INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 310

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by

Walter Trockel

September 1999

University of Bielefeld

33501 Bielefeld, Germany

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Walter Trockel*
IMW, Bielefeld University

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Abstract

This paper follows van Damme (1986) in presenting a Meta Bargaining approach
that justifies the Nash bargaining solution. But in contrast to van Damme's proce-
dure our Meta Bargaining game is universal in the sense that all bargaining solutions
are allowed as strategic choices in the Meta Bargaining game. Also our result holds
true for any number n of players.

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1 Introduction

The idea to support (implement, realize, justify) axiomatic solutions of cooperative games by Nash equilibria of non-cooperative games goes back to the work of John Nash (1951, 1953). Virtual (or asymptotic) support results for the Nash solution have been provided by Nash (1953) in his smoothed demand game and by Binmore, Rubinstein and Wolinsky (1986). Rigorous versions of Nash's somewhat vague treatment have been given later on by Binmore (1987), van Damme (1991) and by Osborne and Rubinstein (1990). Direct implementations are contained in Howard (1992) and Trockel (1999b). Another specific direct support of the Nash solution in 2-person bargaining games is provided by van Damme's (1986) Meta Bargaining game. A modification of this result is contained in Naeve-Steinweg (1997).

The idea behind the Meta Bargaining approach is it that players have fixed fairness standards by which they choose their preferred bargaining solution independently of the concrete bargaining game. Van Damme's Meta Bargaining game admits only specific solutions satisfying some axioms as possible strategic choices. While "conflicts will not always be resolved by our [his] procedure if 'perverse' solutions are allowed that do not satisfy RS" [risk sensitivity], our game is universal in that it admits all solutions. Also our result holds for any number n of players.

2 The Meta Bargaining Game

The Meta Bargaining game we are going to define combines van Damme's (1986) idea of using sets of solutions as strategy sets with the game in Proposition 1 of Trockel (1999), which established support to the Nash solution by its unique Nash equilibrium. First we formalize the general framework. Next we recall Trockel's (1999b) game. Then we construct our corresponding Meta Bargaining game.

For notational convenience we restrict to the case of two-person bargaining games. The same reasoning, however, can be used for the case of general n-person bargaining games as the method by which the Meta Bargaining game is deduced from the game in Trockel (1999b), is independent of the specific n, and the proof there holds true for any $n \in \mathbb{N}$.

A two-person bargaining game is a non empty, compact convex and comprehensive subset S of $\mathbb{R}_+^2$ which is interpreted as the set of payoff vectors which the players are able to obtain by cooperation. In case of non-cooperation the players receive their respective coordinate of the threat point $d \in S$. Like van Damme we restrict ourselves to the case of $d = 0 \in \mathbb{R}^2$. Moreover we normalize S such that $proj_i(S) = S_i = [0, 1], i = 1, 2$. These conventions are no restrictions from the cardinalist point of view, where utilities of players are only determined up to positive affine transformations.

For any bargaining game S the set $\partial S := \{x \in S | (\{x\} + \mathbb{R}^2_+) \cap S = \{x\}\}$ is the (strong) Pareto efficient boundary of S.

This is exactly the framework used by van Damme (1986). Although our analysis can be performed for this framework with some additional notational and terminological effort we restrict this framework for convenience in the following way.

We assume that for any $x \in \partial S$ there exists a unique normal vector $p(x) \in \mathbb{R}^2_{++}$ such that the inner product $p(x) \cdot x = 1$. We call this vector $p(x)$ the *efficiency price system* associated with x. The set of all such bargaining games is denoted by Σ. Now any mapping $f : \Sigma \to \mathbb{R}^2 : S \mapsto f(S) \in S$ is a *bargaining solution*.

Two bargaining solutions f, g are called (S, i)-equivalent for given $S \in \Sigma, i = 1, 2$ if $f(S)_i = g(S)_i$. The (S, i)-equivalence class of a bargaining solution f is denoted $[f]_{S,i}, i = 1, 2$.

We denote by F the set of all bargaining solutions. Notice, that van Damme (1986), as well as Naeve-Steinweg (1997), denotes by F a much smaller subset of the set of all bargaining solutions, all elements of which satisfy certain axioms.

The *Nash solution* is denoted f^N and is defined by $\{f^N(S)\} := \arg \max_{x \in S} x_1 \cdot x_2$.

For any $S \in \Sigma$ a game $\Gamma(S) = (F, F; U_1(\cdot, S), U_i(\cdot, S))$ with $U_i(\cdot, S) : F \times F \to \mathbb{R}, i = 1, 2$ is called a *Meta Bargaining game*.

Now consider for any $S \in \Sigma$ the following two-person game in strategic form

$$G(S) := ([0, 1], [0, 1], \pi_1(\cdot, S), \pi_2(\cdot, S))$$

where $\pi_i(\cdot, S), i = 1, 2$ is defined as follows.

Let $x = (x_1, x_2) \in [0, 1] \times [0, 1]$. Define $y^i(x), i = 1, 2$ by $\{y^i(x)\} = \partial S \cap (\{x_i\} \times \mathbb{R}^2_+)$. If $x \in \partial S$ then $x = y^1(x) = y^2(x)$. For any $y \in \partial S$ define $z_i(y) := \min(y_i, \frac{1}{2p_i(y)}), i = 1, 2$.

Now we define $\pi_i(\cdot, S), i = 1, 2$ by

$$\pi_i(x, S) := \begin{cases} z_i(y^i(x)) & x \notin S \\ x_i & x \in S \end{cases} \tag{1}$$

The game $G(S)$ is a modification of Nash's (1953) simple demand game. It generates the same payoffs as Nash's game for consistent strategy choices $x \in S$ but distinguishes payoffs of the two players in a more subtle way in case of inconsistent strategy choices. For a thorough discussion and interpretation see Trockel (1999b).

