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The Max-Convolution Approach to Equilibrium Analysis

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Abstract: We consider a general competitive market model for trading various commodities. Commodities can be either perfectly divisible like water or inherently indivisible like houses. Money as a medium of exchange is treated as a perfectly divisible commodity and is always present in the market. There are a finite number of traders. Each trader is initially endowed with several units of each commodity and some amount of money. Traders' preferences depend on the bundle of commodities and the quantity of money they hold. All preferences are assumed to be quasi-linear in money. Using the max-convolution approach, we demonstrate that the market has a Walrasian equilibrium if and only if the potential market value function is concave with respect to the total initial endowment of commodities. This result holds uniformly true for both the divisible goods market and the indivisible goods market. We then offer several sufficient conditions on each individual trader in both markets.

Keywords: Market, indivisibility, Walrasian equilibrium, concave function, potential market value, max-convolution.

JEL classification: C6, C62, C68, D4, D41, D46, D5, D50, D51.

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1 Introduction

In this paper we consider a general competitive market model for trading various commodities. Commodities can be either perfectly divisible like water or inherently indivisible like houses. Money as a medium of exchange is treated as a perfectly divisible commodity and is always present in the market. There are a finite number of traders. Each trader is initially endowed with several units of each commodity and some amount of money. Traders' preferences depend on the bundle of commodities and the quantity of money they hold. All preferences are assumed to be quasi-linear in money. Using the max-convolution approach, we demonstrate that the market has a Walrasian equilibrium if and only if the potential market value function is concave with respect to the total initial endowment of commodities. Surprisingly, this result holds uniformly true for both the divisible goods market and the indivisible goods market. We then offer several sufficient conditions on each individual trader in both markets.

Due to quasilinearity in money, for the divisible goods market, the current model is less general than the standard Arrow-Debru model in case all utility functions are assumed to be strictly increasing. But the current model being simple does have several advantages. First, we are able to provide a simple argument free of fixed point theorems for the existence of a Walrasian equilibrium in a divisible goods market with rich enough structure. Our arguments are elementary and accessible to college students. Secondly, this model allows us to study both the divisible goods market and the indivisible goods market, simultaneously, although our attention will be focused on the latter market. In recent years, we have seen growing interest in studying the indivisible goods market. Several models have been investigated by Bikhchandani and Mamer (1997), Laan, Talman and Yang (1997,2002), Ma (1998), Bevia, Quinzii and Silva (1999), Gul and Stacchett (1999), Yang (2000,2001), Danilov, Koshevoy and Murota (2001), and Murota and Tamura (2002), which extend the earlier models introduced by Shapley and Scarf (1974), Gale (1984), Quinzii (1984), Svensson (1984), and Kaneko and Yamamoto (1986) in one way or another. We point out that the quasilinearity in money is a standard assumption in most of the recent literature; see for example, Bikhchandani and Mamer (1997), Ma (1998), Bevia et al. (1999), Gul and

Stacchett (1999), Yang (2001), and Murota and Tamura (2002). Thirdly, the reservation value functions of traders over goods can be very general and are not restricted to monotonically increasing functions. In fact, they can be any concave functions. Therefore, the current model can also apply to the financial market where there are many risky assets and one riskless asset and the utility function of each investor over risky assets may have a satiation portfolio. Satiation refers to the situation where there is an optimal portfolio beyond which the increased return of holding more assets may not be sufficient to offset the increased risk. Finally, the assumption of quasilinearity in money enables us to use the max-convolution approach to analysize the model. Given two functions f_1 and f_2 defined on $\mathbb{R}^n$, the new function f defined as $f(x) = \sup_y\{f_1(x-y) + f_2(y)\}$ is called a max-convolution of f_1 and f_2. This functional operation is one of the techniques originally developed in convex analysis. See for example, Rockafellar (1970). Using this approach leads us to establish a natural connection between equilibrium and concavity.

The rest of the article is organized as follows. In Section 2 we introduce the market model and basic concepts. In Section 3 we establish two necessary and sufficient conditions for the existence of an equilibrium in the model. Finally, in Section 4, we provide several sufficient conditions on the behavior of each individual trader.

