INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 342

Perfectly Fair Allocations with Indivisibilities

by

Ning Sun and Zaifu Yang

December 2002

University of Bielefeld

33501 Bielefeld, Germany

Perfectly Fair Allocations with Indivisibilities [1]

Ning Sun,[2] Zaifu Yang [3]

Abstract: One set of n objects of type I, another set of n objects of type II, and an amount M of money is to be completely allocated among n agents in such a way that each agent gets one object of each type with some amount of money. We propose a new solution concept to this problem called a perfectly fair allocation. It is a refinement of the concept of fair allocation. An appealing and interesting property of this concept is that every perfectly fair allocation is envy free, Pareto optimal, and income fair. It is also shown that a perfectly fair allocation gives each agent what he likes best, and that a fair allocation need not be perfectly fair. Furthermore, we give a necessary and sufficient condition for the existence of a perfectly fair allocation. Precisely, we show that there exists a perfectly fair allocation if and only if the valuation matrix is an optimality preserved matrix. Optimality preserved matrices are a class of new and interesting matrices. We also derive two fundamental properties of optimality preserved matrices and identify several easily verifiable conditions for a valuation matrix to be an optimality preserved matrix. Furthermore, an extension of the basic model is discussed.

[1]This research is supported by Grants-in-Aid of the Japanese Ministry of Education, Culture, Sports, Science and Technology, and the second author is also supported by the Alexander von Humboldt Foundation. An earlier version of this paper was circulated as Discussion Paper No. 1318 (2001) of the Cowles Foundation for Research in Economics, Yale University. In the current version, a completely new section, namely, Section 4 is added. We are very grateful to Professors Satoru Fujishige and Herbert Scarf for their constructive comments. In particular, we would like to thank Professor Herve Moulin for his critical comments which led us to find several new results included in Section 4. Of course, we are solely responsible for any remaining deficiency.

[2]N. Sun, Department of Management, Faculty of System Science and Technology, Akita Prefectural University, Honjo City, Akita 015-0055, Japan. E-mail: sun@akita-pu.ac.jp

[3]Z. Yang, Institute of Mathematical Economics, University of Bielefeld, 33615 Bielefeld, Germany. E-mail:zyang@wiwi.uni-bielefeld.de; and Faculty of Business Administration, Yokohama National University, Yokohama 240-8501, Japan. E-mail: zyang@business.ynu.ac.jp

Keywords: Perfectly fair allocation, indivisibility, discrete optimization, multi-person decision, existence theorem, optimality preserved matrix.

JEL-code: D3, D31, D6, D61, D63, D7, D74.

1 Introduction

The subject of this paper is the distribution of a collection of objects (such as houses and cars) and an amount of money among a group of people. It is concerned with fairness, equity, justice, and efficiency of such distributions. These problems arise naturally in many situations, and are both difficult and controversial. Recall that given an allocation, we say agent i envies agent j if agent i prefers the bundle of agent j to his or her own. An allocation is *envy free* or *equitable* if no agent envies any other. An allocation is *fair* if it is both equitable and Pareto optimal. Furthermore, an allocation is *income fair* if the potential income of every agent is the same. As it has been noted, the concept of fairness may not exactly correspond to the everyday notion of fairness. In fact, how we define equity, fairness, and justice has been, and remains a most provocative question in the course of mankind's endless quest for equity, fairness, and justice. The goal of this paper is to propose a new solution concept to a class of fair allocation problems and to investigate what conditions can ensure the existence of such a solution which is envy free, Pareto optimal, and income fair.

The study of fair division problem can date back at least to Steinhaus (1948). But most of the literature has evovled from Foley (1967) in which the concept of envy free allocation is precisely formulated. A major defect of this concept is that an envy free allocation may not be efficient (i.e., Pareto optimal). Various criteria on equity and justice are discussed in Rawls (1971), Pazner and Schmeidler (1978). Furthermore, in Varian (1974) a general formulation of fair division of divisible goods is given. He proved the existence of an envy free and efficient allocation by imposing certain conditions on the model.

The fair allocation problem of indivisible objects is investigated by Svensson (1983), and further studied by Maskin (1987), Alkan, Demange and Gale (1991), Su (1999), and

Yang (2001). In these papers it is shown that in an economy if each agent consumes only one indivisible object and there is a divisible good (say money), then the set of envy free and efficient allocations is not empty under certain mild conditions. In these models a fundamental assumption in common is that each agent has no use for more than one indivisible object. As noted by Svensson (1983) this assumption leads to a nice conclusion that an envy free allocation must also be efficient. Unfortunately, this property does not automatically carry into more general situations where agents are allowed to consume more than one indivisible object. Sun and Yang (2000) have recently developed a more general model in which there are no restrictions on the agents' consumption of indivisible objects. A sufficient condition is introduced for the existence of an envy free and efficient allocation. On the other hand, algorithmic procedures have been proposed by Aragones (1995), Klijn (2000), and Haake, Raith and Su (2002) to find an efficient and envy free allocation in a setting where agents have quasi-linear utilities in money. Furthermore, Alkan et al. (1991) and Tadenuma and Thomson (1991) have given two different sets of criteria for selecting desirable envy free allocations when there exist multiple envy free allocations.

In this paper we consider the following problem: One set of n objects of type I, another set of n objects of type II, and an amount M of money, are to be completely allocated among n agents in such a way that each agent gets one object of each type with some amount of money. We propose a new solution concept to this problem called a perfectly fair allocation. It is a refinement of the concept of fair allocation. An appealing and interesting property of this concept is that every perfectly fair allocation is Pareto optimal. It is also shown that a perfectly fair allocation is envy free, income fair and gives each agent what he likes best, and that a fair allocation need not be perfectly fair. Furthermore, we give a necessary and sufficient condition for the existence of a perfectly fair allocation. To be more precise, we show that there exists a perfectly fair allocation if and only if the valuation matrix is an optimality preserved matrix. We also derive two fundamental properties of optimality preserved matrices and introduce several easily verifiable conditions for a matrix to be an optimality preserved matrix. We stress that optimality preserved matrices are a class of new and interesting matrices and might be worth being studied in their own right. An extension of the model is also discussed.

The rest of the paper is organized as follows. In Section 2 basic concepts are introduced, and the formal model is defined. In Section 3 we introduce the concept of optimality preserved matrix and establish the existence theorem of perfectly fair allocations. In Section 4 we derive two fundamental properties of optimality preserved matrices and introduce several easily verifiable conditions for a matrix to be an optimality preserved matrix. Finally in Section 5 an extension of the basic model is discussed and existence results are derived.

2 The model of perfectly fair allocation

We first introduce some notation. Let I_k be the set of first k positive integers and $\mathbb{R}^k$ the k-dimensional Euclidean space.

