_i^*,b]$ for each $i\in N$.

Lemma 2.1. There exist $i\in N$ and $x\in L$ such that $v_i(x)\geq x$.

Choose $i\in N$ with the maximal x_i^*; in the case of non-uniqueness, choose the maximal $r_i(x_i^*)$ among them. Suppose that

$$v_i(x)<x \text{ for all } x>x_i^* \tag{3.3}$$

(otherwise, the lemma is already true). For each $x>x_i^*$ and $j\in N$, we have $r_j(x)\leq x_i^*$ (by (3.2) because x_i^* is maximal), hence $r_{-i}(x)\leq x_i^*$. Now if $r_i(x_i^*)>x_i^*$, then for any $x\in]x_i^*,r_i(x_i^*)[$ we have $v_i(x)=r_i(r_{-i}(x))\geq r_i(x_i^*)>x$, contradicting (3.3); therefore, $r_i(x_i^*)\leq x_i^*$.

The choice of i implies that $r_j(x_i^*)\leq x_i^*$ for every $j\in N$, so $x_i^*\in L$ and $r_{-i}(x_i^*)\leq x_i^*$. If $r_{-i}(x_i^*)<x_i^*$, then $v_i(x_i^*)\geq x_i^*$ by (3.2); if $r_{-i}(x_i^*)=x_i^*$, then $r_i(x_i^*)\geq x_i^*$ by the choice of i, so $r_i(x_i^*)=x_i^*$ and $v_i(x_i^*)=x_i^*$. In either case, Lemma 2.1 is proved.

For each $i\in N$, we define

$$y_i = \sup \{x\in[a,b] \mid v_i(x)\geq x\} \tag{3.4}$$

and choose i maximizing y_i; for simplicity, we assume $i=1$. Now we define $z_1=y_1$ and $z_j=r_j(z_1)$ for $j\in I(1)$. Reasoning as in the standard proof of Tarski's theorem, we can see that $v_1(z_1)=z_1$, i.e. (3.1) is satisfied for $i=1$. Lemma 2.1 implies

$$z_j \leq z_1 \text{ for all } j\in I(1). \tag{3.5}$$

Thus, if $I(1)\cup\{1\}=N$, (3.1) is satisfied for all $i\in N$.

Otherwise, we define a new system Σ^1 by the set $N^1=N\setminus\{1\}\setminus I(1)$, the same interval $[a,b]$ and, for all $i\in N^1$, sets $I^1(i)=I(i)\cap N^1$ and functions

$$r_i^{\,1}(x)=r_i(\max\{x, \max_{j\in I(i)\cap I(1)} z_j\}).$$

All the previous constructions, when applied to the system Σ^1, will be distinguished by the superscript ^{1}. Now we take the largest of $y_i^{\,1}$, $i\in N^1$, (defined by (3.4) with v_i replaced with $v_i^{\,1}$), assume it to be $y_2^{\,1}$, and define $z_2=y_2^{\,1}$, $z_j=r_j^{\,1}(z_2)$ for $j\in I^1(2)$. Inequality (3.5) for Σ^1 takes the form $z_j\leq z_2$ for all $j\in I^1(2)=I(2)\setminus I(1)$.

Lemma 2.2. $z_2\leq z_1$.

Suppose the contrary. If $I(1)\cap I(2)$ is empty, then for each $j\in I^1(2)=I(2)$ we have $r_j^{\,1}(z_2)=r_j(z_2)$ because $z_2>z_1\geq z_i$ for any $i\in I(1)\cap I(j)$. Therefore, $z_2=v_2^{\,1}(z_2)=v_2(z_2)$, contradicting the choice of z_1 as the maximum of y_j defined by (3.4).

If $I(1)\cap I(2)$ is not empty, we still have $r_j^{\,1}(z_2)=r_j(z_2)$ for all $j\in I(2)\setminus I(1)$. For $j\in I(1)\cap I(2)$, we have $z_j=r_j(z_1)\geq r_j(z_2)$. Combining both relations, we have $z_2=v_2^{\,1}(z_2)=r_2^{\,1}(r_{-2}^{\,1}(z_2))\leq r_2(r_{-2}(z_2))=v_2(z_2)$. This again contradicts the choice of z_1.

Lemma 2.2 is proved.

Now we define a new system Σ^2 by $N^2=N^1\setminus\{2\}\setminus I(2)$, the same $[a,b]$, $I^2(i)=I(i)\cap N^2$ and

$$r_i^{\,2}(x)=r_i(\max\{x, \max_{j\in I(i)\cap(I(1)\cup I(2))} z_j\})$$

for all $i\in N^2$, and repeat the same procedure for it, finding $z_3=\max_{i\in N^2} y_i^{\,2}$

and defining $z_j = r_j^2(z_3)$ for $j \in I^2(3)$. Eventually, we have z_i defined for all $i \in N$ and only have to show that they form a fixed point for Σ.

On each step of the process, we have just one element $i \in N$ for which a fixed point of v_i (to be more precise, of v_i^s) was chosen; let us call such elements basic. By our simplifying assumption, the basic elements form the subset $\{1,2,...,m\} \subseteq N$, where m is the total number of steps. It is easy to see that (3.1) is automatically satisfied for each basic element i. To establish (3.1) for non-basic i, we have to look at them a bit closer.