By Proposition 1 in Trockel (1999) for any $S \in \Sigma$ the game $G(S)$ has a unique Nash

equilibrium, which coincides with $f^N(S)$. Notice that $f^N(S)$ coincides with the unique equilibrium strategy profile as well as with the resulting unique payoff vector.

Next we derive a specific Meta Bargaining game $\Gamma^G(S)$ from the game $G(S)$. For this purpose we define $U_i^G(\cdot; S)$ by

$$U_i^G(f, g; S) = \pi_i((f(S)_1, g(S)_2); S), i = 1, 2.$$

$\Gamma^G(S)$ is then defined as $\Gamma^G(S) := (F, F; U_1^G(\cdot; S), U_2^G(\cdot; S))$.

3 Equilibria of the Meta Bargaining Game

As for any $S \in \Sigma$ the game $G(S)$ has as its unique Nash equilibrium the point $f^N(S) = (f^N(S)_1, f^N(S)_2)$ the Nash equilibria of $\Gamma^G(S)$ are given by the set $[f^N]_{S,1} \times [f^N]_{S,2}$. All these equilibria result in the same payoff vector $f^N(S)$. We collect this insight in the following

> Proposition:
>
> For any $S \in \Sigma$ the pair $(f, g) \in F \times F$ is a Nash equilibrium if and only if $f(S)_1 = f^N(S)_1$ and $g(S)_2 = f^N(S)_2$. The unique equilibrium payoff vector is $f^N(S)$.

Although any S has many equilibria it is only (f^N, f^N), which is an equilibrium for **each** $S \in \Sigma$. The situation is exactly as in van Damme (1986) apart from the fact that we allow for **all** bargaining solutions to be strategies.

4 Relation to Mechanism Theory

The support result provided by our Proposition can be seen as belonging to the realm of the Nash program. The question then is how it is related to implementation in the sense of mechanism theory.

Mechanism theory is concerned with game forms (= mechanisms) rather than with games. That means that support results have to be established simultaneously for a whole class of games all of which are induced from the same game form by different populations of players.

Formally one has to find a suitable factorization of the players' payoff functions into an outcome function from the strategy space to an outcome space and the players' utility

functions on the outcome space. But to find a suitable outcome space is only one problem. A second one is to represent the solution that is to be implemented by a suitably defined social choice rule.

The relation of the Nash program to mechanism theory has been thoroughly discussed in Trockel (1999b). See also Dagan and Serrano (1998) and Serrano (1997). The latter paper contains the "embedding principle", a method which allows it to transform any support result of the Nash program into a proper implementation result in the sense of mechanism theory.

Applying that embedding principle to our present Proposition we get the following mechanism theoretic implementation. The same method would apply to van Damme's result thereby justifying his use of the term "mechanism".

$A := (\mathbb{R}^2)^\Sigma$ is the *outcome space*. That means that any bargaining solution considered as a behavioral norm constitutes a social state.

Let

$$ev : \Sigma \times A \to \mathbb{R}^2 : (S, f) \mapsto f(S),$$

and

$$ev_S : A \to \mathbb{R}^2 : f \mapsto ev_S(f) = f(S).$$

We define an *outcome function h* by

$$h : F \times F \to A : (f, g) \mapsto (proj_1 \circ ev(\cdot, f), proj_2 \circ ev(\cdot, g)).$$

Next we derive from the game $G(S)$ a specific Meta bargaining game $\Gamma^G(S)$.

For this purpose we define $U_i^G(\cdot, S), i = 1, 2$ by $U_i^G(f, g; S) := \pi_i^S \circ h(f, g)$

For the purpose of Nash implementation we still have to represent the Nash bargaining solution by a suitable social choice rule. This must be a correspondence defined on a class of profiles of preferences or utility functions associating with any such profile a subset of the outcome space A. This subset is interpreted as a set of socially desirable states given the prevailing preference profile.

By identifying any $S \in \Sigma$ with the map ev_S which associates with any social state $f \in A$ the utility vector $ev_S(f) = (f_1(S), f_2(S))$ of the two players in the bargaining game S derived from the solution f we can define our *Nash social choice rule* by

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$$[f^N]_{(\cdot)} : \Sigma \Rightarrow A : S = ev_S \Rightarrow [f^N]_S := [f^N]_{S,1} \times [f^N]_{S,2}$$

It is then an immediate consequence of the proposition in Trockel (1999b) that $[f^N]_{(\cdot)}$ can be Nash implemented by h.

In particular, one gets

$$U_i^G(f^N, f^N; S) = \pi_i^S \circ h(f^N, f^N) = \pi_i^S(h(f^N, f^N)) = \pi_i^S(proj_1 \circ f^N, proj_2 \circ f^N) = \pi_i^S(f^N) = f^N, i = 1, 2.$$

This chain of equalities still holds true if the two first arguments of U_i^G are replaced, respectively, by arbitrary members of $[f^N]_{S,1}$ and $[f^N]_{S,2}$.

5   Concluding Remarks

Our results show that a social planner not knowing the specific population of players who may favour whatever bargaining solution they want, has a mechanism available which causes any possible player population to agree on the Nash solution, once they play the Meta Bargaining game derived for them from the mechanism.

It would be interesting to see whether also other bargaining solutions, for instance the Kalai-Smorodinsky or the Maschler-Perles solution, could be implemented by suitable Meta Bargaining mechanisms.

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