Our model consists of a finite number (n) of agents, denoted by I_n, the same number of indivisible objects of type I, denoted by $\mathcal{O}_1$, the same number of indivisible objects of type II, denoted by $\mathcal{O}_2$, and a fixed amount of money, denoted by M. One might think of $\mathcal{O}_1$ and $\mathcal{O}_2$ as the collections of houses and cars, respectively. For ease of notation, let $\mathcal{O}_1 = \mathcal{O}_2 = I_n$. Here M can be any real number. If M is negative, this will be the case in cost sharing problems. Here money will be treated as a perfectly divisible good. It is assumed that each agent demands or consumes exactly one of the indivisible objects of each type and a certain amount of money. The preference relation of each agent $i \in I_n$ can be represented by a utility function $u_i : \mathcal{O}_1 \times \mathcal{O}_2 \times \mathbb{R} \mapsto \mathbb{R}$. Throughtout the paper it will be assumed that $u_i(h, c, m)$ is a nondecreasing and continuous function in money (i.e., in m).

A feasible allocation is a 3-tuple of vectors $(\pi, \rho, z = (x, y))$ where $\pi = (\pi(1), \cdots, \pi(n))$ and $\rho = (\rho(1), \cdots, \rho(n))$ are the permutations of the elements in $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, and where $\sum_{i=1}^{n}(x_i + y_i) = M$. Thus, at a feasible allocation, all objects and money will be completely distributed to the agents in a way that every agent gets exactly one indivisible object of each type and a certain amount of money. More precisely, each agent i receives a bundle of goods $(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(j)})$ consisting of object $\pi(i)$ of type I and object $\rho(i)$ of type II and the amount $x_{\pi(i)} + y_{\rho(i)}$ of money. If $x_{\pi(i)} + y_{\rho(i)} < 0$, then agent i pays others the amount $|x_{\pi(i)} + y_{\rho(i)}|$ of money.

Let $T = \{z = (x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n}(x_j + y_j) = M\}$ be the $(2n-1)$-dimensional hyperplane and let $\Theta = \{\pi \mid \pi = (\pi(1), \cdots, \pi(n))$ is a permutation of $I_n\}$. Thus a feasible allocation (π, ρ, z) is merely an element of $\Theta \times \Theta \times T$.

We can now introduce the major solution concept of the paper.

Definition 2.1 *A feasible allocation (π, ρ, z) is a perfectly fair allocation if it holds*

$$u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}) \geq u_i(\pi(j), \rho(k), x_{\pi(j)} + y_{\rho(k)}),\ \forall i, j, k \in I_n.$$

Recall that a feasiable allocation is envy free or equitable if no agent prefers any other agent's bundle to his own. Clearly a perfectly fair allocation must be an equitable allocation but the reverse is not true in general. Furthermore, a perfectly fair allocation gives each agent what he likes best. The concept of perfectly fair allocation can be also explained as follows. An auctioneer chooses a compensation scheme vector $z = (x, y) \in T$ for the pairs of objects in $\mathcal{O}_1 \times \mathcal{O}_2$ in such a way that every agent can pick up a pair of house and car with their compensation which he likes best without conflicting his interest with any other's. The following concept is a familiar one.

Definition 2.2 *A feasible allocation is efficient or Pareto optimal if there is no other feasible allocation which makes everyone at least as well as before and at least one agent strictly better off.*

The problem of the concept of equitable allocation lies in the fact that it is not necessarily efficient. The following example indicates that an equitable allocation indeed need not be efficient.

Example 1. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M is equal to zero. Both agents have quasi-linear utilities in money (i.e., $u_i(h, c, m) = \alpha(i, h, c) + m$, $i = 1, 2$) and the values of the agents for the different pairs of house and car are given in Table 1.

In this example when agent 1 gets house $h1$ and car $c2$ with 1\$ and agent 2 gets house $h2$ and car $c1$ by paying 1\$, this allocation is equitable but not Pareto optimal, because another allocation in which agent 1 gets house $h2$ and car $c2$ by paying 0.5\$ and agent 2 gets house $h1$ and car $c1$ with 0.5\$ makes both agents strictly better off.

Table 1: The values of objects for both agents

$\alpha(1,h,c)$	$C1$	$C2$	$\alpha(2,h,c)$	$c1$	$c2$
$h1$	2	3	$h1$	3	2
$h2$	4	5	$h2$	4	4

One of the most appealing and interesting properties of perfectly fair allocation is that it is also efficient as shown below.

Theorem 2.3 *Every perfectly fair allocation is Pareto optimal.*

Proof: Let (π, ρ, z) be a perfectly fair allocation. Then it follows that

$$u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}) \geq u_i(\pi(j), \rho(k), x_{\pi(j)} + y_{\rho(k)}), \forall i, j, k \in I_n; \tag{2.1}$$

Now suppose to the contrary that (π, ρ, z) is not efficient. Then there would exist a feasible allocation $(\bar{\pi}, \bar{\rho}, \bar{z})$ weakly preferred by all agents and strictly preferred by at least one agent. That is, it holds

$$u_i(\bar{\pi}(i), \bar{\rho}(i), \bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)}) \geq u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}), \forall i \in I_n; \tag{2.2}$$

and there is some $j \in I_n$ satisfying

$$u_j(\bar{\pi}(j), \bar{\rho}(j), \bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > u_j(\pi(j), \rho(j), x_{\pi(j)} + y_{\rho(j)}). \tag{2.3}$$

Inequalities (2.1), (2.2) and (2.3) imply that for all $i \in I_n$,

$$u_i(\bar{\pi}(i), \bar{\rho}(i), \bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)}) \geq u_i(\bar{\pi}(i), \bar{\rho}(i), x_{\bar{\pi}(i)} + y_{\bar{\rho}(i)}),$$

and

$$u_j(\bar{\pi}(j), \bar{\rho}(j), \bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > u_j(\bar{\pi}(j), \bar{\rho}(j), x_{\bar{\pi}(j)} + y_{\bar{\rho}(j)}).$$

Since $u_i(j, k, \cdot)$, $i, j, k \in I_n$, are nondecreasing in money, we have that for all $i \in I_n$,

$$\bar{x}_{\bar{\pi}(i)} + \bar{y}_{\bar{\rho}(i)} \geq x_{\bar{\pi}(i)} + y_{\bar{\rho}(i)},$$

and

$$\bar{x}_{\bar{\pi}(j)} + \bar{y}_{\bar{\rho}(j)}) > x_{\bar{\pi}(j)} + y_{\bar{\rho}(j)}.$$

This implies that

$$M = \sum_{j=1}^{n}(\bar{x}_j + \bar{y}_j) > \sum_{j=1}^{n}(x_j + y_j) = M,$$

yielding a contradiction. Therefore, (π, ρ, z) must be efficient as well. □

In addition, we will show that the concept of perfectly fair allocation has yet another remarkable property, namely, it is consistent with income-fairness. The concept of income-fair allocation is suggested by Pazner and Schmeidler (1978). This concept can be reformulated in the present model as follows. Given an allocation $(\pi, \rho, (x, y))$, we construct a pure exchange economy $E(\pi, \rho, (x, y))$ in which the bundle $(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)})$ is viewed as agent i's initial endowment. We say that an allocation $(\pi, \rho, (x, y))$ is *an income fair allocation* if there exists a vector $(p^1, p^2, p^3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that $(\pi, \rho, (x, y))$ is a competitive equilibrium allocation, (p^1, p^2, p^3) is a competitive equilibrium price vector for the economy $E(\pi, \rho, (x, y))$, and the potential income is the same for every agent.