So let $i \in N$ be non-basic; then $z_i = r_i^{k-1}(z_k)$, where $1 \leq k \leq m$ and $i \notin I(s)$ for any $s<k$. By definition,

$$r_i^{k-1}(z_k) = r_i(\max\{z_k, \max_{j \in I(i) \cap (I(1) \cup ... \cup I(k-1))} z_j\});$$

on the other hand, repeatedly applying (3.5) and Lemma 2.2 along the process of defining z_j, we obtain $z_j \leq z_k$ for any $j \in I(i) \setminus (I(1) \cup ... \cup I(k\text{-}1) \cup \{k\})$. Thus $z_i = r_i^{k-1}(z_k) = r_i(\max_{j \in I(i)} z_j)$, i.e. exactly (3.1).

The Fundamental Lemma is proved.

Remark. The observation that the superposition of two decreasing functions is increasing, used by Vives (1990), could also be used to prove Theorem 1 while there seems to be no way to prove Theorem 2 with Milgrom and Roberts's reversing trick. On the other hand, where the trick works, it certainly provides the most elegant proof.

Let us now derive Theorem 2 from the Fundamental Lemma.

Denote, for each $i \in N$, $Y_i = f_i(X_i) \subseteq \mathbb{R}$ and $S_i = F_i(X_{-i})$. All Y_i are compact in their intrinsic topologies by our assumption; therefore, there exist $a = \min_{i \in N} \min Y_i$ and $b = \max_{i \in N} \max Y_i$. It is easy to check that all S_i are also compact in their intrinsic topologies.

Indeed, for any $i \in N$, $A \subseteq S_i$, and $j \in I(i)$, we denote $a^o = \sup A$, $A_j = \{a \in A \mid \exists x_{-i}\ a = F(x_{-i}) = f_j(x_j)\} \subseteq Y_j$, and $a^j = \sup_j A_j$, where $\sup_j$ means the least upper bound in Y_j, existing because of the compactness of Y_j in its intrinsic topology

(a° is the "genuine" supremum). Then we have $A=\bigcup_{j\in I(i)}A_j$, so $a^{\circ}\le\max_{j\in I(i)}a^j$; let $a^+=\min\{a^j \mid a^j\ge a^{\circ}\}$. Since $a^+\ge a^{\circ}$, it is an upper bound for A; since it is minimal, it is the least upper bound.

By our other assumption, $r_i(x_{-i})=q_i(\max_{j\in I(i)} f_j(x_j))$ with q_i defined and decreasing on S_i. Now for each $d\in[a,b]$ we define $\pi_i(d)=\sup_i\{s\in S_i \mid s\le d\}$, where $\sup_i$ means the least upper bound in S_i, and finally, define a mapping ρ_i: $[a,b] \to [a,b]$ by $\rho_i(d)=f_i\circ q_i\circ\pi_i(d)$. Each ρ_i is decreasing, so the Fundamental Lemma is applicable implying the existence of a fixed point $\xi\in[a,b]^N$ such that $\xi_i=\rho_i(\max_{j\in I(i)}\xi_j)$. Denoting $\sigma_i=\max_{j\in I(i)}\xi_j$ and $x_i^{\circ}=q_i\circ\pi_i(\sigma_i)$, we have $\xi_i=f_i\circ q_i\circ\pi_i(\sigma_i)=f_i(x_i^{\circ})$, hence $\sigma_i\in S_i$, hence $\pi_i(\sigma_i)=\sigma_i$; therefore, $x_i^{\circ}=q_i(\max_{j\in I(i)}f_j(x_j^{\circ}))=r_i(x_{-i}^{\circ})$. Theorem 2 is proved.

Remark. If we reverse the order on all X_i and replace each f_i with $-f_i$, then the maximum aggregation will be transformed into the minimum one. Thus the exact analogue of Theorem 2 is valid for the latter too.

Naturally, the mutuality condition is not necessary in any sense: if the restrictions $\langle I(i)\rangle_{i\in N}$ are described by a graph without odd cycles, a fixed point exists by Theorem 1. At the moment I can only demonstrate that the condition cannot simply be dropped.

Example 2. Let $N=\{1,2,3\}$, $I(1)=\{2,3\}$, $I(2)=\{1,3\}$, $I(3)=\{2\}$, $[a,b]=[0,3]$,

$$r_1(x)=\begin{cases}2, & x\ge 2,\\ 3, & x<2,\end{cases}\qquad r_2(x)=\begin{cases}0, & x=3,\\ 1, & x<3,\end{cases}\qquad r_3(x)=\begin{cases}1, & x\ge 1,\\ 2, & x<1.\end{cases}$$

It is easy to see that the system has no fixed point in the sense of (3.1): if $z_2=0$, then $z_3=r_3(z_2)=2$, $z_1=r_1(\max\{z_2,z_3\})=2$, so $z_2=r_2(\max\{z_1,z_3\})=1\ne z_2$; if $z_2=1$, then $z_3=r_3(z_2)=1$, $z_1=r_1(\max\{z_2,z_3\})=3$, so $z_2=r_2(\max\{z_1,z_3\})=0\ne z_2$.

Returning to the example with players in a circle at the end of the previous section, we see that if, additionally, each player's utility is only affected by the maximal (or minimal) of the choices of the neighbours, an equilibrium exists for odd n too.

4. Additive Aggregation

Theorem 3. Suppose we have a system of decreasing reactions where, for each $i \in N$, there is an increasing function f_i: $X_i \to \mathbb{R}$ such that $f_i(X_i)$ is compact in the Euclidean topology and r_i is decreasing w.r.t. $F_i(x_{-i}) = \Sigma_{j \neq i} f_j(x_j)$. Then there exists a fixed point x^o satisfying (1.1).

