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Reconfirming the Prenucleolus

by

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Reconfirming the Prenucleolus*

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Abstract

By means of an example it is shown that the prenucleolus is not the only minimal solution that satisfies nonemptiness, Pareto optimality, covariance, the equal treatment property and the reduced game property, even if universe of players is infinite. This example also disproves a conjecture of Gurvich et al. Moreover, we prove that the prenucleolus is axiomatized by nonemptiness, covariance, the equal treatment property, and the reconfirmation property, provided the universe of players is infinite.

1 Introduction and Notation

The prenucleolus and the prekernel are widely accepted solutions for cooperative transferable utility games. Introduced as auxiliary solutions of the prebargaining set, they became important solutions in their own rights, heavily supported by the fact that they can be justified by simple and intuitive axioms. Both are closely related, because they share many properties and because one, the prenucleolus, is a subsolution of the other. Indeed, both solutions are *nonempty* (NE), *Pareto optimal* (PO), *covariant under strategic equivalence* (COV), *anonymous* (AN), and satisfy the *equal treatment property* (ETP)

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and the *reduced game property* (RGP). One additional property for each of the solutions suffices to characterize it: The prenucleolus is *single-valued* (SIVA) and the prekernel satisfies the *converse reduced game property* (CRGP). In fact, the prenucleolus is axiomatized by means of SIVA, COV, AN, and RGP (see Sobolev (1975)), whereas the prekernel is axiomatized by NE, PO, COV, ETP, RGP, and CRGP (see Peleg (1986)). AN can be replaced by ETP in Sobolev's axiomatization (see Orshan (1993)) and CRGP in Peleg's axiomatization can be replaced by a *maximality principle*: The prekernel is the maximum solution that satisfies the remaining axioms NE, PO, COV, ETP, and RGP. In view of the fact that the prenucleolus of a game is a distinguished special point of the prekernel, the following questions arise in a natural way:

(1) Is it possible to replace SIVA by NE and a *minimality principle*, i.e., is the prenucleolus the minimum (or at least the **unique** minimal) solution that satisfies NE, PO, COV, ETP, and RGP?

(2) Is it possible to find an intuitive axiom playing the rôle of a "minimality principle" that characterizes the prenucleolus, if the "maximality principle" CRGP is replaced by this axiom, i.e., is the prenucleolus characterized by NE, PO, COV, ETP, RGP, and some additional intuitive axiom?

Answers to the Questions (1) and (2) are presented in this paper which is organized as follows:

Precise definitions of properties and of solutions are recalled in the current section. In Section 2 we show that the answer to Question (1) is negative. Indeed, we construct a solution $\widehat{\sigma}$ which is **finite-valued** and satisfies NE, PO, COV, AN, ETP, RGP, and which does not contain the prenucleolus (see Theorem 2.7). Thus, $\widehat{\sigma}$ contains a minimal subsolution having the desired properties (see Corollary 2.8). This result yields a new aspect of the impact of SIVA in Sobolev's and in Orshan's axiomatization of the prenucleolus. Hence it "reconfirms" the prenucleolus in the sense that it implicitly reveals a part of the special character of this solution.

As a byproduct, Theorem 2.7 disproves the following conjecture raised by Gurvich, Menshikova, and Menshikov (1994): Any TU game is the reduced game of a "huge" game, the prekernel of which consists of the prenucleolus only, with respect to the prenucleolus of the "huge" game. If this conjecture was true, then a proof of it would also yield a new proof of Sobolev's or Orshan's result, because any solution with the desired properties is a subsolution of the prekernel.

In Section 3, Question (2) is answered. Since 1993 we know that there is a suitable axiom which yields the desired characterization. This axiom requires from any member of the solution of every game that every element of the solution of the reduced game with respect to every coalition, when combined with the restriction of the initial element to the complement coalition, establishes a member of the solution of the initial game. Recently Hwang and Sudhölter (2001) called this intuitive axiom the *reconfirmation property* (RCP), which reflects a natural interpretation of the property, and they employed RCP in an axiomatization of the core, which emphasizes the importance of this axiom. Hence, the prenucleolus is characterized by NE, COV, ETP, RGP, and RCP. Note that PO can be deduced from the first four axioms. Surprisingly it turns out that RGP is **not** needed in this characterization, that is, the prenucleolus is axiomatized by NE, COV, ETP, and RCP (see Theorem 3.4).

We think that this axiomatization gives a new insight into the character of the prenucleolus and that it reinforces RCP as a significant property which plays an important rôle in some characterizations of "classical" solutions like the core and the prenucleolus. Of course, this result also demonstrates the "power" of RCP, when combined with ETP. Indeed, AN, even together with PO and RGP, does not replace ETP in Theorem 3.4, because, for example, the *positive* core (see Definition 2.1) satisfies NE, PO, AN, COV, RGP, and RCP. Also it should be remarked that, for single-valued solutions, RGP and RCP are equivalent. If single-valuedness is not required, then there are important solutions which satisfy RCP, but violate RGP. For example, the least core satisfies NE, PO, AN, COV, and RCP, but it violates RGP. Especially this fact demonstrates that RCP stands for itself as an interesting property, independently of RGP.