2 The Unified Market Model

First, we introduce some notation. The set I_k denotes the set of the first k positive integers. The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$. We will use W^n to denote either $\mathbb{R}^n$ or $\mathbb{Z}^n$. The vector $\mathbf{0}$ denotes the vector of zeros. The vector $e(i)$, $i \in I_n$, is the ith unit vector of $\mathbb{R}^n$. Furthermore, $x \cdot y$ means the inner product of vectors x and y.

Consider a market for trading various commodities. In the market there are m agents (or traders), n commodities, and a perfectly divisible good called money. The set of all agents will be denoted by $T = \{1, 2, \cdots, m\}$. Each agent i is initially endowed with a bundle $\omega^i \in W^n_+$ of goods and some amount m_i of money. Let ω stand for the total goods in the market, i.e., $\omega = \sum_{i\in A}\omega^i$. Thus, for each good $h = 1, \cdots, n$, there are ω_h units available in the market. It is understood that $\omega_h > 0$ for every $h = 1, \cdots, n$. Each agent i's preferences over goods and money are quasilinear: that is, the utility of agent i holding c units of money and the bundle x of goods can be expressed as

$$u_i(x, c) = V_i(x) + c,$$

where $V_i(x)$ is the reservation value $V_i(x)$, the quantity of money that agent i valuates the bundle x of goods.

For each $i \in T$, the reservation value function $V_i : W^n \mapsto \mathbb{R}$ is assumed to be bounded from above. This assumption plainly states that no good is infinitely desirable. Finally, each agent i is assumed to have a sufficient amount m_i of money in the sense that $m_i \geq \sup_{x \in W^n} V_i(x) - V_i(\omega^i)$. This means that if the prices of a bundle x of goods is no greater than the reservation value $V_i(x)$, then agent i can afford to buy the bundle x. Since V_i is bounded above, m_i is finite. This market model will be represented by $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, W^n)$, where the commodities space W^n is the market indicator. That is, if $W^n = \mathbb{R}^n$, then the market is a divisible goods market; if $W^n = \mathbb{Z}^n$, then the market is an indivisible goods market where goods can be houses, cars, and computers. Note that money is always present in both markets.

A family $(x^1, x^2, \cdots, x^m)$ of bundles $x^i \in W^n$ is called a (feasible) allocation if $\sum_{i \in A} x^i = \omega$. An allocation $(x^1, x^2, \cdots, x^m)$ is *socially efficient* if it is an optimal solution of the following problem:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} V_i(y^i) \\ s.t. & \sum_{i=1}^{m} y^i = \omega \\ & y^i \in W^n,\ i = 1, 2, \cdots, m. \end{array} \tag{2.1}$$

A price vector $p \in \mathbb{R}^n$ indicates a price (units of money) for each good. Suppose that the goods are exchanged on the market at a price vector $p \in \mathbb{R}^n$. If agent i buys a bundle x of goods, then he has to pay $p \cdot x$. When a price vector $p \in \mathbb{R}^n$ is given, then each agent i will optimally choose bundle of goods under his budget constraints. That is, the demand of goods by agent i is defined by

$$D_i(p) = \{x \mid (V_i(x) + p(\omega^i - x)) = \max\{V_i(y) + p(\omega^i - y) \mid p \cdot y \leq m_i + p \cdot \omega^i,\ \ y \in W^n\}\}.$$

Note that $m_i \geq \sup_{x \in W^n} V_i(x) - V_i(\omega^i)$ for every $i \in T$. This implies that the budget constraint $p \cdot y \leq m_i + p \cdot \omega^i$ is redundant. Thus, the set $D_i(p)$ can be simplified as

$$D_i(p) = \{x \mid (V_i(x) - p \cdot x) = \max\{V_i(y) - p \cdot y \mid \ y \in W^n\}\}.$$

A tuple $((x^1, x^2, \cdots, x^m); p)$ is a *Walrasian equilibrium* if p is a vector in $\mathbb{R}^n$; and if $x^i \in D_i(p)$ for every $i \in T$; and if $\sum_{i \in T} x^i = \omega$. The allocation $(x^1, x^2, \cdots, x^m)$ will be called an equilibrium allocation. Thus, at equilibrium, each agent gets his best bundle of goods under his budget constraint and moreover market is clear. The following simple lemma indicates that a free market mechanism will lead to a socially efficient allocation of scarce resources.