Just as in Theorem 2, the key role is played by a particular case.

Fundamental Lemma on Additive Aggregation. Assume given a finite set N, a real number $c>0$, and, for each $i \in N$, a decreasing function r_i: $[0,(n\text{-}1)c] \to [0,c]$. Then there exists a vector $x^o \in [0,c]^N$ such that

$$x_i^o = r_i(\Sigma_{j \neq i} x_j^o) \qquad \text{for all } i \in N. \qquad (4.1)$$

Proof of the Fundamental Lemma

Let us introduce necessary notations first. Throughout the proof, the variable x denotes a vector from $[0,c]^N$ with coordinates x_i; the inequality $x'' \geq x'$ is understood coordinate-wise, $x'' > x'$ means Pareto dominance ($\geq$ everywhere with $>$ somewhere); t is a real from $[0,nc]$ (a total); z is a pair $<t,x>$, we always assume $z'=<t',x'>$, $z''=<t'',x''>$, etc. unless explicitly defined otherwise. We extend each function r_i to the whole $[0,nc]$ by $r_i(s)=0$ for $(n\text{-}1)c<s\leq nc$, and define $B_i(t)=\{x_i \mid x_i=r_i(t\text{-}x_i)\}$, $B(t)=\Pi_{i \in N} B_i(t)$, $B=\{z \mid x \in B(t)\}$.

Obviously, x forms a fixed point, i.e. satisfies (4.1), if and only if $<\Sigma_{i \in N} x_i, x> \in B$. Following Novshek (1985), we start with a relaxed version of the condition:

$$\Sigma_{i \in N} x_i \leq t. \qquad (4.2)$$

Now denote C the set of $z \in B$ satisfying (4.2), C contains the point $<nc,0,...,0>$ and so is not empty, and denote D the closure of C in the Euclidean topology of $\mathbb{R}^{n+1}$ (which may be defined by the norm $|z|=\max\{|t|,$

$\max_{i\in N} |x_i| \}$).

Loosely speaking, we will search for $z\in C$ satisfying (4.2) as an equality by trying to minimize t and maximize (in the Pareto sense) x. The implementation of the idea is by no means straightforward. It makes no sense to compare x-components of $z\in C$ with different t-components directly; we have to learn how to "translate" x from t to t' first. Then a (strict) partial order P on D is defined, consisting of three components, one of them upper semi-continuous, two others without any good topological properties but simple enough by themselves. This combination of properties allows us to prove the existence of a maximizer of P over D; this maximizer turns out to be a maximizer of P over C. Finally, every such maximizer must satisfy the equality needed.

Since each r_i is monotonic, $r_i(s+0) = \lim_{s'\to s, s'>s} r_i(s') = \liminf_{s'\to s} r_i(s')$ and similarly $r_i(s-0)$ are well defined for every $s\in]0,nc[$. Denote $B_i^-(t)=\{x_i \mid r_i(t-x_i+0)\le x_i\}\supseteq B_i(t)$; $z\in D$ implies $x_i\in B_i^-(t)$ for each $i\in N$.

Let $i\in N$, $x_i\in B_i^-(t)$, $t'>t$; for any $y_i\le x_i$, we define $g(y_i)=r_i(t'-y_i)$; $g(y_i)\le x_i$ because $x_i\in B_i^-(t)$. Thus g maps $[0,x_i]$ into itself and is increasing. By Tarski's theorem, there exists the greatest fixed point of g; we denote it $\tau_i(t,x_i;t')\in B_i(t')$. It could also be defined as the maximum of the set $B_i(t')\cap[0,x_i]$ or as the ordinate of the first, after $<t-x_i, r_i(t-x_i)>$, point of the graph of r_i where the "cumulative reaction" is t'; without a reference to Tarski's theorem, however, its very existence could be unclear. For $t'=t$, we can use the same definition if $r_i(t-x_i)\le x_i$, again obtaining $\tau_i(t,x_i;t)\in B_i(t)$; if $x_i<r_i(t-x_i)$, we have to define $\tau_i(t,x_i;t)=x_i$, in which case it does not belong to $B_i(t)$. Thus $\tau_i(t,x_i;t)=x_i$ unless $r_i(t-x_i)<x_i$.

The vector form of the translation, $\tau(t,x;t')$ (or $\tau(z;t')$), is defined coordinate-wise. We will use it for $z\in D$; since t increases and x decreases, each translation still belongs to D (even to C if $t<t'$ or $x_i\ge r_i(t-x_i)$ for all $i\in N$). The following properties of τ are easy to verify:

$\tau(z;t') \leq x$ for any $z=(t,x) \in D$, $t \leq t'$;

$\tau(z;t'') \leq \tau(z;t')$ whenever $t \leq t' \leq t''$;

$\tau(t,x';t^*) \leq \tau(t,x'';t^*)$ whenever $x' \leq x''$, $t \leq t^*$;

$\tau(z;t'') = \tau(t',\tau(z;t');t'')$ whenever $t \leq t' \leq t''$.