Up to the end of this section some relevant definitions and results from Maschler, Peleg, and Shapley (1972) and Peleg (1986) are recalled. Let U be a universe of players containing, without loss of generality, $1, \ldots, k$ whenever $|U| \geq k$. A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N \subseteq U$ is finite and $v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$. For any game (N, v) let

$$X(N,v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \text{ and } \mathcal{I}^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible* and *Pareto optimal* feasible payoffs (*preimputations*), respectively. We use $x(S) = \sum_{i \in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$. For disjoint coalitions $S, T \in 2^N$ let $(x_S, x_T) = x_{S \cup T}$. For $x \in \mathbb{R}^N,\ S \subseteq N$, and distinct players $k, l \in N$ let

$$e(S,x,v) = v(S) - x(S) \text{ and } s_{kl}(x,v) = \max_{S \subseteq N \setminus \{l\} : k \in S} e(S,x,v)$$

denote the *excess* of S and the *maximal surplus* of k over l, respectively, at x with respect to (w.r.t.) (N, v). The *prekernel* of (N, v) is given by

$$\mathcal{K}^*(N,v) = \{x \in \mathcal{I}^*(N,v) \,|\, s_{kl}(x,v) = s_{lk}(x,v)\ \forall k \in N,\ l \in N \setminus \{k\}\}.$$

For $X \subseteq \mathbb{R}^N$ let $\mathcal{N}((N,v);X)$ denote the *nucleolus* of (N, v) w.r.t. X, i.e. the set of members of X that lexicographically minimize the nonincreasingly ordered vector of excesses of the coalitions (see Schmeidler (1969)). It is well-known that the nucleolus w.r.t. $X(N,v)$ is a singleton, the unique element of which is called the *prenucleolus* of (N, v) and is denoted by $\nu(N, v)$.

In general, a *solution* σ associates with each game (N, v) a subset of $X(N, v)$. Let σ be a solution. Then σ

(1) is *covariant under strategic equivalence* (COV), if for all games $(N,v), (N,w)$ satisfying $w = \beta v + z$ for some $\beta > 0, z \in \mathbb{R}^N$ the equation $\sigma(N,w) = \beta\sigma(N,v) + z$ holds. (Here we use the convention which identifies $z \in \mathbb{R}^N$ with the *additive* coalitional function, again denoted by z, on the player set N defined by $z(S) = \sum_{i \in S} z_i$ for all $S \in 2^N$. Also note, that the games v and w are called *strategically equivalent*.);

(2) is *nonempty* (NE), if $\sigma(N,v) \neq \emptyset$ for every game (N,v);

(3) is *Pareto optimal* (PO), if $\sigma(N,v) \subseteq \mathcal{I}^*(N,v)$ for every game (N,v);

(4) is *single-valued* (SIVA), if $|\sigma(N,v)| = 1$ for every game (N,v);

(5) is *anonymous* (AN), if the following condition is satisfied for all games (N,v) and (M,w). If $\pi : N \to M$ is a bijection such that $\pi v = w$, then $\sigma(M,w) = \pi(\sigma(N,v))$ (In this case the games (N,v) and (M,w) are *isomorphic.*);

(6) satisfies the *equal treatment property* (ETP), if for every game (N,v), for every $x \in \sigma(N,v)$, $x_k = x_l$ for all *substitutes* $k,l \in N$ (k and l are *substitutes*, if $v(S \cup \{k\}) = v(S \cup \{l\})\ \forall S \subseteq N \setminus \{k,l\}$.);

(7) satisfies the *reduced game property* (RGP), if for every game (N,v), $\emptyset \neq S \subseteq N$, and every $x \in \sigma(N,v)$, $x_S \in \sigma(S, v^{S,x})$ (The *reduced game* $(S, v^{S,x})$ is defined by $v^{S,x}(\emptyset) = 0$, $v^{S,x}(S) = v(N) - x(N \setminus S)$, and $v^{S,x}(T) = \max_{Q \subseteq N \setminus S}(v(T \cup Q) - x(Q))$ for $\emptyset \neq T \subsetneq S$);

(8) satisfies the *converse reduced game property* (CRGP), if for every game (N,v) with $|N| \geq 2$ the following condition is satisfied for every $x \in \mathcal{I}^*(N,v)$: If, for every $S \subseteq N$ with $|S| = 2$, $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N,v)$;

(9) satisfies the *reconfirmation property* (RCP), if for every game (N,v), $\emptyset \neq S \subseteq N$, for every $x \in \sigma(N,v)$ and $y \in \sigma(S, v^{S,x})$, $(y, x_{N \setminus S}) \in \sigma(N,v)$.

For interpretations and discussions, in particular of the variants 7, 8, and 9 of the reduced game property, see Peleg (1986) and Hwang and Sudhölter (2001).

2 Non-Uniqueness of a Minimal Solution

This section is devoted to the construction of a solution $\widehat{\sigma}$ which satisfies NE, COV, PO, ETP, RGP, and which does not contain the prenucleolus as a subsolution. Moreover, as it is finite-valued, the solution contains a minimal subsolution satisfying the axioms. As a byproduct the constructive proof provides an example which disproves the conjecture of Gurvich, Menshikova, and Menshikov (1994) mentioned in Section 1. Throughout this section we shall assume that $|U| \geq 4$.

First a specific *coalition structure*, i.e., a partition of the set of players, is defined. Let (N,v) be a game and $x = \nu(N,v)$. Define the binary relation $\sim_v$ on N by

$$k \sim_v l \Leftrightarrow k = l \text{ or } (k \neq l \text{ and } s_{kl}(x,v) \leq 0)$$

and note that $\sim_v$ is reflexive and transitive. It is also symmetric (an equivalence relation), because $x \in \mathcal{K}^*(N,v)$. Let $\mathcal{T}(N,v)$ denote the set of equivalence classes of $\sim_v$. In Section 3 of Maschler, Peleg, and Shapley (1972) the collection $\mathcal{T}(N,v)$ of coalitions is the partition, which corresponds to the smallest excess strictly greater than 0, of the *profile* generated by the prekernel element $x = \nu(N,v)$. (Of course $\mathcal{T}(N,v) = \{N\}$ in the case that there is no coalition of positive excess.)