Lemma 2.4 *Every perfectly fair allocation is an income fair allocation.*

Proof: Let $(\pi, \rho, (x, y))$ be a perfectly fair allocation. Now define an economy in which agent i initially owns the bundle $(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)})$. Let $p^1 = -x$, $p^2 = -y$, and $p^3 = 1$. Then the vector (p^1, p^2, p^3) is a competitive equilibrium price vector for the economy since for every agent i, perfect-fairness implies that

$$u_i(\pi(i), \rho(i), x_{\pi(i)} + y_{\rho(i)}) \geq u_i(\pi'(i), \rho'(i), x_{\pi'(i)} + y_{\rho'(i)}), \ \forall \pi', \rho' \in \Theta.$$

In the economy, the potential income $I(i) = x_{\pi(i)} + y_{\rho(i)} + p^1_{\pi(i)} + p^2_{\rho(i)} = 0$ for all $i \in I_n$. Thus, $(\pi, \rho, (x, y))$ is an income fair allocation. □

The following example shows that an efficient and envy free (i.e. fair) allocation may not be income fair. Thus, the concept of perfectly fair allocation is indeed a proper refinement of the concept of fair allocation.

Example 2. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M is equal to zero. The values of the agents for the different pairs of house and car are given in Table 2, and utility functions are given by $u_i(h, c, m) = \alpha(i, h, c) + m$, $i = 1, 2$.

In this example there is only one envy free and efficient allocation, namely, agent 1 gets house $h1$ and car $c1$ and $x\$$ with $2 \leq x \leq 2.5$, and agent 2 gets house $h2$ and car $c2$ by paying $x\$$. Suppose that this allocation is income fair. Then for agent 2, the following system of inequalities must have a solution.

$$\begin{aligned} 5 + x_2 + y_2 &\geq x_1 + y_1 \\ 5 + x_2 + y_2 &\geq 4.5 + x_1 + y_2 \\ 5 + x_2 + y_2 &\geq 4.5 + x_2 + y_1 \\ x_1 + y_1 &= -(x_2 + y_2) \\ x_1 + y_1 &= x \\ 2 \leq x &\leq 2.5 \end{aligned}$$

It follows from the second and third inequalities that $x \leq 0.5$, yielding a contradiction to the sixth inequality. Thus this fair allocation is not income fair and therefore is not a perfectly fair allocation, either.

Table 2: The values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$
$h1$	5	4.5	$h1$	0	4.5
$h2$	4.5	9	$h2$	4.5	5

In the following two sections we will establish several existence theorems for perfectly fair allocations in the case that agents have quasi-linear utilities in money. Relaxing the assumption of quasi-linearity in money still poses a difficult challenge to us.

3 Existence theorems

Given an $n \times n \times n$ trimatrix $A = (\alpha(i, h, c))$, an assignment $(\pi, \rho) \in \Theta \times \Theta$ is an *optimal assignment* if $\sum_{i \in I_n} \alpha(i, \pi(i), \rho(i)) \geq \sum_{i \in I_n} \alpha(i, \tau(i), \gamma(i))$ for every $(\tau, \gamma) \in \Theta \times \Theta$. Similarly, given an $n \times n$ matrix $B = (\beta(i, o))$, we call an assignment $\pi \in \Theta$ an *optimal assignment* if $\sum_{i \in I_n} \beta(i, \pi(i)) \geq \sum_{i \in I_n} \beta(i, \tau(i))$ for every $\tau \in \Theta$.

When we restrict to the case where every agent has quasi-linear utilities in money, then the model described in Section 2 can be simply represented as $\mathcal{E} = ((\alpha(i,h,c)), n, M)$ where $(\alpha(i,h,c))$ is an $n \times n \times n$ trimatrix, n is the number of agents, and M is the total amount of money. Recall that $\alpha(i,h,c)$ is the value of a pair of house h and car c to agent i. We call $(\alpha(i,h,c))$ the *valuation matrix*. Furthermore, for a specific model where objects are only houses or cars, we will simply represent such a model by $\mathcal{E} = ((\beta(i,o)), n, M)$, where $(\beta(i,o))$ is an $n \times n$ matrix, n is the number of agents, and M is the total amount of money: $\beta(i,o)$ is the value of object o to agent i.

Recall the following duality theorem from linear programming, which has been used by Shapley and Shubik (1972), and Alkan et al. (1991) for related models.

Lemma 3.1 *Let $B = (\beta(i,o))$ be an $n \times n$ matrix. If $\pi \in \Theta$ is an optimal assignment, there exist two n-vectors v and w such that*

$$v_i + w_o \geq \beta(i,o), \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$v_i + w_{\pi(i)} = \beta(i,\pi(i)), \ \forall i \in I_n.$$

Lemma 3.2 *Given a model $\mathcal{E} = ((\beta(i,o)), n, M)$, then there exists at least one optimal assignment with respect to the matrix $(\beta(i,o))$. For each optimal assignment π, there exists a distribution n-vector x of money M such that (π, x) is an efficient and envy free allocation.*

Proof: The first statement is obvious, since there are only a finite number of assignments. The second statement can be seen as follows. Since π is an optimal assignment, it follows from Lemma 3.1 that there exists v and w such that

$$v_i + w_o \geq \beta(i,o), \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$v_i + w_{\pi(i)} = \beta(i,\pi(i)), \ \forall i \in I_n.$$

From the above inequalities we obtain

$$\beta(i,\pi(i)) - w_{\pi(i)} \geq \beta(i,o) - w_o, \ \ \forall i \in I_n, o \in \mathcal{O}_1.$$

Let $y_i = -w_i$, $\delta = (M - \sum_{i \in I_n} y_i)/n$, and $x_i = y_i + \delta$ for each $i \in I_n$. Define $x = (x_1, \cdots, x_n)$. Then we have

$$\beta(i,\pi(i)) + x_{\pi(i)} \geq \beta(i,o) + x_o, \ \ \forall i \in I_n, o \in \mathcal{O}_1$$

and

$$\sum_{i \in I_n} x_i = M.$$

Thus, (π, x) is an efficient and fair allocation. □

Using the same argument of the above lemma or Theorem 4.1 of Sun and Yang (2000), we have

Theorem 3.3 *Given a model* $\mathcal{E} = ((\alpha(i,h,c)), n, M)$*, then there exists at least one optimal assignment with respect to the matrix* $(\alpha(i,h,c))$*. For each optimal assignment* (π, ρ)*, there exists a distribution* $2n$*-vector* (x,y) *of money* M *such that* $(\pi, \rho, (x,y))$ *is an efficient and envy free allocation.*

As Example 2 indicates that perfectly fair allocations may not always exist, this motivates a natural question: Under what circumstance does a perfectly fair allocation exist? The remaining section is to present a necessary and sufficient condition for the existence of a perfectly fair allocation.