Lemma 2.1 *Suppose that the allocation $(x^1, x^2, \cdots, x^m)$ is an equilibrium allocation. Then it must be socially efficient.*

Proof: Since $(x^1, x^2, \cdots, x^m)$ is an equilibrium allocation. Then there must exist $p \in \mathbb{R}^n$ such that $((x^1, x^2, \cdots, x^m); p)$ is a Walrasian equilibrium. Then we have for all $i \in T$ and all $y \in W^n$ it holds

$$V_i(x^i) - p \cdot x^i \geq V_i(y) - p \cdot y.$$

Let $(y^1, y^2, \cdots, y^m)$ be any allocation such that $\omega = \sum_{i=1}^m y^i$ with $y^i \in W^n$. Then we have

$$V_i(x^i) - p \cdot x^i \geq V_i(y^i) - p \cdot y^i,$$

for all $i \in T$. It follows that $\sum_{i=1}^m V_i(x^i) \geq \sum_{i=1}^m V_i(y^i)$. This implies that $(x^1, x^2, \cdots, x^m)$ is an optimal solution of problem (2.1) and so it is socially efficient. □

The above lemma shows that an equlibrium allocation is indeed interesting and appealing. Unfortunately, an equilibrium allocation may not always exist, as the following example demonstrates. In an indivisible goods market, there are two traders 1 and 2 and two indivisible goods in the market. Trader 1 initially has one unit of each indivisible good and five dollars, while trader 2 has only one unit of the indivisible good 1 and ten dollars. Their reservation value functions are given by $V_1((x_1, x_2)) = 0$ if $x_1 = 0$ or $x_2 = 0$ for all $(x_1, x_2) \in \mathbf{Z}_+^2$, and $V_1((x_1, x_2)) = 10$ for all integrs $x_1 > 0$ and $x_2 > 0$; $V_2((0,0)) = 0$, $V_2((0, x_2)) = 10$ for all integers $x_2 \geq 1$, $V_2((1,0)) = 10$, $V_2((x_1, 0)) = 20$ for all integers $x_1 \geq 2$, and $V_2((x_1, x_2)) = 20$ for all integers $x_1 \geq 1$ and $x_2 \geq 1$. Clearly, V_1 and V_2 are weakly increasing and bounded and their marginal utilities are weakly decreasing. Obviously, the prices of the goods must be nonnegative. We have to consider the following four possibilities. In case the equilibrium prices $p_1 = 0$ and $p_2 = 0$, then a possible equilibrium allocation must be such that trader 1 gets one unit of each indivisible good and trader 2 gets one unit of the indivisible good 1. But this cannot be an equilibrium since trader 2 would demand one unit of each indivisible good, or two units of the indivisible good 1. In case the equilibrium prices $p_1 = 0$ and $p_2 > 0$, then a possible equilibrium allocation must be such that trader 2 gets two units of the indivisible good 1 and trader 1 gets one unit of the indivisible good 2. But this cannot be an equilibrium since trader 1 would demand one unit of each indivisible good, or nothing. In case the equilibrium prices $p_1 > 0$ and $p_2 = 0$, then a possible equilibrium allocation must be such that trader 2 gets one unit of each indivisible good and trader 1 gets one unit of the indivisible good 1. But this cannot be an equilibrium since trader 1 would demand nothing. In case the equilibrium prices $p_1 > 0$ and $p_2 > 0$, then one possible equilibrium allocation must be such that trader 1 gets one unit of each indivisible good and trader 2 gets one unit of the indivisible good 1. But this cannot be an equilibrium since for trader 2 we would have $10 - p_1 \geq 20 - p_1 - p_2$ and $10 - p_1 \geq 20 - 2p_1$, which imply $p_1 \geq 10$ and $p_2 \geq 10$. Thus, trader 1 would not demand one unit of each indivisible good, yielding a contradiction. Another possible equilibrium allocation must be such that trader 2 gets all indivisible goods and trader 1 gets nothing. But this cannot be an equilibrium either since trader 2 would just like to demand one unit of each indivisible good. In summary, the market has no equilibrium. Other non-existence examples for indivisible goods markets can be found in Bikhchandani and Mamer (1997), Ma (1998), and Bevia et al. (1999), Yang (2001).