Definition 2.1 *Let (N, v) be a game and $x = \nu(N, v)$. Let $t^+ = \max\{t, 0\}$ denote the positive part of the real number t. The* **positive core** *of (N, v) is the set*

$$\mathcal{C}^+(N, v) = \{y \in X(N, v) \mid (e(S, x, v))^+ = (e(S, y, v))^+ \ \forall S \subseteq N\}.$$

Note that the positive core was first mentioned by Orshan (1994), who fully described this solution in the three-person case. Though this solution may be regarded as an interesting core extension, in the present paper it serves as an auxiliary solution only.

Lemma 2.2 *The positive core satisfies* RGP *and* RCP.

Though this lemma is contained in Lemma 2.2 of Sudhölter (1993), we shall present a proof which proceeds analogously to the proof of RGP of the prenucleolus due to Peleg (1988).

Proof: Let (N, v) be a game, $x \in \mathcal{I}^*(N, v)$ and $\emptyset \neq S \subseteq N$. For $\alpha \in \mathbb{R}$ define

$$\mathcal{D}(\alpha, x, v) = \{S \subseteq N \mid e(S, x, v) \geq \alpha\} \cup \{\emptyset, N\}.$$

Then Definition 2.1 can be formulated as

$$x \in \mathcal{C}^+(N, v) \Leftrightarrow \mathcal{D}(\alpha, x, v) = \mathcal{D}(\alpha, \nu(N, v), v) \ \forall \alpha > 0. \tag{2.1}$$

According to Kohlberg (1971) the prenucleolus is characterized by the equivalence

$$x = \nu(N, v) \Leftrightarrow \left(\forall \alpha \in \mathbb{R} \ \forall y \in \mathbb{R}^N : \begin{array}{c} y(N) = 0, \ y(T) \geq 0 \ \forall T \in \mathcal{D}(\alpha, x, v) \\ \Rightarrow y(T) = 0 \ \forall T \in \mathcal{D}(\alpha, x, v) \end{array} \right). \tag{2.2}$$

By (2.1) and (2.2) the positive core is characterized by the equivalence

$$x \in \mathcal{C}^+(N, v) \Leftrightarrow \left(\forall \alpha > 0 \ \forall y \in \mathbb{R}^N : \begin{array}{c} y(N) = 0, \ y(T) \geq 0 \ \forall T \in \mathcal{D}(\alpha, x, v) \\ \Rightarrow y(T) = 0 \ \forall T \in \mathcal{D}(\alpha, x, v) \end{array} \right). \tag{2.3}$$

Note that $x_S \in \mathcal{I}^*(N, v^{S,x})$ and

$$\mathcal{D}(\alpha, x_S, v^{S,x}) = \{S \cap T \mid T \in \mathcal{D}(\alpha, x, v)\} \ \forall \alpha \in \mathbb{R}. \tag{2.4}$$

By (2.4), for any $\alpha \in \mathbb{R}$, the set

$$\{y_S \in \mathbb{R}^S \mid y_S(S) = 0, \ y_S(Q) \geq 0 \ \forall Q \in \mathcal{D}(\alpha, x_S, v^{S,x})\}$$

is the projection of

$$\{y \in \mathbb{R}^N \mid y(N) = 0, \ y_i = 0 \ \forall i \in N \setminus S, \ y(T) \geq 0 \ \forall T \in \mathcal{D}(\alpha, x, v)\},$$

thus RGP is implied by (2.3). In order to show RCP let $x \in \mathcal{C}^+(N, v)$ and $z \in \mathcal{C}^+(S, v^{S,x})$. By RGP, $x_S \in \mathcal{C}^+(S, v^{S,x})$, thus $\mathcal{D}(\alpha, x_S, v^{S,x}) = \mathcal{D}(\alpha, z, v^{S,x}) \ \forall \alpha > 0$ by (2.1). Hence, by (2.4) applied to $(z, x_{N \setminus S})$, $\mathcal{D}(\alpha, (z, x_{N \setminus S}), v) = \mathcal{D}(\alpha, x, v)$ for every $\alpha > 0$. Thus (2.3) finishes the proof. **q.e.d.**

Remark 2.3 Let (N, v) be a game and $x = \nu(N, v)$. By the definition of the positive core we obtain

$$\forall k, l \in N,\ k \neq l: \begin{pmatrix} s_{kl}(x,v) > 0 \Rightarrow s_{kl}(x,v) = s_{kl}(y,v) \text{ and} \\ s_{kl}(x,v) \leq 0 \Leftrightarrow s_{kl}(y,v) \leq 0 \end{pmatrix} \forall y \in \mathcal{C}^+(N,v).$$

Let (N, v) be a game. For any total order $\preceq$ of $\mathcal{T} = \mathcal{T}(N,v)$, let us say $\mathcal{T} = \{T_1, \ldots, T_t\}$ with $T_1 \prec \cdots \prec T_t$, recursively define $\alpha^v_{\preceq} \in \mathbb{R}^{\mathcal{T}}$ (We abbreviate $\alpha^v_{\preceq}$ by α if there is no danger of confusion.) by

$$\alpha(T_1) = \min\{y(T_1) \mid y \in \mathcal{C}^+(N,v)\} \text{ and} \tag{2.5}$$
$$\alpha(T_i) = \min\{y(T_i) \mid y \in \mathcal{C}^+(N,v),\ y(T_j) = \alpha(T_j)\ \forall j = 1, \ldots, i-1\} \tag{2.6}$$

for all $i = 2, \ldots, t$ and put

$$\mathcal{I}^*_{\preceq}(N,v) = \{z \in \mathbb{R}^N \mid z(T) = \alpha^v_{\preceq}(T)\ \forall T \in \mathcal{T}(N,v)\}. \tag{2.7}$$