Condition 3.4 *The trimatrix* $(\alpha(i,h,c))$ *has the following property: For every* $i \in I_n$*, it holds*

$$\alpha(i,h1,c1) + \alpha(i,h2,c2) = \alpha(i,h1,c2) + \alpha(i,h2,c1),$$
$$\forall h1, h2 \in \mathcal{O}_1, c1, c2 \in \mathcal{O}_2.$$

Condition 3.5 *The trimatrix* $(\alpha(i,h,c))$ *has the following property: For every* $i \in I_n$*, there exist two* n *vectors* $H^i = (H^i(1), \cdots, H^i(n))$ *and* $C^i = (C^i(1), \cdots, C^i(n))$ *such that it holds*

$$\alpha(i,h,c) = H^i(h) + C^i(c), \ \ \forall h \in \mathcal{O}_1, c \in \mathcal{O}_2.$$

Lemma 3.6 *Conditions 3.4 and 3.5 are equivalent.*

Proof: Condition 3.5 clearly implies Condition 3.4. Now we prove that Condition 3.4 implies Condition 3.5. From Condition 3.4, we see that

$$\alpha(i,1,1)+\alpha(i,h,c)=\alpha(i,1,c)+\alpha(i,h,1) \qquad \text{for all } h\in\mathcal{O}_1 \text{ and } c\in\mathcal{O}_2.$$

Thus we obtain that

$$\alpha(i,h,c)-\alpha(i,h,1)=\alpha(i,1,c)-\alpha(i,1,1) \qquad \text{for all } h\in\mathcal{O}_1 \text{ and } c\in\mathcal{O}_2.$$

For each $h\in\mathcal{O}_1$ and $c\in\mathcal{O}_2$, let

$$H^i(h)=\alpha(i,h,1), \quad \text{and} \quad C^i(c)=\alpha(i,1,c)-\alpha(i,1,1).$$

Then we have that $\alpha(i,h,c) = H^i(h)+C^i(c)$ for all $h\in\mathcal{O}_1$ and $c\in\mathcal{O}_2$. That is, Condition 3.5 holds. □

Definition 3.7 *Given an $n\times n\times n$ trimatrix $A=(\alpha(i,h,c))$ and an assignment $(\pi,\rho)\in\Theta\times\Theta$, the following process is called an $\mathcal{M}$-transformation of A from (π,ρ) if each element $\alpha(i,h,c)$ except for $\alpha(i,\pi(i),\rho(i))$, $i\in I_n$ is added with a nonnegative number $\delta(i,h,c)$ so that the new trimatrix $T=(\alpha(i,h,c)+\delta(i,h,c))$ satisfies Condition 3.4, where $\delta(i,\pi(i),\rho(i))=0$ for each $i\in I_n$.*

The trimatrix T above will be called an $\mathcal{M}$-matrix resulted from (π,ρ).

Definition 3.8 *An $n\times n\times n$ trimatrix $(\alpha(i,h,c))$ is an optimality preserved matrix if there exist an optimal assignment $(\pi,\rho)\in\Theta\times\Theta$ and an $\mathcal{M}$-transformation from (π,ρ) such that (π,ρ) is still an optimal assignment in the $\mathcal{M}$-matrix resulted from (π,ρ).*

Obviously, a trimatrix satisfying Condition 3.4 is an optimality preserved matrix. We are now ready to introduce the main existence result of this paper which states a necessary and sufficient condition for the existence of a perfectly fair allocation.

Theorem 3.9 *Given a model $\mathcal{E}=((\alpha(i,h,c)),n,M)$, there exists a perfectly fair allocation if and only if the valuation trimatrix $(\alpha(i,h,c))$ is an optimality preserved matrix.*

Proof: Since $(\alpha(i,h,c))$ is an optimality preserved matrix, then there exist an optimal assignment $(\pi,\rho) \in \Theta \times \Theta$ and an $\mathcal{M}$-transformation from (π,ρ) such that (π,ρ) is still an optimal assignment in the $\mathcal{M}$-matrix resulted from (π,ρ). Let $T = (\bar{\alpha}(i,h,c))$ be the $n \times n \times n$ $\mathcal{M}$-matrix resulted from (π,ρ). So we have

$$\bar{\alpha}(i,h,c) \geq \alpha(i,h,c)$$

$$\bar{\alpha}(i,\pi(i),\rho(i)) = \alpha(i,\pi(i),\rho(i))$$

for all $i \in I_n$, $(h,c) \in \mathcal{O}_1 \times \mathcal{O}_2$, and

$$\sum_{i\in I_n} \bar{\alpha}(i,\pi(i),\rho(i)) \geq \sum_{i\in I_n} \bar{\alpha}(i,\tau(i),\gamma(i)), \forall(\tau,\gamma) \in \Theta \times \Theta. \tag{3.4}$$

Since T satisfies Condition 3.4, then there exist two n-vectors H^i and C^i for each $i \in I_n$ so that $\bar{\alpha}(i,h,c) = H^i(h)+C^i(c)$ for every $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. Then we can rewrite equation (3.4) as

$$\sum_{i\in I_n}(H^i(\pi(i)) + C^i(\rho(i)) \geq \sum_{i\in I_n}(H^i(\tau(i)) + C^i(\gamma(i))), \forall(\tau,\gamma) \in \Theta \times \Theta. \tag{3.5}$$

It follows from equation (3.5) that

$$\begin{array}{rcl} \sum_{i\in I_n} H^i(\pi(i)) & \geq & \sum_{i\in I_n} H^i(\tau(i)) \\ \sum_{i\in I_n} C^i(\rho(i)) & \geq & \sum_{i\in I_n} C^i(\gamma(i)) \end{array}$$

for all $(\tau,\gamma) \in \Theta \times \Theta$. By Lemma 3.2 there exist two n-vectors x and y such that $\sum_{i\in I_n} x_i = M/2$, $\sum_{i\in I_n} y_i = M/2$, and

$$\begin{array}{rcl} H^i(\pi(i)) + x_{\pi(i)} & \geq & H^i(j) + x_j \\ C^i(\rho(i)) + y_{\rho(i)} & \geq & C^i(l) + y_l \end{array}$$

for all $i,j,l \in I_n$. It follows that

$$\begin{array}{rcl} \alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} & = & \bar{\alpha}(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \\ & = & H^i(\pi(i)) + C^i(\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \\ & \geq & H^i(j) + x_j + C^i(l) + y_l \\ & = & \bar{\alpha}(i,j,l) + x_j + y_l \\ & \geq & \alpha(i,j,l) + x_j + y_l \end{array}$$

for all $i, j, l \in I_n$. Thus $(\pi, \rho, (x, y))$ is a perfectly fair allocation.