Now we can define the following binary relations on D:

$z'' P_1 z'$ iff $\exists t^*$ such that $t^* > t'$, $t^* \geq t''$, and $\tau(z'';t^*) > \tau(z';t^*)$;

$z'' P_2 z'$ iff $t'' < t'$ and $\tau(z'';t') \geq \tau(z';t')$;

$z'' P_3 z'$ iff $t'' \leq t'$ and $\tau(z'';t') > \tau(z';t')$;

$z'' P z'$ iff $z'' P_1 z'$ or $z'' P_2 z'$ or $z'' P_3 z'$ (i.e. $P = P_1 \cup P_2 \cup P_3$).

The anti-reflexivity of P is obvious. It is also transitive, but we only need the transitivity of P_1.

Lemma 3.1. For any $z,z',z'' \in D$, $z'' P_1 z'$ and $z' P_1 z$ imply $z'' P_1 z$.

The definition of P_1 associates with each pair $<z',z>$ and $<z'',z'>$ an appropriate t^*. Let t^{**} be the greatest of the two t^*'s. We obviously have $\tau(z'';t^{**}) \geq \tau(z';t^{**}) \geq \tau(z;t^{**})$, and one of the inequalities is strict (i.e. Pareto dominance). Thus $z'' P_1 z$ and the lemma is proved.

Lemma 3.2. P_1 is upper semi-continuous, i.e. $z'' P_1 z'$ implies the existence of an open neighbourhood U of z' in D such that $z'' P_1 z$ for all $z \in U$.

Let us assume $\tau(z';t^*) = x^* < \tau(z'',t^*)$ (with $t' < t^*$, $t'' \leq t^*$) and define $\delta = (t^*-t')/2$, $U = \{z \in D \mid \mid z-z' \mid < \delta\}$. Let $z \in U$, $i \in N$; from $t < t'+\delta$ and $x_i > x_i'-\delta$, we easily obtain $t-x_i < t'-x_i'+2\delta \leq t^*-x_i^*$ (since $x_i^* \leq x_i'$), hence

$$x_i \geq r_i(t-x_i+0) \geq r_i(t^*-x_i^*) = x_i^*. \quad (4.3)$$

On the other hand, for any $y_i \leq x_i$, we have $t^*-y_i > t^*-x_i'-\delta > t'-x_i'$, hence $r_i(t^*-y_i) \leq r_i(t'-x_i'+0) \leq x_i'$; therefore, $B_i(t^*) \cap [0,x_i] \subseteq B_i(t^*) \cap [0,x_i']$. The maximum of the latter set is, by definition, $\tau_i(t',x_i';t^*) = x_i^*$. Combining this fact with (4.3), we obtain $\tau(z;t^*) = x^*$, hence $z'' P_1 z$.

Remark. P_2 is lower semi-continuous, but this is of no help to us.

A maximizer of P over D is $z \in D$ such that $z' P z$ is impossible for any $z' \in D$. The upper semi-continuity of P_1 implies the existence of a maximizer of P_1 over any compact set, but we need more than that.

Lemma 3.3. For every $z' \in D$, one of the three statements is true:

(i) z' is a maximizer of P_1 over D;

(ii) there exists a maximizer z'' of P_1 over D such that $z'' P_1 z'$;

(iii) there exists a sequence $z^{(n)} \to z^o$ such that $z^{(1)} P_1 z'$, $z^{(n+1)} P_1 z^{(n)}$ for $n=1,2,...$ and z^o is a maximizer of P_1 over D.

Remark. For a general upper semi-continuous relation, nothing more can be asserted. In our case, it seems likely that (iii) implies $z^o P_1 z'$, i.e. (ii), but I have not checked this carefully.

Suppose none of the statements holds and denote $Z=\{z \in D \mid zP_1 z'\}$. For each $z \in Z$, let $U_z=\{z'' \in D \mid zP_1 z''\}$; by Lemma 3.2, each U_z is open; by the negation of (ii), they cover Z because a maximizer of P_1 over Z is also a maximizer of P_1 over D. Since D, being a subset of $\mathbb{R}^{n+1}$, has a countable base of open sets, a countable family of U_z also covers Z (the Lindelöf theorem, see e.g. Kuratowski, 1966, p. 54). Denote X the set of corresponding $z \in Z$.

Now we apply Zorn's Lemma (Kuratowski, 1966, p. 27) to show the existence of a maximizer of P_1 over X. Consider a P_1-chain $Y \subseteq X$; if it has the greatest element, it is bounded; otherwise, it must be infinite. Since Y is countable, we may pick a sequence $z^{(n)} \in Y$ such that $z^{(n+1)} P_1 z^{(n)}$ for $n=1,2,...$ and for every $z \in Y$ there exists n such that $z^{(n)} P_1 z$ (Birkhoff, 1967, Theorem VIII.22, p.200). Since D is compact, we may assume $z^{(n)} \to z^o \in D$ without restricting generality. Since (iii) does not hold, there exists $z^* \in D$ such that $z^* P_1 z^o$; since P_1 is upper semi-continuous, $z^* P_1 z^{(n)}$ for all n big enough. Therefore, z^* is an upper bound for Y.

By Zorn's Lemma, there exists a maximizer z'' of P_1 over X, but U_z for $z \in X$ cover all $Z \supseteq X$, so there must exist $z \in X$ such that $zP_1 z''$. This contradiction proves the lemma.

Lemma 3.4. There exists a maximizer of P over D.