Remark 2.4 Let (N, v) be a game. Note that $\alpha^v_{\preceq}$ is well-defined, because $\mathcal{C}^+(N,v)$ is nonempty and compact. Therefore $\mathcal{I}^*_{\preceq}(N,v)$ is a nonempty convex set of preimputations, which, by Equations (2.5), (2.6), and (2.7), intersects the positive core, i.e., we have

$$\mathcal{C}^+_{\preceq}(N,v) := \mathcal{C}^+(N,v) \cap \mathcal{I}^*_{\preceq}(N,v) \neq \emptyset. \tag{2.8}$$

This subset of the positive core can also be expressed as

$$\mathcal{C}^+_{\preceq}(N,v) = \{y \in \mathcal{C}^+(N,v) \mid (y(T_1), \ldots, y(T_t)) \leq_{lex} (z(T_1), \ldots, z(T_t))\ \forall z \in \mathcal{C}^+(N,v)\}. \tag{2.9}$$

Lemma 2.5 *Let (N, v) be a game and $\preceq$ be a total order of $\mathcal{T}(N,v)$. Then the nucleolus of (N, v) w.r.t. $\mathcal{I}^*_{\preceq}(N,v)$ is a singleton, which belongs to $\mathcal{C}^+(N,v)$.*

Proof: By (2.8), $\mathcal{C}^+_{\preceq}(N,v) \neq \emptyset$. Hence

$$\mathcal{N}\big((N,v); \mathcal{I}^*_{\preceq}(N,v)\big) = \mathcal{N}\big((N,v); \mathcal{C}^+_{\preceq}(N,v)\big) \tag{2.10}$$

by the definition of the positive core. By (2.9) the latter set of preimputations is a nonempty, convex, and compact set. Thus $|\mathcal{N}((N,v); \mathcal{C}^+(N,v))| = 1$ (see Schmeidler (1969)). **q.e.d.**

Definition 2.6 *For any game (N, v) and any total order $\preceq$ of $\mathcal{T}(N,v)$ let $\nu_{\preceq}(N,v)$ be the unique member of $\mathcal{N}\big((N,v); \mathcal{I}^*_{\preceq}(N,v)\big)$. The solution $\widehat{\sigma}$ is defined by*

$$\widehat{\sigma}(N,v) = \{\ \nu_{\preceq}(N,v)\ \mid\ \preceq \text{ is a total order of } \mathcal{T}(N,v)\}.$$

Theorem 2.7 *The solution $\widehat{\sigma}$ satisfies* NE, PO, COV, ETP, RGP, *and* AN. *Moreover, then the prenucleolus is not a subsolution of $\widehat{\sigma}$.*

Proof: Let (N,v) be a game and $\preceq$ be a total order of $\mathcal{T} := \mathcal{T}(N,v)$.

(1) We show that $\widehat{\sigma}$ satisfies NE, AN, PO, and COV. The vector $x := \nu_{\preceq}(N,v)$ is a member of $\widehat{\sigma}(N,v)$, thus $\widehat{\sigma}$ satisfies NE. A bijection $\pi : N \to M$ maps $\mathcal{T}$ bijectively to $\mathcal{T}(M,\pi v)$, because the prenucleolus satisfies AN. Thus $\widehat{\sigma}$ satisfies AN as well. Moreover, $\widehat{\sigma}$ satisfies PO, because it is a subsolution of $\mathcal{C}^+$. Let $\beta > 0$, $z \in \mathbb{R}^N$, $w := \beta v + z$, and $y := \beta x + z$. Then $\mathcal{T}(N,w) = \mathcal{T}$, because the prenucleolus satisfies COV. Moreover, by COV of the positive core (which can be shown in a straightforward way), the equation

$$\mathcal{C}^+_{\preceq}(N,w) = \beta \mathcal{C}^+_{\preceq}(N,v) + z$$

is valid, thus $\nu_{\preceq}(N,w) = \beta \nu_{\preceq}(N,v) + z = y$ by (2.10). Hence $\widehat{\sigma}$ satisfies COV.

(2) In order to show ETP and RGP a *derived* game $(N, v_{\preceq})$ is defined by

$$v_{\preceq}(S) = \begin{cases} v(S) & , \text{ if } S \in 2^N \setminus \mathcal{T}, \\ \alpha^v_{\preceq}(T) & , \text{ if } S = T \text{ for some } T \in \mathcal{T}. \end{cases}$$