3 Equilibrium Existence Theorems

In this section we will establish several existence theorems for the market model. Recall that ω is the total initial endowment of goods. We define the following potential market value function on W^n:

$$R(x) = \sup\{\sum_{i\in T} V_i(x^i) \mid \sum_{i\in T} x^i = x, x^i \in W^n\}.$$

$R(x)$ is the maximal market value that can be achieved by all the trades with the resource vector x.

Now we are ready to present our first existence theorem which gives a necessary and sufficient condition for the existence of a Walrasian equilibrium.

Theorem 3.1 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, W^n)$, there exists a Walrasian equilibrium if and only if the following system of linear inequalities has a solution $p \in \mathbb{R}^n$*

$$p \cdot (x - \omega) \ \geq \ R(x) - R(\omega), \ \forall x \in W^n.$$

Proof: Suppose that $((x^{1*}, x^{2*}, \cdots, x^{m*}); p^*)$ is a Walrasian equilibrium. Then we have for all $i \in T$ and all $y \in W^n$ it holds

$$V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(y) - p^* \cdot y. \tag{3.2}$$

It follows from Lemma 2.1 that $\sum_{i=1}^m V_i(x^{i*}) = R(\omega)$. For any $x \in W^n$, then there must exist $x^i \in W^n$ with $\sum_{i=1}^m x^i = x$ such that $\sum_{i=1}^m V_i(x^i) = R(x)$. It follows from (3.2) that

$$V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(x^i) - p^* \cdot x^i, \ i = 1, 2, \cdots, n.$$

Therefore we have

$$p^* \cdot (x - \omega) \geq R(x) - R(\omega), \ \forall x \in W^n.$$

On the other hand, suppose that $p^* \in \mathbb{R}^n$ is a solution of the following system

$$p \cdot (x - \omega) \ \geq \ R(x) - R(\omega), \ \forall x \in W^n.$$

Let $(x^{1*}, x^{2*}, \cdots, x^{m*})$ be any allocation so that $\omega = \sum_{i=1}^m x^{i*}$ and $R(\omega) = \sum_{i=1}^m V_i(x^{i*})$ with $x^{i*} \in W^n$. Note that such allocation always exists by the definition of $R(x)$. We will show that $((x^{1*}, x^{2*}, \cdots, x^{m*}); p^*)$ is a Walrasian equilibrium. For any agent i and any $y \in W^n$, let $x = \sum_{l \neq i} x^{l*} + y$. By assumption we have $R(\omega) - p^* \cdot \omega \geq R(x) - p^* \cdot x$. By definition of $R(x)$, we have $R(x) \geq \sum_{l \neq i} V_l(x^{l*}) + V_i(y)$. Therefore, it follows that

$$\begin{aligned}\textstyle\sum_{i=1}^m V_i(x^{i*}) - p^* \cdot \sum_{i=1}^m x^{i*} &= R(\omega) - p^* \cdot \omega \\ &\geq R(x) - p^* \cdot x \\ &\geq \textstyle\sum_{l \neq i} V_l(x^{l*}) + V_i(y) - p^* \cdot (\sum_{l \neq i} x^{l*} + y).\end{aligned}$$

The above implies that

$$V_i(x^{i*}) - p^* \cdot x^{i*} \geq V_i(y) - p^* \cdot y.$$

Since i and y are taken arbitrarily, it is clear that $((x^{1*}, x^{2*}, \cdots, x^{m*}), p^*)$ is indeed a Walrasian equilibrium. □

In the above theorem, the equilibrium price of each good may be postive, zero, or even negative. The following lemma gives a rather weak condition to ensure that all goods have positive equilibrium prices.

Lemma 3.2 *Suppose that the market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, W^n)$ has a Walrasian equilibrium. If $R(\omega + e(i)) > R(\omega)$ for all $i \in I_n$, then the equilibrium prices for all goods are positive.*

Clearly, if the reservation value function of each trader is strictly increasing, then all goods must have positive equilibrium prices.