Denote Z^o the set of all maximizers of P_1 over D. By Lemma 3.3, Z^o is not empty; by Lemma 3.2, it is closed. Denote $t^o=\min_{z\in Z^o} t$, $D(t^o)=\{y\in\mathbb{R}^N \mid <t^o,y>\in D\}$ - a compact subset of $\mathbb{R}^N$, and $D^+(t^o)=\{y\in D(t^o) \mid \tau(t^o,y;t^o)=y\}$. $D^+(t^o)$ need not be compact, but it contains limits of all increasing sequences: indeed, if $y\in D(t^o)$ and $\tau_i(t^o,y_i;t^o)\neq y_i$, then $r_i(t^o\text{-}y_i)<y_i$ by the definition of τ_i and $r_i(t^o\text{-}y_i\text{-}0)=y_i$ because $y\in D(t^o)$; therefore, the open interval $]r_i(t^o\text{-}y_i),y_i[$ has no intersection with the range of r_i and y_i cannot be approximated from below. Now we pick $x\in D(t^o)$ such that $<t^o,x>\in Z^o$ and $x^+\in D^+(t^o)$ which is Pareto maximal on $D^+(t^o)$ and satisfies $x^+\geq\tau(t^o,x;t^o)$ (e.g. a maximizer of $\sum_{i\in N} y_i$ over $D^+(t^o)$ under the constraint $y\geq\tau(t^o,x;t^o)$), and denote $z^+=<t^o,x^+>\in D$.

It is easy to see that z^+ is a maximizer of both P_1 and P_3 over D: If $z'P_3z^+$, then $\tau(z';t^o)>\tau(t^o,x^+;t^o)=x^+$ and $\tau(z';t^o)\in D^+(t^o)$ contradict the Pareto maximality of x^+. If $z'P_1z^+$, then $z'P_1<t^o,x>$ because $\tau(z^+;t)\geq\tau(t^o,x;t)$ for every $t>t^o$, contradicting the choice of x.

Let us show z^+ to be a maximizer of P_2 too. Suppose the contrary: there exists $z'\in D$ such that $t'<t^o$ and

$$\tau(z';t^o) \geq \tau(z^+;t^o) = x^+. \qquad (4.4)$$

The definition of t^o implies $z'\notin Z^o$. If $z''P_1z'$, then $t^*>t^o$ (where t^* comes from the definition of P_1) would imply $z''P_1z^+$ while $t^*=t^o$ would imply $z''P_3z^+$; therefore, $t^*<t^o$, hence $z''\notin Z^o$. We see that neither (i) nor (ii) from Lemma 3.3 can hold; therefore, we must have (iii) with $t^{(n)}<t^o$ and $z^o=<t^o,x^o>$. Without restricting generality, we may assume that $t^{(n)}$ monotone increases and each $x_i^{(n)}$ either monotone increases or monotone decreases.

As we have just seen, for t_n^* from the definition of P_1 for $z^{(n+1)}P_1z^{(n)}$, there must be $t_n^*<t^o$, hence $\tau(z^{(n+1)};t^o)\geq\tau(z^{(n)};t^o)\geq\tau(z';t^o)\geq x^+$. Since x^+ is Pareto optimal, we must have equalities here. On the other hand, $\tau(z^{(n+1)};t_n^*)>\tau(z^{(n)};t_n^*)$ implies a strong inequality for some coordinates. Without restricting generality, we may assume that

$$\tau_i(z^{(n+1)};t_n^*) > \tau_i(z^{(n)};t_n^*) \qquad (4.5)$$

for some $i \in N$ and all $n=1,2,...$ Then the definition of τ implies that the sequence $x_i^{(n)}$ is strictly increasing: otherwise, $x_i^{(n)} \geq x_i^{(n+1)}$ would imply $\tau_i(t^{(n)},x_i^{(n)};t) \geq \tau_i(t^{(n+1)},x_i^{(n+1)};t)$ for any $t \geq t^{(n+1)}$, contradicting (4.5). Combining this with $x_i^{(n)} \geq x_i^{+}$ from (4.4), we obtain $x_i^{o} > x_i^{+}$ even though $\tau_i(t^o,x_i^o;t^o)=x_i^{+}$. Therefore, $x^o \notin D^{+}(t)$ and x_i^o cannot be approximated from below, see the argument at the start of the proof of the lemma. This contradiction shows that z' satisfying (4.4) cannot exist, so z^{+} maximizes P over D.

Lemma 3.5. There exists a maximizer of P over C.

Let $z \in D$ be a maximizer of P over D. Suppose first that $x_i < r_i(t-x_i)$ for some $i \in N$ and pick one such i. Then we define $x_i^o = r_i(t-x_i)$, $t^o = t-x_i+x_i^o > t$, $x_j^o = \tau_j(t,x_j;t^o)$ for $j \neq i$. We have $\sum_{j \in N} x_j^o = x_i^o + \sum_{j \neq i} x_j^o \leq t^o - t + \sum_{j \in N} x_j$; therefore, (4.2) for z implies (4.2) for z^o, hence $z^o \in C$. Obviously, $x^o > \tau(z;t^o)$, so $z^o P_1 z$, contradicting the choice of z.

Thus we have to conclude that $x_i \geq r_i(t-x_i)$ for all $i \in N$. Then we define $z^o = \tau(z;t) \in C$ and have $\tau(z;t') = \tau(z^o;t')$ for any $t' \geq t$. Since only these terms participate in the definition of P, $z' P z^o$ for any $z' \in D$ would imply $z' P z$, contradicting the choice of z. (If z was taken from the proof of Lemma 3.4, then $z^o = z$).

Lemma 3.6. If z is a maximizer of P over C, then (4.2) for z is satisfied as an equality.