Then

$$\mathcal{N}\big((N,v); \mathcal{I}^*_{\preceq}(N,v)\big) = \mathcal{N}\big((N,v_{\preceq}); \mathcal{I}^*_{\preceq}(N,v)\big), \tag{2.11}$$

because $v_{\preceq}$ differs from v at most on the partition $\mathcal{T}$, the elements T of which receive a fixed amount $z(T) = \alpha^v_{\preceq}(T)$ by every preimputation z of $\mathcal{I}^*_{\preceq}(N,v)$. The expression of the right hand side of (2.11) is the prenucleolus $\nu((N,v_{\preceq}); \mathcal{T})$ of the game $(N, v_{\preceq})$ with *coalition structure* $\mathcal{T}$. Thus,

$$x := \nu_{\preceq}(N,v) = \nu((N,v_{\preceq}); \mathcal{T}). \tag{2.12}$$

It is well-known that the prenucleolus of a game with coalition structure is a member of the prekernel of the game with coalition structure. For every pair (k,l) of distinct players the equation

$$s_{kl}(x,v) = \begin{cases} s_{kl}(x, v_{\preceq}) & , \text{ if } k,l \in T \text{ for some } T \in \mathcal{T}, \\ s_{kl}(\nu(N,v), v) & , \text{ otherwise}, \end{cases} \tag{2.13}$$

is a consequence of the definition of the derived game and of Lemma 2.2 and Remark 2.3, respectively. Equations (2.12) and (2.13) imply $x \in \mathcal{K}^*(N,v)$. Thus $\widehat{\sigma}$ is a subsolution of the prekernel which satisfies ETP.

In order to show that $\widehat{\sigma}$ satisfies RGP, let $\emptyset \neq S \subseteq N$ and $w = v^{S,x}$. Using the well-known fact

$$s_{kl}(x_S, w) = s_{kl}(x,v) \;\forall k,l \in S \text{ with } k \neq l$$

we obtain

$$\mathcal{T}(S,w) = \{T \cap S \mid T \in \mathcal{T} \text{ and } T \cap S \neq \emptyset\}. \tag{2.14}$$

Let $\preceq^S$ be the total order on $\mathcal{T}(S,w)$ *consistent* with $\preceq$, which is defined by the following requirement. If $T, T' \in \mathcal{T}$ satisfy $T \prec T'$ and $T \cap S \neq \emptyset \neq T' \cap S$, then $T \cap S \prec^S T' \cap S$. By RGP and RCP of the positive core (see Lemma 2.2),

$$\mathcal{C}^+_{\preceq^S}(S,w) = \{y \in \mathbb{R}^S \mid (y, x_{N\setminus S}) \in \mathcal{C}^+_{\preceq}(N,v)\}$$

and, thus,

$$\mathcal{I}^*_{\preceq^S}(S,w) = \{y \in \mathbb{R}^S \mid (y, x_{N\setminus S}) \in \mathcal{I}^*_{\preceq}(N,v)\}.$$

Therefore $(v_{\preceq})^{S,x} = w_{\preceq^S}$. Hence $\nu_{\preceq^S}(S,w) = x_S$ by RGP of the prenucleolus of games with coalition structures (see Theorem 5.2.7 of Peleg (1988)).

(3) In order to show that ν is not a subsolution of $\widehat{\sigma}$ the following "cyclic" 4-person game (M,u) is defined by $M = \{1,\ldots,4\}$ and

$$u(S) = \begin{cases} 1 & , \text{ if } S \in \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}, \\ 0 & , \text{ if } S \in \{\emptyset, M\}, \\ -2 & , \text{ otherwise.} \end{cases}$$

Note that (M,u) is *transitive*. (A game is transitive, if its symmetry group, i.e., the group of permutations of N which do not change the game, is transitive.) Indeed, the cyclic permutation, which maps 1 to 2, 2 to 3, 3 to 4, and 4 to 1, is a symmetry. Hence, by AN and PO, we obtain $\nu(M,u) = 0 \in \mathbb{R}^M$. Therefore, $x \in \mathcal{C}^+(M,u)$, iff $x(S) = 0$ for every $S = \{1,2\},\ldots,\{4,1\}$, and $x(T) \geq -2$ for every $T \subseteq M$. These inequalities show that $\mathcal{C}^+(M,u) = \text{convex hull}\{(-1,1,-1,1),(1,-1,1,-1)\}$. Also, $\mathcal{T}(M,u) = \{\{k\} \mid k \in M\}$. Thus $\widehat{\sigma}(M,u) = \{(-1,1,-1,1),(1,-1,1,-1)\}$. **q.e.d.**

Corollary 2.8 *There is a minimal solution that satisfies* NE, PO, COV, ETP, RGP, AN, *and that does not coincide with the prenucleolus.*

Proof: Let Σ denote the partially ordered set of subsolutions of $\widehat{\sigma}$ which satisfy NE, PO, COV, ETP, RGP, and AN. By Theorem 2.7, $\widehat{\sigma} \in \Sigma$. In order to show that a chain Σ^0 (a subset of comparable elements) has a lower bound we verify that σ^0, defined by $\sigma^0(N,v) = \bigcap_{\sigma\in\Sigma^0} \sigma(N,v)$, belongs to Σ. The solution σ^0 satisfies PO, COV, ETP, RGP, and AN, because all members of the chain satisfy these axioms. Moreover, σ^0 satisfies NE, because any $\sigma \in \Sigma$ is finite-valued. Hence, by Zorn's Lemma, Σ has a minimal element. By Theorem 2.7, $\nu \notin \Sigma$. **q.e.d.**

In Section 4.4 of Gurvich, Menshikova, and Menshikov (1994) the following question is raised: Let (N,v) be a game. Is there any game $(\widetilde{N},\widetilde{v})$ such that a) $N \subseteq \widetilde{N}$, b) $\mathcal{K}^*(\widetilde{N},\widetilde{v})$ is a singleton consisting of the prenucleolus y only, and c) $\widetilde{v}^{N,y} = v$? A positive answer to this question would yield a new proof Sobolev's or Orshan's axiomatization of the prenucleolus. Theorem 2.7 shows that the answer to this question is negative. Moreover, the game (M,u) defined in the last part of the proof of this theorem is an explicit "counter" example. (Note that a one-parameter set of games which contains (M,u) is discussed in Orshan (1994) and a variant of (M,u) is used to prove the main result of Sudhölter and Peleg (2001).)