Now we turn to study the behavior of the market potential value function R. In general, the function $R : W^n \mapsto \mathbb{R}$ is said to be *concave* if, for any points x^1, x^2, $\cdots$, x^l in W^n with any convex parameters λ_1, λ_2, $\cdots$, λ_l, it holds

$$R(\sum_{h=1}^{l} \lambda_h x^h) \geq \sum_{h=1}^{l} \lambda_h R(x^h).$$

In particular, the function $R : W^n \mapsto \mathbb{R}$ is said to be *concave with respect to* ω if ω is a convex combination of points x^1, x^2, $\cdots$, x^l in W^n with convex parameters λ_1, λ_2, $\cdots$, λ_l, then we have

$$R(\omega) \geq \sum_{h=1}^{l} \lambda_h R(x^h).$$

Clearly, if R is a concave function, then R must be a concave function with respect to ω. The other way around is not true. Note that when $W^n = \mathbb{Z}^n$, a concave function R will also be called a discrete concave function.

Our second theorem below establishes a natural connection between Walrasian equilibrium and local concavity. More precisely, it says that the market has a Walrasian equilibrium if and only if the market potential value function R is a concave function with respect to ω.

Theorem 3.3 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, W^n)$, there exists a Walrasian equilibrium if and only if the market potential value function $R : W^n \mapsto \mathbb{R}$ is a concave function with respect to ω.*

Proof: By Theorem 3.1 it is sufficient to show that the market potential value function R is a concave function with respect to ω if and only if the following system of linear inequalities has a solution $p \in \mathbb{R}^n$.

$$p \cdot (x - \omega) \;\; \geq \;\; R(x) - R(\omega), \;\; \forall x \in W^n.$$

Suppose that p^* is a price vector satisfying the above inequalities. Let $x^1, x^2, \cdots, x^l \in W^n$ with convex parameters λ_1, λ_2, $\cdots$, λ_l such that $\omega = \sum_{h=1}^{l} \lambda_h x^h$. Since

$$R(\omega) - p^* \cdot \omega \geq R(x^h) - p^* \cdot x^h, \;\; h = 1, 2, \cdots, l,$$

and $\lambda_h \geq 0$ for $h = 1, 2, \cdots, l$, then we have

$$\lambda_h(R(\omega) - p^* \cdot \omega) \geq \lambda_h(R(x^h) - p^* \cdot x^h), \;\; h = 1, 2, \cdots, l.$$

Since $\sum_{h=1}^{l} \lambda_h = 1$ and $\omega = \sum_{h=1}^{l} \lambda_h x^h$, it follows that

$$R(\omega) \geq \sum_{h=1}^{l} \lambda_h R(x^h).$$

Thus, the potential market value function R is a concave function with respect to ω.

On the other hand, suppose that the potential market value function R is a concave function with respect to ω. Then, by definition, if ω is a convex combination of points x^1, x^2, $\cdots$, x^l in W^n with convex parameters λ_1, λ_2, $\cdots$, λ_l, then we have

$$R(\omega) \geq \sum_{h=1}^{l} \lambda_h R(x^h). \tag{3.3}$$

Now let G be the graph of the function R, i.e.,

$$G = \{(x, R(x)) \mid x \in W^n\}.$$

Let H be the convex hull of the set G, which is a closed convex set. Take an arbitrary point $(\omega, z) \in H$. Then there exist x^1, x^2, $\cdots$, x^l in W^n with convex parameters λ_1, λ_2, $\cdots$, λ_l, such that $\omega = \sum_{h=1}^{l} \lambda_h x^h$ and $z = \sum_{h=1}^{l} \lambda_h R(x^h)$. It follows from (3.3) that

$$(\omega, R(\omega)) \geq (\omega, z).$$

This implies that $(\omega, R(\omega))$ is a boundary point of the set H. The well known separation theorem implies that there exists a nonzero vector $(-p, t) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$-p \cdot \omega + tR(\omega) \geq -p \cdot y + tz$$

for all $(y, z) \in H$. In particular, we have

$$-p \cdot \omega + tR(\omega) \geq -p \cdot x + tR(x), \tag{3.4}$$

for all $x \in W^n$.

Since $\omega_h > 0$ for all $h = 1, 2, \cdots, n$, and ω lies in the interior of W^n, it is easy to see that there does not exists any nonzero vector $p \in \mathbb{R}^n$ such that

$$-p \cdot \omega \geq -p \cdot x, \ \forall x \in W^n.$$

This means that $t \neq 0$. It follows from (3.3) and (3.4) that t can be made positive. Without loss of generality, we may assume $t = 1$. Now the system (3.4) implies

$$p \cdot (x - \omega) \geq R(x) - R(\omega), \ \forall x \in W^n.$$

The proof is complete. □

The following corollary is a straightforward result of Theorem 3.3.