Supposing the contrary, $\sum_{i \in N} x_i = t - \Delta$ with $\Delta > 0$, we denote $s_i = t - x_i$ $(i \in N)$. Now if there exist $i \in N$ and $s_i' \in]s_i - \Delta, s_i[$ such that $r_i(s_i') + s_i' \geq t$, then we may define z^*: $x_i^* = r_i(s_i')$, $t^* = s_i' + x_i^* \geq t$, $x_j^* = \tau_j(t,x_j;t^*)$ for $j \neq i$. We have $x_i^* - x_i = t^* - s_i' - x_i = t^* - (s_i + x_i) + (s_i - s_i') < t^* - t + \Delta$; Therefore, $\sum_{j \in N} x_j^* = x_i^* + \sum_{j \neq i} x_j^* \leq x_i^* - x_i + \sum_{j \in N} x_j < (t^* - t + \Delta) + (t - \Delta) = t^*$, hence $z^* \in C$. Furthermore, $x^* > \tau(z;t^*)$, hence $z^* P_1 z$ or $z^* P_3 z$, according as $t^* > t$ or $t^* = t$, contradicting the choice of z.

Thus we have to assume

$$r_i(s_i') + s_i' < t \text{ for all } i \in N,\ s_i' \in]s_i - \Delta, s_i[. \tag{4.4}$$

We denote $\delta=\Delta/(n+1)$, $x_i^*=r_i(s_i-\delta)$, $t_i^*=x_i^*+s_i-\delta$; by (4.4), we have $x_i\leq x_i^*<x_i+\delta$, $t-\delta\leq t_i^*<t$. Therefore, $\Sigma_{i\in N}x_i^*<\Sigma_{i\in N}x_i+n\delta=t-\delta$. Let us define $t^{**}=\max_{i\in N}\{t_i^*\}<t$, $x_i^{**}=\tau_i(t_i^*,x_i^*;t^{**})$ for all $i\in N$; clearly, $\Sigma_{i\in N}x_i^{**}\leq\Sigma_{i\in N}x_i^*<t^{**}$, so $z^{**}\in C$. On the other hand, by (4.4), $\tau(z^{**};t)=x$, so $z^{**}P_2z$. This contradiction proves Lemma 3.6 and, therefore, the Fundamental Lemma.

Remark 1. The proof of Lemma 3.5 and Lemma 3.6 together imply that every maximizer of P over D actually belongs to C and is therefore associated with a fixed point. The converse is not true: there may exist $z',z''\in C$ such that both satisfy (4.2) as an equality but $z''P_1z'$ (P_2 or P_3 are impossible here).

Remark 2. If z^o is a maximizer of P over C, then $q(t)=\tau(z^o;t)$ for $t\in[t^o,nc]$ is a selection from the correspondence $B(t)$ exactly of the type that Novshek (1985) constructs for the case of simple configurations. In this respect the above proof is even closer to Novshek's original argument than that of Kukushkin (1994). In the general case, however, there is no way to "construct" such a selection directly, without knowing z^o first; so we have a pure existence theorem.

Let us now derive Theorem 3 from the Fundamental Lemma.

Denote, for each $i\in N$, $Y_i=f_i(X_i)$, $S_i=\Sigma_{j\neq i}Y_j$ (note that each Y_i and S_i are compact subsets of $\mathbb{R}$), $a_i=\min Y_i$, $b_i=\max Y_i$, $a_{-i}=\Sigma_{j\neq i}a_j$, $b_{-i}=\Sigma_{j\neq i}b_j$, and $c=\max_{i\in N}(b_i-a_i)$; by our assumption, $r_i(x_{-i})=q_i(\Sigma_{j\neq i}f_j(x_j))$ with q_i decreasing. For each $d\in[0,(n-1)c]$, we define $\pi_i(d)=\max\{s_i\in S_i \mid s_i\leq a_{-i}+d\}$, and, finally, define a mapping ρ_i: $[0,(n-1)c]\to[0,c]$ by $\rho_i(d)=f_i\circ q_i\circ\pi_i(d)-a_i$. Each ρ_i is decreasing, so the Fundamental Lemma is applicable implying the existence of a fixed point $\xi\in[0,c]^N$ such that $\xi_i=\rho_i(\Sigma_{j\neq i}\xi_j)$. Denoting $\sigma_i=\Sigma_{j\neq i}\xi_j$ and $x_i^o=q_i\circ\pi_i(\sigma_i)$, we have $\xi_i+a_i=f_i\circ q_i\circ\pi_i(\sigma_i)=f_i(x_i^o)$, hence $\Sigma_{j\neq i}f_j(x_j^o)=a_{-i}+\sigma_i$, hence $\pi_i(\sigma_i)=\Sigma_{j\neq i}f_j(x_j^o)$ (by the definition of π_i); therefore, $x_i^o=q_i(\Sigma_{j\neq i}f_j(x_j^o))=r_i(x_{-i}^o)$. Theorem 3 is proved.

Remark. If we assume $X_i\subseteq\mathbb{R}$ and $f_i(x_i)=x_i$ ($i\in N$), we obtain the theorem of

Kukushkin (1994) as a corollary of Theorem 3.

Unfortunately, Theorem 3 provides no information about the players in a circle considered in the two previous sections: the proof relies on the presence of all x_j in the sum in (4.2).

To finish with additivity, let us show that the straightforward multi-dimensional analogue of Theorem 3 is not valid.