3 An Axiomatization of the Prenucleolus

This section is devoted to show that the prenucleolus is axiomatized by NE, COV, ETP, and RCP, provided $|U| = \infty$. We shall use the following theorem.

Theorem 3.1 (Orshan (1993)) *The unique solution that satisfies* SIVA, COV, ETP, *and* RGP *is the prenucleolus, provided* $|U| = \infty$.

Let σ be a solution. Let Γ^2 be the set of games with at most two persons. The following lemmata are useful.

Lemma 3.2 *Assume that* $|U| \geq 2$ *and that* σ *satisfies* NE, COV, ETP, *and* RCP. *Then* $\sigma(N, v) = \nu(N, v)$ *for every* $(N, v) \in \Gamma^2$.

Proof: Let (M, u) be a two-person game. By NE there exists $x \in \sigma(M, u)$. By ETP and COV there exists $a \in \mathbb{R}$ such that $x_i = u(\{i\}) + a$ for $i \in M$. Let $i \in M$ and $u^i = u^{\{i\},x}$. By NE there exists $y \in \sigma(\{i\}, u^i)$. By COV, $y^\alpha := \alpha(y - u^i(\{i\})) + u^i(\{i\}) \in \sigma(\{i\}, u^i)$ for every $\alpha > 0$. By RCP, $(y^\alpha, x_{M\setminus\{i\}}) \in \sigma(M, u)$, thus $y^\alpha = x_i$ for all $\alpha > 0$. We conclude that $y^\alpha = y$ for all $\alpha > 0$, hence the proof is complete. **q.e.d.**

For any game (N, v) and any $x \in \mathbb{R}$ let $\mu(x, v) = \max_{S \subseteq N} e(S, x, v)$.

Lemma 3.3 *Under the assumptions of Lemma 3.2 and the additional assumption that* $|U| = \infty$, *the following assertions are valid:*

(1) σ *satisfies* PO.

(2) Let (N, v) *be a game, let* $x \in \sigma(N, v)$, *and let* $i \in N$. *Then there exist* $S^i, S^{-i} \subseteq N$ *such that* $i \in S^i$, $i \notin S^{-i}$, *and* $e(S^i, x, v) = e(S^{-i}, x, v) = \mu(x, v)$.

Proof: Let (N, v) be a game. As $|U| = \infty$ we may assume that $N \subseteq U \setminus \{1, 2\}$ and that $\widetilde{N} = \{1, 2\} \cup N \subseteq U$. By NE there exists $x \in \sigma(N, v)$. Define $\widetilde{v}(S)$, $S \subseteq \widetilde{N}$, by $\widetilde{v}(S) = v(S \setminus \{1, 2\})$. (That is, $(\widetilde{N}, \widetilde{v})$ arises from (N, v) by adding the two null-players 1 and 2.) By NE there exists $\widetilde{x} \in \sigma(\widetilde{N}, \widetilde{v})$. By ETP, $\widetilde{x}_1 = \widetilde{x}_2$.

Claim 1: $\widetilde{x}(\widetilde{N}) = \widetilde{v}(\widetilde{N})$: Let $\widetilde{v}^1 = \widetilde{v}^{\{1\},\widetilde{x}}$. By Lemma 3.2, $\sigma(\{1\}, \widetilde{v}^1) = \{\widetilde{v}^1(\{1\})\}$. By RCP, $(\widetilde{v}^1(\{1\}), \widetilde{x}_{N\setminus\{1\}}) \in \sigma(\widetilde{N}, \widetilde{v})$. By ETP, $\widetilde{v}^1(\{1\}) = \widetilde{x}_2 = \widetilde{x}_1$, thus $\widetilde{x}(\widetilde{N}) = \widetilde{v}(\widetilde{N})$.

Claim 2: $\widetilde{x}_1 = \widetilde{x}_2 = 0$: For every $i \in N$ let $\widetilde{v}^{i,1} = \widetilde{v}^{\{1,i\}}$. Assume the contrary. Then two cases may occur:

(1) $\widetilde{x}_1 = \widetilde{x}_2 < 0$: Then $e(\{1\}, \widetilde{x}, \widetilde{v}) = -\widetilde{x}_1 > 0$, thus $\mu(\widetilde{x}, \widetilde{v}) > 0$. Let $S \subseteq \widetilde{N}$ attain $\mu(\widetilde{x}, \widetilde{v})$. By Claim 1, $\emptyset \neq S \neq \widetilde{N}$. By our assumption, $\{1, 2\} \subseteq S$. Let $i \in N \setminus S$. By Lemma 3.2, $\widetilde{y} := \nu(\{1, i\}, \widetilde{v}^{i,1}) \in \sigma(\{1, i\}, \widetilde{v}^{i,1})$. Then $\widetilde{y}_1 > \widetilde{x}_1$. By RCP we have $(\widetilde{y}, \widetilde{x}_{\widetilde{N}\setminus\{1,i\}}) \in \sigma(\widetilde{N}, \widetilde{v})$. A contradiction to ETP is obtained, because $\widetilde{y}_1 > \widetilde{x}_1 = \widetilde{x}_2$.