Now suppose that $(\pi, \rho, (x, y))$ is a perfectly fair allocation. Then it holds that

$$\alpha(i, \pi(i), \rho(i)) + x_{\pi(i)} + y_{\rho(i)} \geq \alpha(i, h, c) + x_h + y_c$$

for all $i \in I_n$, $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. It is readily seen that (π, ρ) is an optimal assignment with respect to $(\alpha(i, h, c))$. Let $A_i = \alpha(i, \pi(i), \rho(i)) + x_{\pi(i)} + y_{\rho(i)}$ for each $i \in I_n$. Let $d_i(h, c) = A_i - \alpha(i, h, c) - x_h - y_c$ for every $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. Clearly, $d_i(h, c) \geq 0$. Furthermore, $d_i(\pi(i), \rho(i)) = 0$ for all $i \in I_n$. Let $H^i(h) = A_i - x_h$ and $C^i(c) = -y_c$. Now define $\bar{\alpha}(i, h, c) = \alpha(i, h, c) + d_i(h, c)$. Clearly $\bar{\alpha}(i, h, c) = H^i(h) + C^i(c)$ and $\bar{\alpha}(i, \pi(i), \rho(i)) = \alpha(i, \pi(i), \rho(i))$ for all $i \in I_n$. Thus $(\bar{\alpha}(i, h, c))$ satisfies Condition 3.4. Furthermore, for any $(\tau, \gamma) \in \Theta \times \Theta$, we have

$$\begin{aligned}
\textstyle\sum_{i \in I_n} \bar{\alpha}(i, \pi(i), \rho(i)) &= \textstyle\sum_{i \in I_n} \alpha(i, \pi(i), \rho(i)) \\
&= \textstyle\sum_{i \in I_n} (A_i - x_{\pi(i)} - y_{\rho(i)}) \\
&\geq \textstyle\sum_{i \in I_n} (\bar{\alpha}(i, \tau(i), \gamma(i)) + x_{\tau(i)} + y_{\gamma(i)} - x_{\pi(i)} - y_{\rho(i)}) \\
&= \textstyle\sum_{i \in I_n} \bar{\alpha}(i, \tau(i), \gamma(i)) - \sum_{i \in I_n} x_{\pi(i)} \\
&\quad - \textstyle\sum_{i \in I_n} y_{\rho(i)} + \sum_{i \in I_n} x_{\tau(i)} + \sum_{i \in I_n} y_{\gamma(i)} \\
&= \textstyle\sum_{i \in I_n} \bar{\alpha}(i, \tau(i), \gamma(i)).
\end{aligned}$$

This means that $(\alpha(i, h, c))$ is an optimality preserved matrix. This completes the proof. □

One can easily verify that the matrix $(\alpha(i, h, c))$ in Example 1 is an optimality preserved matrix and thus there exists a perfectly fair allocation, whereas the matrix $(\alpha(i, h, c))$ in Example 2 is not an optimality preserved matrix and therefore there is no perfectly fair allocation in the example.

To make the reader more acquainted with optimality preserved matrices, we give one more example.

Example 3. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M. The values of the agents for the different pairs of house and car are given in Table 3.

Table 3: The values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$
$h1$	4	$\underline{5}$	$h1$	3	4.5
$h2$	4	0	$h2$	$\underline{5}$	5

The matrix $(\alpha(i,h,c))$ is an optimality preserved matrix. This can be seen from the optimal assignment $((1,2),(2,1))$ which is underlined in the tables 3 and 4. The transformation operations are indicated in Table 4.

Table 4: The changed values of objects for both agents

$\alpha(1,h,c)$	$c1$	$c2$	$\alpha(2,h,c)$	$c1$	$c2$
$h1$	4	$\underline{5}$	$h1$	$3+1.5$	4.5
$h2$	4	$0+5$	$h2$	$\underline{5}$	5

4 An investigation on optimality preserved matrices

In this section we will derive two fundamental properties of optimality preserved matrices and introduce several easily verifiable conditions for a matrix to be an optimality preserved matrix.

In Definition 3.8, an optimality preserved matrix is directly associated with a particular optimal assignment of the trimatrix. This raises the following question. When a trimatrix has many optimal assignments, do we need to check all optimal assignments in order to verify if the trimatrix is an optimality preserved matrix ? The following theorem tells us that whether a trimatrix is an optimality preserved matrix is independent of any particular optimal assignment. In other words, it is sufficient to check (an arbitrarily chosen) one optimal assignment in verifying the preserved optimality. Thus, preserved optimality is a fundamental intrinsic property of trimatrices.

Theorem 4.1 *Whether a trimatrix is an optimality preserved matrix does not depend on the choice of a particular optimal assignment of the trimatrix.*

Proof: Suppose that the $n \times n \times n$ trimatrix $V = (\alpha(i,h,c))$ is an optimality preserved matrix associated with the optimal assignment $(\pi, \rho) \in \Theta \times \Theta$. Now let $(\tau, \gamma) \in \Theta \times \Theta$ be another optimal assignment different from (π, ρ). It is sufficient to show that the trimatrix V is also optimality preserved matrix associated with (τ, γ). Since both (π, ρ) and (τ, γ) are optimal assignments of the matrix V, we have

$$\sum_{i \in I_n} \alpha(i, \pi(i), \rho(i)) = \sum_{i \in I_n} \alpha(i, \tau(i), \gamma(i)) \geq \sum_{i \in I_n} \alpha(i, \pi'(i), \rho'(i)) \tag{4.6}$$

for every $(\pi', \rho') \in \Theta \times \Theta$. It follows from Definition 3.8 that there exists a new $n \times n \times n$ trimatrix $\bar{V} = (\bar{\alpha}(i,h,c))$ such that $\bar{\alpha}(i, \pi(i), \rho(i)) = \alpha(i, \pi(i), \rho(i))$ for all $i \in I_n$, $\bar{\alpha}(i,h,c) \geq \alpha(i,h,c)$ for all i, h, c, and (π, ρ) is also an optimal assignment of the trimatrix $\bar{V}$. Thus, we have

$$\sum_{i \in I_n} \alpha(i, \pi(i), \rho(i)) \geq \sum_{i \in I_n} \bar{\alpha}(i, \tau(i), \gamma(i)) \geq \sum_{i \in I_n} \alpha(i, \tau(i), \gamma(i)).$$

Combining with inequality (4.6), we have

$$\sum_{i \in I_n} \alpha(i, \tau(i), \gamma(i)) \geq \sum_{i \in I_n} \bar{\alpha}(i, \pi'(i), \rho'(i))$$

for $(\pi', \rho') \in \Theta \times \Theta$. Therefore, (τ, γ) is also an optimal assignment of the trimatrix $\bar{V}$. Moreover, $\bar{\alpha}(i, \tau(i), \gamma(i)) = \alpha(i, \tau(i), \gamma(i))$ for all $i \in I_n$. Now, itt is readily seen that V is an optimality preserved matrix associated with (τ, γ). □

Let (π, ρ) be an optimal assingment of an $n \times n \times n$ trimatrix $V = (\alpha(i,h,c))$. The value $\max V = \sum_{i \in I_n} \alpha(i, \pi(i), \rho(i))$ will be called *the social value of* V. The next theorem gives another fundamental property of an optimality preserved matrix. Namely, it says that preserved optimality of a trimatrix $V = (\alpha(i,h,c))$ is totally symmetric with respect to i, h, and c.