Example 3. Let us consider three mappings r_i: $\mathbb{R}^2 \to \mathbb{R}^2$ (i=1,2,3):

$$r_1(s_1,s_2) = \begin{cases} (1,0), & s_2 \leq 1, \\ (0,0), & s_2 > 1, \end{cases} \qquad r_2(s_1,s_2) = \begin{cases} (0,2), & s_2 \leq 0, \\ (0,0), & s_2 > 0, \end{cases}$$

$$r_3(s_1,s_2) = \begin{cases} (0,1), & s_1 \leq 0, \\ (0,0), & s_1 > 0. \end{cases}$$

All the three are decreasing, but no vector $x^o = \langle x_1^o, x_2^o, x_3^o \rangle$ from the Cartesian product of their ranges X_i can satisfy (4.1). Actually this situation is equivalent to that of Example 1: x_1 reacts to x_2, x_2 reacts to x_3, and x_3 to x_1. The extra dimension makes the restriction imposed by additivity futile. A small modification of r_i can make them strictly decreasing in each variable without a fixed point emerging.

5. Lexicographic Preorders

Let there be a system of decreasing reactions with N={1,2,3,4}, $X_i \subseteq \mathbb{R}$ ($i \in N$), and each function r_i: $X_{-i} \to X_i$ decreasing w.r.t. the lexicographic preorder ϑ_i described as follows. The players are arranged in a circle, and if the choices of the neighbours of player i at x_{-i}'' Pareto dominate those at x_{-i}', then $x_{-i}'' \vartheta x_{-i}'$; only if the choices of the neighbours are the same, the choice of the opposite player matters. Thus for i=1 we have: $r_1(x_2'',x_3'',x_4'') \leq r_1(x_2',x_3',x_4')$ if $(x_2'',x''_4) > (x_2',x_4')$, in the Pareto sense, or if $(x_2'',x_4'') = (x_2',x_4')$ and $x_3'' > x_3'$. And similarly for the others: every odd player reacts to choices of even players first and only then takes into account the choice of the odd fellow, whereas even players react to odd choices first. Without the "lexicographical additions" we would have a situation covered by Theorem 1 - a graph without an odd cycle. With them, it needs a special investigation.

Theorem 4. Every system of decreasing reactions with $N=\{1,2,3,4\}$, $X_i \subseteq \mathbb{R}$ $(i \in N)$ each compact in its intrinsic topology, and reactions decreasing w.r.t. preorders ϑ_i just described has a fixed point satisfying (1.1).

Fix x_2 and x_4; for players 1 and 3, we have a duopoly with decreasing reactions, which must have a fixed point $<q_1(x_2,x_4),q_3(x_2,x_4)>$ such that

$$\begin{cases} q_1(x_2,x_4)=r_1(x_2,x_4;q_3(x_2,x_4)), \\ q_3(x_2,x_4)=r_3(x_2,x_4;q_1(x_2,x_4)). \end{cases} \tag{5.1}$$

Since r_i are decreasing w.r.t. ϑ_i, both q_1 and q_3 are decreasing on $X_2 \times X_4$.

Quite similarly, for each x_1,x_3, there exist $q_2(x_1,x_3)$, $q_4(x_1,x_3)$ such that

$$\begin{cases} q_2(x_1,x_3)=r_2(x_1,x_3;q_4(x_1,x_3)), \\ q_4(x_1,x_3)=r_4(x_1,x_3;q_2(x_1,x_3)) \end{cases} \tag{5.2}$$

and both q_2 and q_4 are decreasing on $X_1 \times X_3$.

Now the system $<N,X_i,q_i>$ satisfies the assumptions of Theorem 1; therefore, there exists a fixed point $<z_i>$ satisfying

$$z_i=q_i(z_j,z_k) \tag{5.3}$$

(where j and k are the neighbours of i). Combining (5.3) with (5.1) and (5.2) for z, we obtain (1.1).

Theorem 5. Suppose there are given three sets $X_i \subseteq \mathbb{R}$ $(i=1,2,3)$ compact in their intrinsic topologies, there is an increasing function f: $X_3 \rightarrow \mathbb{R}$, and there are three functions r_i: $X_{-i} \rightarrow X_i$ such that $r_3(x_1,x_2)$ is decreasing in both arguments (not necessarily strictly), $r_1(x_2,x_3)$ is lexicographically decreasing in the sense that $r_1(x_2'',x_3'') \leq r_1(x_2',x_3')$ if $f(x_3'')>f(x_3')$, or if

$f(x_3'')=f(x_3')$ and $x_2''>x_2'$, or if $f(x_3'')=f(x_3')$, $x_2''=x_2'$ and $x_3''>x_3'$, while $r_2(x_1,x_3)$ is decreasing w.r.t. a similar ordering: first $f(x_3)$ matters, then x_1, and only then x_3. Then there exists a fixed point satisfying (1.1).