(2) $\widetilde{x}_1 = \widetilde{x}_2 > 0$: Then $e(\widetilde{N}\setminus\{1\}, \widetilde{x}, \widetilde{v}) = \widetilde{x}_1 > 0$ by Claim 1, thus $\mu(\widetilde{x}, \widetilde{v}) > 0$. Let $S \subseteq \widetilde{N}$ attain $\mu(\widetilde{x}, \widetilde{v})$. By Claim 1, $\emptyset \neq S \neq \widetilde{N}$. By our assumption, $\{1, 2\} \cap S = \emptyset$. Let $i \in S$. By Lemma 3.2, $\widetilde{y} := \nu(\{1, i\}, \widetilde{v}^{i,1}) \in \sigma(\{1, i\}, \widetilde{v}^{i,1})$. Then $\widetilde{y}_1 < \widetilde{x}_1$. By RCP, $(\widetilde{y}, \widetilde{x}_{\widetilde{N}\setminus\{1,i\}}) \in \sigma(\widetilde{N}, \widetilde{v})$. A contradiction to ETP is obtained, because $\widetilde{y}_1 < \widetilde{x}_1 = \widetilde{x}_2$.

Now assertion (1) of our lemma can be deduced. By Claim 2, $(N, \widetilde{v}^{N,\widetilde{x}}) = (N, v)$, thus $\widetilde{x} := (0, 0, x) \in \sigma(\widetilde{N}, \widetilde{v})$ by RCP. By Claim 1, $x(N) = v(N)$.

In order to prove assertion (2), we have to show that the following conditions are satisfied:

$$N = \bigcup\{S \subseteq N \mid e(S, x, v) = \mu(x, v)\} \quad (3.1)$$

$$\emptyset = \bigcap\{S \subseteq N \mid e(S, x, v) = \mu(x, v)\} \quad (3.2)$$

Assume the contrary. Then two cases may occur:

(1) There exists $i \in N \setminus \bigcup\{S \subseteq N \mid e(S, x, v) = \mu(x, v)\}$: By Lemma 3.2,

$$\widetilde{y} = \nu(\{1, i\}, \widetilde{v}^{\{1,i\},\widetilde{x}}) \in \sigma(\{1, i\}, \widetilde{v}^{\{1,i\},\widetilde{x}}).$$

The fact that $\widetilde{y}_1 > 0 = \widetilde{x}_2$, is in contradiction to ETP.

(2) There exists $i \in \bigcap\{S \subseteq N \mid e(S, x, v) = \mu(x, v)\}$: By Lemma 3.2,

$$\widetilde{y} = \nu(\{1, i\}, \widetilde{v}^{\{1,i\},\widetilde{x}}) \in \sigma(\{1, i\}, \widetilde{v}^{\{1,i\},\widetilde{x}}).$$

The fact that $\widetilde{y}_1 < 0 = \widetilde{x}_2$, is in contradiction to ETP. **q.e.d.**

Now the main theorem of this section can be proved.

Theorem 3.4 *The prenucleolus is the unique solution that satisfies* NE, COV, ETP, *and* RCP, *provided* $|U| = \infty$.

Proof: The prenucleolus satisfies the desired properties. Indeed, it satisfies RCP, because RCP and RGP are equivalent for single-valued solutions. To show the opposite direction let σ be a solution that satisfies the desired axioms. By Lemma 3.2, σ satisfies PO.

In view of Theorem 3.1 it suffices to show that σ satisfies SIVA. Let (N, v) be a game. Take a disjoint copy $N^* \subseteq U$ of N, i.e.,

$$N \cap N^* = \emptyset \text{ and } N \to N^*,\ i \mapsto i^* \text{ is a bijection.}$$

Choose any real number α satisfying $\alpha > (n^2 + n)\max_{P,Q \subseteq N}(v(P) - v(Q))$ and define a "replicated" game $(N \cup N^*, \hat{v})$ by

$$\hat{v}(S \cup T^*) = \begin{cases} v(S) & , \text{ if } T = S, \\ -\alpha & , \text{ otherwise,} \end{cases}$$

where $S, T \subseteq N$. Let $z \in \sigma(N \cup N^*, \hat{v})$. It is the aim to show that the reduced game (N, u) w.r.t. N and z (defined by $u = \hat{v}^{N,z}$) is given by

$$u(S) = \hat{v}(S \cup S^*) - z(S) = v(S) - z(S)\ \forall S \subseteq N \quad (3.3)$$

In order to prove (3.3) first note that for any $i \in N$ the players i and i^* are substitutes. Hence, by ETP, $z_i = z_{i^*}$ for all $i \in N$.

Claim 1: For all $i \in N$, $z_i \geq \min_{P,Q \subseteq N}(v(P) - v(Q))$:

Assume, on the contrary, that there exists $i_0 \in N$ such that $z_{i_0} < \min_{P,Q \subseteq N}(v(P) - v(Q))$. Choose any coalition $S \cup T^*$ attaining $\mu(z, \hat{v})$. In view of the fact that

$$e(\{i_0, i_0^*\}, z, \hat{v}) = v(\{i_0\}) - 2z_{i_0} > -z_{i_0} > 0,$$

the maximal excess cannot be attained by $\emptyset$ or by $N \cup N^*$. By Lemma 3.3, Claim 1 is shown as soon as $i_0 \in S$ is verified.