Corollary 3.4 *Given a market model $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, W^n)$, where the parameters m_i, ω^i, $i \in T$, are changing, there exists a Walrasian equilibrium for every $\omega = \sum_{i \in T} \omega^i$ if and only if the market potential value function $R : W^n \mapsto \mathbb{R}$ is a concave function.*

The above two theorems hold uniformly true for both the divisible goods market and the indivisible goods market. Let us return to the previous non-existence example. For this example, both reservation value functions V_1 and V_2 are discrete concave functions on $\mathbb{Z}^2_+$. We have $R((1,1)) = 20$, $R((2,1)) = 20$, and $R((3,1)) = 30$. Because $R((2,1)) = 20 < (R((1,1)) + R((3,1))/2 = 25$, the function R is not concave with respect to $\omega = (2,1)$ and thus the market has no equilibrium.

Note that the conditions stated in both theorems above are imposed on the collective behaviors of all traders. In the next section we will provide several sufficient conditions on the behaviors of each individual trader.

4 Max-Convolution Preservable Functions

In this section we will identify agents' reservation value functions for the existence of Walrasian equilibrium. For this purpose, we will introduce a class of max-convolution preservable functions. Let $f_i : W^n \mapsto \mathbb{R}$, $i = 1, \cdots, m$ with $m \geq 2$, be any class of functions bounded from above. Define a new function $R : W^n \mapsto \mathbb{R}$ as

$$R(x) = \sup\{\sum_{i=1}^{m} f_i(x^i) \mid \sum_{i=1}^{m} x^i = x, x^i \in W^n\}.$$

The function R will be called a *max-convolution function*. Suppose every function f_i exhibits some given property A. Then functions f_i, $i = 1, \cdots, m$, are said to be *max-convolution A preservable* if the function R preserves property A. We also say that property A is max-convolution preservable. In particular, we are intereseted in max-convolution concave preservable functions. As shown in the previous section, if each reservation value function V_i is max-convolution concave preservable, then the market has a Walrasian equilibrium.

When the commodities space W^n is $\mathbb{R}^n$, then we have the following lemma; see e.g., Rockafellar (1970).

Lemma 4.1 *Let $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ be continuous concave functions for $i = 1, \cdots, m$. Then functions f_i, $i = 1, \cdots, m$, are max-convolution concave preservable.*

As a consequence, we have that every divisible goods market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{R}^n)$, has a Walrasian equilibrium if reservation value functions $V_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in T$, are continuous and concave.

In general, it is reasonable to assume that the reservation value function of each trade over goods is weakly increasing. But for financial goods, this may not be the case. For example, consider a financial market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{R}^n)$ where money is a riskless asset and all other n goods are risky assets. In this case, the reservation value function of each trader may not be monotone and could be any concave function with an unconstrained global maximum at some portfolio. In a mean-variance model, the reservation value function takes the standard form: $V_i(x) = V_i(\rho, \sigma) = V_i(x \cdot s^i, \frac{1}{2}x \cdot T_i x)$, where $s^i \in \mathbb{R}^n$ is the perceived mean vector of trader i, and $T_i \in \mathbb{R}^{n \times n}$ is the perceived covariance matrix of trader i. In general, these matrices T_i and vectors s^i may differ cross traders. It is assumed that the matrix T_i is symmetric positive definite; V_i is a strictly concave C^1 function of x with

$$\frac{\partial V_i(\rho, \sigma)}{\partial \rho} > 0, \quad \frac{\partial V_i(\rho, \sigma)}{\partial \sigma} < 0.$$

Thus, V_i is a mean-variance utility function and is an increasing function of the expected return ρ and a decreasing function of the expected variance σ. It is not difficult to show that trader i has an unconstrained global maximum at some portfolio x. Such an situation is called *a satiation portfolio.* Since V_i, $i \in T$, are continuous, concave and bounded from above, the financial market clearly has an equilibrium.

When the commodities space W^n is $\mathbb{Z}^n$, things become much more complicated. The non-existence example in Section 2 indicates that not every discrete concave function is max-convolution preservable. Clearly, both functions V_1 and V_2 are discrete concave functions on $\mathbb{Z}^2_+$. But, the max-convolution function R of V_1 and V_2 fails to be discrete concave as shown at the end of the previous section.