Let us assume the convention $i,j\in\{1,2\}$, $i\neq j$; denoting $V=f(X_3)$, we define the following functions for each $v\in V$:

$$\xi_i^-(v) = \inf_{x_3\in f^{-1}(v)} \inf_{x_3\in X_3} r_i(x_j;x_3) = \inf_{x_3\in f^{-1}(v)} r_i(\max X_j;x_3),$$

$$\xi_i^+(v) = \sup_{x_3\in f^{-1}(v)} \sup_{x_3\in X_3} r_i(x_j;x_3) = \sup_{x_3\in f^{-1}(v)} r_i(\min X_j;x_3),$$

$$\xi_3^-(v) = r_3(\xi_1^+(v),\xi_2^+(v)),\ \xi_3^+(v) = r_3(\xi_1^-(v),\xi_2^-(v)),$$
$$g^-(v) = f(\xi_3^-(v)),\ g^+(v) = f(\xi_3^+(v)).$$

Each function $\xi_i^-(v)$, $\xi_i^+(v)$, for $i=1,2$, is decreasing; moreover, $v'<v''$ implies $\xi_i^-(v')\geq\xi_i^+(v'')$ because $r_i(x_j';x_3')\geq r_i(x_j'';x_3'')$ for any x_j', x_j'', x_3', x_3'' such that $f(x_3')=v'$, $f(x_3'')=v''$. Therefore, $\xi_3^-(v)$, $\xi_3^+(v)$, $g^-(v)$ and $g^+(v)$ are increasing, and $v'<v''$ implies $g^+(v')\leq g^-(v'')$. So the correspondence $\tau(v)=[g^-(v),g^+(v)]$ satisfies the assumptions of Theorem 1 from d'Orey (1996), hence there exists $v^\circ\in V$ such that $g^-(v^\circ)=\{v^\circ\}=g^+(v^\circ)$.

Now for each $k=1,2,3$, we define $Y_k=X_k\cap[\xi_k^-(v^\circ),\xi_k^+(v^\circ)]$; by our definitions, we have $f(Y_3)=\{v^\circ\}$ and $r_k(Y_{-k})\subseteq Y_k$ for $k=1,2,3$. Given $y_2\in Y_2$, we have two decreasing mappings, $r_1(y_2,\cdot)$ and $r_3(\cdot,y_2)$, between Y_1 and Y_3. Therefore, there exists a fixed point $y_1=q_1(y_2)$, $y_3=q_3^{(2)}(y_2)$ such that $q_1(y_2)=r_1(y_2,q_3^{(2)}(y_2))$, $q_3^{(2)}(y_2)=r_3(q_1(y_2),y_2)$. Now $y_2'<y_2''$ implies $r_1(y_2',y_3')\geq r_1(y_2'',y_3'')$ for any y_3',y_3'' because $f(y_3')=f(y_3'')=v^\circ$; therefore, $q_1(\cdot)$ is decreasing. Similarly, there exist, for each $y_1\in Y_1$, $q_2(y_1)\in Y_2$ and $q_3^{(1)}(y_1)$ such that $q_2(y_1)=r_2(y_1,q_3^{(1)}(y_1))$ and $q_3^{(1)}(y_1)=r_3(y_1,q_2(y_1))$; $q_2(\cdot)$ is also decreasing. Now we again have two decreasing mappings, $q_1(\cdot)$ and $q_2(\cdot)$, between two sets Y_1 and Y_2, and again have a fixed point y_1°, y_2° such that $y_1^\circ=q_1(y_2^\circ)$ and $y_2^\circ=q_2(y_1^\circ)$. Define $y_3^\circ=q_3^{(1)}(y_1^\circ)=r_3(y_1^\circ,y_2^\circ)=$

$q_3^{(2)}(y_2^{\circ})$. Obviously, y_1°, y_2°, y_3° constitute the fixed point needed.

6. Conclusion: Open Questions

Since Theorem 1 gives a necessary and sufficient condition, there does not seem to be much room for extensions. Besides, the necessity is established with so simple an example that really nothing is left. As to the sufficiency part, if one is prepared to restrict oneself to finite sets, again everything is clear, due to that wonderful result of Roddy (1994): any finite sets with the fixed point property will do. For infinite sets, complete lattices seem to be the most safe solution: for all I know, the analogue of Roddy's theorem for this case is not yet established. In any case, the remaining problem belongs to the theory of fixed points for increasing mappings, see e.g. Fofanova et al. (1996) and references there.

An obvious open question about Theorem 2 is how to describe graphs of admissible dependencies for which the theorem remains true. The examples investigated so far do not inspire much hope for a compact solution. Another interesting question concerns multi-dimensional versions: the maximum aggregation can be defined on a lattice. Finally, the result "should" be extendible to reactions decreasing w.r.t. leximax (or leximin) ordering, but I have no idea how to do this at the moment.

Theorem 3 asks for an extension to reactions decreasing w.r.t. partial sums under some (mutuality?) conditions on admissible dependencies; however, a new technique for proofs seems to be needed. If Theorem 2 is extended to leximax and Theorem 3 to partial sums, the suspicion that they can be derived from the same general theorem might become overwhelming. Not everything is clear with possible multi-dimensional versions of the theorem despite the counter-example. An essential achievement would present an equilibrium existence result for Bayesian games with additive aggregation and (cardinal) strategic substitutes.

The Euclidean compactness of $f_i(X_i)$ in Theorem 3 looks like a serious obstacle to the unification of Theorems 2 and 3. If we only assumed each $f_i(X_i)$ compact in its intrinsic topology, their sum might not be compact (even in its intrinsic topology) and the proof would collapse. On the other hand, no counter-example disproving such a modification of the theorem is known at the moment. This seemingly minor technical problem is important for understanding relations, if any, between Novshek's and Tarski's fixed point theorems.

As to the results of Section 5, the main open question about them is whether they are doomed forever to remain queer isolated cases or may be eventually incorporated into a more respectable general theorem.

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