Theorem 4.2 *An $n \times n \times n$ trimatrix $V = (\alpha(i,h,c))$ is an optimality preserved matrix if and only if there exist three $n-$vectors u, v and w, such that $\alpha(i,h,c) \leq u_i + v_h + w_c$ for all i, h and $c \in I_n$, and $\sum_{i \in I_n} u_i + \sum_{h \in I_n} v_h + \sum_{c \in I_n} w_c = \max V$.*

Proof: Sufficiency: Let $\bar{\alpha}(i,h,c) = u_i + v_h + w_c$ for all i, h and c and let $\bar{V} = (\bar{\alpha}(i,h,c))$. Then, we have $\bar{\alpha}(i,h,c) \geq \alpha(i,h,c)$ for all i, h and c. Next, let (π, ρ) be an optimal assingment of $V = (\alpha(i,h,c))$. Thus,

$$\sum_{i\in I_n} \alpha(i,\pi(i),\rho(i)) = \max V = \sum_{i\in I_n} u_i + \sum_{h\in I_n} v_h + \sum_{c\in I_n} w_c = \sum_{i\in I_n} \bar{\alpha}(i,\pi(i),\rho(i)).$$

It follows that $\bar{\alpha}(i,\pi(i),\rho(i)) = \alpha(i,\pi(i),\rho(i))$ for all $i \in I_n$. Moreover, note that for any assingment (π', ρ') of $\bar{V} = (\bar{\alpha}(i,h,c))$, we have that

$$\sum_{i\in I_n} \bar{\alpha}(i,\pi',\rho') = \sum_{i\in I_n} u_i + \sum_{h\in I_n} v_h + \sum_{c\in I_n} w_c = \sum_{i\in I_n} \bar{\alpha}(i,\pi(i),\rho(i)).$$

This says that (π, ρ) is also an optimal assingment of the trimatrix $\bar{V} = (\bar{\alpha}(i,h,c))$. Therefore, by definition, we see that $V = (\alpha(i,h,c))$ is an optimality preserved matrix.

Necessity: By Theorem 3.9, we see that for any social money M, the model $\mathcal{E} = ((\alpha(i,h,c)), n, M)$ has a perfectly fair allocation $(\pi, \rho, (x,y))$. That is, $\alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)} \geq \alpha(i,h,c) + x_h + y_c$ for all i, h and c. Now let $u_i = \alpha(i,\pi(i),\rho(i)) + x_{\pi(i)} + y_{\rho(i)}$ for all i, and $v = -x$, $w = -y$. Then we have that $\alpha(i,h,c) \leq u_i - x_h - y_c = u_i + v_h + w_c$ for all i, h and c. Moreover, $\sum_{i\in I_n} u_i + \sum_{h\in I_n} v_h + \sum_{c\in I_n} w_c = \sum_{i\in I_n}(u_i - x_{\pi(i)} - y_{\rho(i)}) = \sum_{i\in I_n} \alpha(i,\pi(i),\rho(i)) = \max V$. Thus we proved the necessity. □

We remark that although in a model $\mathcal{E} = ((\alpha(i,h,c)), n, M)$ the economic role or meaning of i is totally different those from h and c, whether an $n \times n \times n$ trimatrix $V = (\alpha(i,h,c))$ is an optimality preserved matrix is symmetric with respect to i and h, c.

In the following, we will identify several easily verifiable conditions for a matrix to be an optimality preserved matrix.

Theorem 4.3 *An $n \times n \times n$ trimatrix $V = (\alpha(i,h,c))$ is an optimality preserved matrix, if one of the following conditions holds:*

A1: For every $i \in I_n$, we have

$$\alpha(i,h1,c1) + \alpha(i,h2,c2) = \alpha(i,h1,c2) + \alpha(i,h2,c1),$$
$$\forall h1, h2 \in \mathcal{O}_1, c1, c2 \in \mathcal{O}_2.$$

A2: For every $h \in \mathcal{O}_1$, we have

$$\alpha(i,h,c1) + \alpha(j,h,c2) = \alpha(i,h,c2) + \alpha(j,h,c1),$$
$$\forall i, j \in I_n, c1, c2 \in \mathcal{O}_2.$$

A3: For every $c \in \mathcal{O}_2$, we have

$$\alpha(i, h1, c) + \alpha(j, h2, c) = \alpha(j, h1, c) + \alpha(i, h2, c),$$
$$\forall h1, h2 \in \mathcal{O}_1, i, j \in I_n.$$

Proof: If Condition A1 is satisfied, the result follows immediately from the definition of an optimality preserved matrix.

Next, by Theorem 4.2 we see that whether an $n \times n \times n$ trimatrix $V = (\alpha(i, h, c))$ is an optimality preserved matrix is symmetric with respect to i and h, c. Therefore, if Condition A2 or Condition A3 is satisfied, then V is an optimality preserved matrix. □

Condition A1 indicates that the reservation value function of each agent over pairs of house and car are separable and additive. We remark that although Conditions A1, A2, and A3 are symmetric and similar, they are in fact independent of one another as indicated by the following example.

Example 4. Consider the case in which there are two agents 1, 2 and there are two houses $h1$, $h2$, and two cars $c1$, $c2$, and total money (say, dollar) M. The values of the agents for the different pairs of house and car are given in Table 5. For this example, Condition A2 is satisfied, but Condition A1 is not satisfied nor is Condition A3.

Table 5: The values of objects for both agents

$\alpha(1, h, c)$	c1	c2	$\alpha(2, h, c)$	c1	c2
$h1$	2	4	$h1$	1	3
$h2$	7	8	$h2$	5	6

Therefore, Conditions A2 and A3 provide two different classes of optimality preserved matrices in which the reservation value function of each agent over pairs of house and car are not necessarily separable and additive.

Theorem 4.4 *An $n \times n \times n$ trimatrix $V = (\alpha(i, h, c))$ is an optimality preserved matrix, if one of the following conditions holds:*

B1: There exists an element $\pi \in \Theta$ such that for each $i \in I_n$, we have

$$\alpha(i, h, c) \leq \frac{1}{2}\alpha(i, h, \pi(h)) + \frac{1}{2}\alpha(i, \pi^{-1}(c), c)$$

for all $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$.

B2: There exists an element $\pi \in \Theta$ such that for each $h \in \mathcal{O}_1$, we have

$$\alpha(i,h,c) \leq \frac{1}{2}\alpha(c,h,\pi(c)) + \frac{1}{2}\alpha(\pi^{-1}(i),h,i)$$

for all $i \in I_n$, $c \in \mathcal{O}_2$.