Fortunately, Murota (1998) and Murota and Shioura (1999) have recently introduced a class of remarkable discrete concave functions which have several fundamental combinatorial properties similar to those of continous concave functions. A function $f : \mathbb{Z}^n \mapsto \mathbb{R}$ is called an *$M^\natural$-concave* if it satisfies the following condition:

For any $x, y \in \mathbb{Z}^n$ and $k \in \text{supp}^+(x-y)$ with $\text{supp}^+(x-y) \neq \emptyset$,

$$\begin{aligned} f(x) + f(y) \ \leq \ & \max[f(x - e(k)) + f(y + e(k)), \\ & \max_{l \in \text{supp}^-(x-y)}\{f(x - e(k) + e(l)) + f(y + e(k) - e(l))\}] \end{aligned}$$

where $\text{supp}^+(x-y) = \{k \in I_n \mid x_k > y_k\}$ and $\text{supp}^-(x-y) = \{k \in I_n \mid x_k < y_k\}$.

The functions $f : \mathbb{Z}^n \mapsto \mathbb{R}$ given as $f(x) = a \cdot x + c$ with $a \in \mathbb{R}^n$, and as $f(x) = \sum_{i=1}^n g_i(x_i)$ where $g_i : \mathbb{Z} \mapsto \mathbb{R}$, $i \in I_n$, are discrete concave, are all simple examples of M$^\natural$-concave function. The reader can easily verify that the reservation value function V_1 in the non-existence example at Section 2 is not M$^\natural$-concave by considering the points $x = (2, 1)$ and $y = (0, 0)$. The following result is due to Murota (1998).

Theorem 4.2 *Let $f_i : \mathbb{Z}^n \mapsto \mathbb{R}$ be M$^\natural$-concave functions for $i = 1, \cdots, m$. Then functions f_i, $i = 1, \cdots, m$, are max-convolution concave preservable.*

As a consequence, we have that every indivisible goods market $\mathcal{M} = (V_i, m_i, \omega^i, i \in T, \mathbb{Z}^n)$, has a Walrasian equilibrium if reservation value functions $V_i : \mathbb{Z}^n \mapsto \mathbb{R}$, $i \in T$, are M$^\natural$-concave.

Finally we will introduce the well-known Kelso and Crawford's gross substitutes condition. Consider a market with m traders and n indivisible objects (or goods), denoted by $N = \{1, 2, \cdots, n\}$. Each trader i has a reservation value function over the objects, denoted by $V_i : 2^N \to \mathbb{R}$, where 2^N is the collection of all subsets of N. It is assumed that $V_i(\emptyset) = 0$ and V_i is weakly increasing. Given a price vector $p \in \mathbb{R}^n$, the demand set $D_i(p)$ of trader i is defined as

$$D_i(p) = \{S \mid (V_i(S) - \textstyle\sum_{h \in S} p_h) = \max\{V_i(T) - \sum_{h \in T} p_h \mid \ T \subseteq N\}\}.$$

For the existence of an equilibrium, Kelso and Crawford (1982) introduced the following condition with respect to $D_i(p)$, known as *gross substitutes*.

(1) For any two price vectors p and q such that $p \leq q$, and any $A \in D_i(p)$, there exists $B \in D_i(q)$ such that $\{i \in A \mid p(i) = q(i)\} \subseteq B$.

Fujishige and Yang (2000) have recently shown that a reservation value function $V : 2^N \mapsto \mathbb{R}$ satisfies the gross substitutes condition if and only if V is M$^\natural$-concave.

Note that the M$^\natural$-concave function is now specified on a set function. It reads as follows: a set function $f : 2^N \to \mathbb{R}$ is an M$^\natural$-concave function if for each $S, T \subseteq N$ and $s \in S \setminus T$ with $S \setminus T \neq \emptyset$ the function f satisfies

$$f(S) + f(T) \leq \max[f(S-s) + f(T+s), \max_{t \in T \setminus S}\{f(S-s+t) + f(T+s-t)\}].$$

In the above formula, we read $S - s$ and $T + s$ as $S \setminus \{s\}$ and $T \cup \{s\}$, respectively.

The reader can find more applications of M$^\natural$-concave functions in Danilov et al. (2001) and Murota and Tamura (2002).

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