Assume, on the contrary, $i_0 \notin S$. If $i_0 \notin T$, then the observation

$$e(S \cup \{i_0\} \cup T^* \cup \{i_0^*\}, z, \hat{v}) - e(S \cup T^*, z, \hat{v}) = \begin{cases} -2z_{i_0} & > 0, \text{ if } S \neq T, \\ v(S \cup \{i_0\}) - v(S) - 2z_{i_0} & > 0, \text{ if } S = T \end{cases}$$

yields the desired contradiction in this case. If $i_0 \in T$, then the observation

$$e(S \cup \{i_0\} \cup T^*, z, \hat{v}) - e(S \cup T^*, z, \hat{v}) = \begin{cases} -z_{i_0} & > 0, \text{ if } S \neq T \setminus \{i_0\}, \\ v(S \cup \{i_0\}) + \alpha - z_{i_0} & > 0, \text{ if } S = T \setminus \{i_0\} \end{cases}$$

yields the desired contradiction.

Claim 2: $z_i \leq n \max_{P,Q \subseteq N}(v(P) - v(Q)) \; \forall i \in N$:

Let $i_0 \in N$ be a player. Observe that

$$v(N) = 2z(N) = 2z_{i_0} + 2z(N \setminus \{i_0\}) \geq 2z_{i_0} + 2(n-1) \min_{P,Q \subseteq N}(v(P) - v(Q))$$

by PO, ETP, and Claim 1. Thus our claim follows immediately.

Now the proof can be finished. Put $\widetilde{S} = \{i \in N \mid z_i < 0\}$ and observe that

$$u(S) = \max\{v(S) - z(S), -\alpha - z(\widetilde{S})\} \; \forall \emptyset \neq S \subsetneqq N. \tag{3.4}$$

Let S be a nontrivial ($\emptyset \neq S \subsetneqq N$) coalition. Then

$$v(S) - z(S) \geq v(S) - (n-1) \max_{P,Q \subseteq N}(v(P) - v(Q)) \geq -n \max_{P,Q \subseteq N}(v(P) - v(Q))$$

and

$$-\alpha - z(\widetilde{S}) \leq -\alpha + n^2 \max_{P,Q \subseteq N}(v(P) - v(Q)) < -n \max_{P,Q \subseteq N}(v(P) - v(Q)),$$

where the last inequality is implied by the definition of α. Hence u is given by

$$u(S) = v(S) - z(S) \; \forall S \subseteq N.$$

By NE there exists $x \in \sigma(N, v)$. COV implies $x - z_N \in \sigma(N, u)$, thus $(x - z_N, z_{N^*}) \in \sigma(N \cup N^*, \hat{v})$ by RCP. ETP implies $x - z_N = z_N$, thus $x = 2z_N$ is the unique member of $\sigma(N, v)$. **q.e.d.**

Four examples are presented which show that each of the axioms (1) NE, (2) COV, (3) ETP, and (4) RCP is logically independent of the remaining axioms in Theorem 3.4.

Let (N, v) be a game. Let σ^i, $i = 1, 2, 4$, be defined by

$$\begin{aligned} \sigma^1(N,v) &= \emptyset, \\ \sigma^2(N,v) &= \{x \in \mathcal{I}^*(N,v) \mid x_i = x_j \ \forall i, j \in N\}, \\ \sigma^4(N,v) &= \mathcal{K}^*(N,v). \end{aligned}$$

Let $\preceq$ be a total order of U. For every finite set $N \subseteq U$ let $\preceq_N$ be the restriction of $\preceq$ to N and let $\leq_{lex}^N$ be the induced lexicographical order on $\mathbb{R}^N$. Then σ^3 is defined by

$$\sigma^3(N,v) = \{x \in \mathcal{C}^+(N,v) \mid x \leq_{lex}^N y \ \forall y \in \mathcal{C}^+(N,v)\}.$$

It is straightforward to check that σ^i, $i = 1, \ldots, 4$, satisfies all properties except the i-th one. If $|U| \geq 4$, then none of the solutions coincides with the prenucleolus.

It should be remarked that σ^4 can be replaced by the Shapley value. Then the examples also show that Sobolev's and Orshan's characterizations of the prenucleolus are, in fact, axiomatizations.

The prekernel σ^4 satisfies NE, COV, AN, ETP, and RGP. Thus RCP cannot be replaced by RGP in Theorem 3.4. The positive core (see Definition 2.1) satisfies NE, COV, AN, RGP, and RCP, hence ETP cannot be replaced by AN (as in the "classical") axiomatization. Another well-known solution is the least core. The *least core* of a game (N, v) is defined by

$$\mathcal{LC}(N,v) = \{x \in \mathcal{I}^*(N,v) \mid \max_{\emptyset \neq S \subsetneqq N} e(S,x,v) = \max_{\emptyset \neq S \subsetneqq N} e(S, \nu(N,v), v)\}.$$

It is well-known and easy to verify that $\mathcal{LC}$ satisfies NE, COV, and AN. It also satisfies RCP, because the maximal excess of nontrivial coalitions in a reduced game is not larger than the maximal excess of nontrivial coalitions in the game. The least core does not satisfy RGP.

Remark 3.5 The infinity assumption on $|U|$ in Theorems 3.4is crucial. Indeed, if $|U| < \infty$, then define for any game (N, v),

$$\sigma(N,v) = \begin{cases} \{\nu(N,v)\} & , \text{ if } N \subsetneqq U, \\ \{x \in \mathcal{K}^*(N,v) \mid x_S = \nu(S, v^{S,x}) \ \forall \emptyset \neq S \subsetneqq N\} & , \text{ if } N = U. \end{cases}$$

Then σ satisfies all axioms of Theorem 3.4. Also, there are examples which show that the prenucleolus is a proper subsolution of σ when $|U| \geq 4$.

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