B3: There exists an element $\pi \in \Theta$ such that for each $c \in \mathcal{O}_2$, we have

$$\alpha(i,h,c) \leq \frac{1}{2}\alpha(i,\pi(i),c) + \frac{1}{2}\alpha(\pi^{-1}(h),h,c)$$

for all $i \in I_n$, $h \in \mathcal{O}_1$.

Proof: Suppose that Condition B1 is satisfied. Without loss of generality, we may assume that $\pi(i) = i$ for all $i \in I_n$. Then Condition B1 can be simplified as follows. For every $i \in I_n$, it holds

$$\alpha(i,h,c) \leq [\alpha(i,h,h) + \alpha(i,c,c)]/2$$

for all $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. Now we define a new $n \times n \times n$ trimatrix $\bar{V} = (\bar{\alpha}(i,h,c))$, where

$$\bar{\alpha}(i,h,c) = [\alpha(i,h,h) + \alpha(i,c,c)]/2$$

for every $i \in I_n$, $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$. By definition, we have $\bar{\alpha}(i,h,h) = \alpha(i,h,h)$ for all $i,h \in I_n$, and $\bar{\alpha}(i,h,c) \geq \alpha(i,h,c)$ for all $i,h,c \in I_n$. Now let $(\pi,\rho) \in \Theta \times \Theta$ be an optimal assignment of the trimatrix $\bar{V}$. Then we have

$$\begin{aligned} \textstyle\sum_{i \in I_n} \bar{\alpha}(i,\pi(i),\rho(i)) &= \textstyle\sum_{i \in I_n}[\alpha(i,\pi(i),\pi(i)) + \alpha(i,\rho(i),\rho(i))]/2 \\ &= \tfrac{1}{2}\textstyle\sum_{i \in I_n} \alpha(i,\pi(i),\pi(i)) + \tfrac{1}{2}\textstyle\sum_{i \in I_n} \alpha(i,\rho(i),\rho(i)). \end{aligned}$$

It follows that both (π,π) and (ρ,ρ) are also optimal assignments of the trimatrix $\bar{V}$. Since $\alpha(i,\pi(i),\pi(i)) = \bar{\alpha}(i,\pi(i),\pi(i))$ for all $i \in I_n$, it is clear that (π,π) is also an optimal assignment of the trimatrix V. Furthermore, for every $i \in I_n$, we have

$$\bar{\alpha}(i,h1,c1) + \bar{\alpha}(i,h2,c2) = \bar{\alpha}(i,h1,c2) + \bar{\alpha}(i,h2,c1),$$
$$\forall h1,h2 \in \mathcal{O}_1, c1,c2 \in \mathcal{O}_2.$$

By definition, the trimatrix V is an optimality preserved matrix.

By using Conditions A1 and A2 in Theorem 4.3, we can demonstrate Conditions B2 and B3 following the proof for Condition B1. □

Condition B1 indicates that the reservation value function of each agent over pairs of houses and cars exhibits some kind of local convexity. Again, we remark that although Conditions B1, B2, and B3 are symmetric and similar, they are in fact independent of one another. One can easily verify this fact by giving an example. Furthermore, it is also easy to check that Conditions A1, A2, A3, B1, B2, and B3 are independent of one another.

We can further extend the above theorem by allowing any convex parameter θ.

Theorem 4.5 *An $n \times n \times n$ trimatrix $V = (\alpha(i, h, c))$ is an optimality preserved matrix, if one of the following conditions holds:*

B1': There exists an element $\pi \in \Theta$ and a real number $0 \le \theta \le 1$ such that for each $i \in I_n$, we have

$$\alpha(i, h, c) \le \theta\alpha(i, h, \pi(h)) + (1-\theta)\alpha(i, \pi^{-1}(c), c)$$

for all $h \in \mathcal{O}_1$, $c \in \mathcal{O}_2$.

B2': There exists an element $\pi \in \Theta$ and a real number $0 \le \theta \le 1$ such that for each $h \in \mathcal{O}_1$, we have

$$\alpha(i, h, c) \le \theta\alpha(c, h, \pi(c)) + (1-\theta)\alpha(\pi^{-1}(i), h, i)$$

for all $i \in I_n$, $c \in \mathcal{O}_2$.

B3': There exists an element $\pi \in \Theta$ and a real number $0 \le \theta \le 1$ such that for each $c \in \mathcal{O}_2$, we have

$$\alpha(i, h, c) \le \theta\alpha(i, \pi(i), c) + (1-\theta)\alpha(\pi^{-1}(h), h, c)$$

for all $i \in I_n$, $h \in \mathcal{O}_1$.

The above results demonstrate that the class of optimality preserved matrices is fairly large and rich.

5 An extension

In this section we consider an extension of the previous model. Suppose there are m different types of objects. There are n objects of each type, denoted by $\mathcal{O}_j$, $j \in I_m$. For example, one might think of $\mathcal{O}_1$ as the collection of houses, of $\mathcal{O}_2$ as cars, of $\mathcal{O}_3$ as trucks, and so on. The utility function of each agent is defined as $u_i : \mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_m \times \mathbb{R} \mapsto \mathbb{R}$ which is assumed to be a nondecreasing and continuous function in money. Then we can extend the definition of perfectly fair allocation as follows.

Definition 5.1 *An allocation $(\pi^1, \cdots, \pi^m, x^1, \cdots, x^m)$ is a perfectly fair allocation if it holds that $\pi^j \in \Theta$, $j \in I_m$, $\sum_{i \in I_n} \sum_{j \in I_m} x_i^j = M$, and*

$$\begin{aligned} u_i(\pi^1(i), \cdots, \pi^m(i), x^1_{\pi^1(i)} + \cdots + x^m_{\pi^m(i)}) \;&\geq\; u_i(h_1, \cdots, h_m, x^1_{h_1} + \cdots + x^m_{h_m}) \\ &\forall i \in I_n,\ h_j \in \mathcal{O}_j, j \in I_m. \end{aligned}$$

One can show that every perfectly fair allocation is envy free, Pareto optimal, and income fair. To obtain an existence result, once again we will focus our attention on the case where every agent has quasi-linear utilities in money. In this case we can represent the model by $\mathcal{E} = ((\alpha(i, h_1, h_2, \cdots, h_m)), n, M)$ where the matrix $(\alpha(i, h_1, \cdots, h_m))$ is an n^{m+1}-matrix, n is the number of agents and M is the total amount of money. Each entry $\alpha(i, h_1, \cdots, h_m)$ represents the value of the combination of objects h_1, h_2, $\cdots$, h_m to agent i.

Conditions 3.4 and 3.5 can be appropriately modified as follows:

Condition 5.2 *The n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ has the following property: For every $i \in I_n$ and $j, k \in I_m$ with $1 \leq j < k \leq m$, it holds*

$$\begin{aligned} &\alpha(i, h_1, \cdots, h_j, \cdots, h_k, \cdots, h_m) + \alpha(i, h_1, \cdots, h'_j, \cdots, h'_k, \cdots, h_m) \\ &= \alpha(i, h_1, \cdots, h_j, \cdots, h'_k, \cdots, h_m) + \alpha(i, h_1, \cdots, h'_j, \cdots, h_k, \cdots, h_m), \\ &\forall\ (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m\ h'_j \in \mathcal{O}_j,\ h'_k \in \mathcal{O}_k \end{aligned}$$

Condition 5.3 *The n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ has the following property: For every $i \in I_n$ and $j \in I_m$, there exists an n-vector $H_i(j) = (H_i(j,1), \cdots, H_i(j,n))$ such that it holds*

$$\alpha(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j), \quad \forall\ (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m.$$

Lemma 5.4 *Conditions 5.2 and 5.3 are equivalent.*

Proof: That Condition 5.3 implies Condition 5.2 is obvious. Now we prove that Condition 5.2 implies Condition 5.3 by induction. We have proved the case of $m = 2$ in Section

3. Suppose that the case of $m-1$ is true. Now let us prove the case of m. It follows from Condition 5.2 and the assumption that for every $i \in I_n$, $j \in I_m \setminus \{m\}$, and each fixed $h_m \in \mathcal{O}_m$ there exists an n-vector $H_i'(j, h_m) = (H_i'(j, 1, h_m), \cdots, H_i'(j, n, h_m))$ such that

$$\alpha(i, h_1, \cdots, h_{m-1}, h_m) = \sum_{j=1}^{m-1} H_i'(j, h_j, h_m), \quad \forall (h_1, \cdots, h_{m-1}) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_{m-1}.$$

Note that: for each fixed $j \in I_m$, $(H_i'(j, h_j, h_m))$ can be looked as a trimatrix for all $i \in I_n$, $h_j \in \mathcal{O}_j$, and $h_m \in \mathcal{O}_m$. Recall that from Condition 5.2 we have: for every $j (\neq m) \in I_m$

$$\begin{aligned} &\alpha(i, h_1, \cdots, h_j, \cdots, h_m) + \alpha(i, h_1, \cdots, h_j', \cdots, h_m') \\ &= \alpha(i, h_1, \cdots, h_j, \cdots, h_m') + \alpha(i, h_1, \cdots, h_j', \cdots, h_m). \end{aligned}$$

This implies that

$$H_i'(j, h_j, h_m) + H_i'(j, h_j', h_m') = H_i'(j, h_j, h_m') + H_i'(j, h_j', h_m),$$

for all $h_j, h_j' \in \mathcal{O}_j$, and $h_m, h_m' \in \mathcal{O}_m$. Then by Lemma 3.6, we see that for each $j(\neq m) \in I_m$ there exist two n-vectors $H_i(j)$ and $H_i'(j)$ such that $H_i'(j, h_j, h_m) = H_i(j, h_j) + H_i'(j, h_m)$ for all $h_j \in \mathcal{O}_j$ and $h_m \in \mathcal{O}_m$. Define $H_i(m) = \sum_{j=1}^{m-1} H_i'(j)$. Then we obtain that

$$\alpha(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j), \quad \forall (h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m.$$

This says that Condition 5.3 is true for the case of m. □

Definition 5.5 *Given an n^{m+1}-matrix $A = (\alpha(i, h_1, \cdots, h_m))$ and an assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$, the following process is called an $\mathcal{M}$-transformation of A from $(\pi^1, \cdots, \pi^m)$ if each element $\alpha(i, h_1, \cdots, h_m)$ except for $\alpha(i, \pi_1(i), \cdots, \pi_m(i))$, $i \in I_n$ is added with a nonnegative number $\delta(i, h_1, \cdots, h_m)$ so that the new n^{m+1}-matrix $T = (\alpha(i, h_1, \cdots, h_m) + \delta(i, h_1, \cdots, h_m))$ satisfies Condition 5.2.*

The n^{m+1}-matrix T above will be called an $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$.

Definition 5.6 *An n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix if there exist an optimal assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$ and an $\mathcal{M}$-transformation from $(\pi^1, \cdots, \pi^m)$ such that $(\pi^1, \cdots, \pi^m)$ is still an optimal assignment in the $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$.*

Clearly, an n^{m+1}-matrix satisfying Condition 5.2 is an optimality preserved matrix.

Having these preparations, we can now establish the following existence theorem on this more general model. Here we render a complete proof, which we believe will provide some additional insight into the problem, although some part of the proof is similar to that given in Theorem 3.9.

Theorem 5.7 *Given a model $\mathcal{E} = ((\alpha(i, h_1, \cdots, h_m)), n, M)$, there exists a perfectly fair allocation if and only if the valuation n^{m+1}-matrix $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix.*

Proof: Since $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix, then there exist an optimal assignment $(\pi^1, \cdots, \pi^m) \in \Theta \times \cdots \times \Theta$ and an $\mathcal{M}$-transformation from $(\pi^1, \cdots, \pi^m)$ such that $(\pi^1, \cdots, \pi^m)$ is still an optimal assignment in the $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$. Let $T = (\bar{\alpha}(i, h_1, \cdots, h_m))$ be the n^{m+1} $\mathcal{M}$-matrix resulted from $(\pi^1, \cdots, \pi^m)$. So we have

$$\bar{\alpha}(i, h_1, \cdots, h_m) \geq \alpha(i, h_1, \cdots, h_m)$$

$$\bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) = \alpha(i, \pi^1(i), \cdots, \pi^m(i))$$

for all $i \in I_n$, $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$, and

$$\sum_{i \in I_n} \bar{\alpha}(i, \pi^1(i), \cdots, \pi^m(i)) \geq \sum_{i \in I_n} \bar{\alpha}(i, \tau^1(i), \cdots, \tau^m(i)), \tag{5.7}$$

for all $(\tau^1, \cdots, \tau^m) \in \Theta \times \cdots \times \Theta$. Since T satisfies Condition 5.2, then for every $i \in I_n$ and $j \in I_m$ there exists an n-vector $H_i(j)$ so that $\bar{\alpha}(i, h_1, \cdots, h_m) = \sum_{j \in I_m} H_i(j, h_j)$ for every $(h_1, \cdots, h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Then we can rewrite equation (5.7) as

$$\sum_{i \in I_n} \sum_{j \in I_m} H_i(j, \pi^j(i)) \geq \sum_{i \in I_n} \sum_{j \in I_m} H_i(j, \tau^j(i)), \tag{5.8}$$

for all $(\tau^1, \cdots, \tau^m) \in \Theta \times \cdots \times \Theta$. It follows from equation (5.8) that

$$\sum_{i \in I_n} H_i(j, \pi^j(i)) \geq \sum_{i \in I_n} H_i(j, \tau^j(i))$$

for every $j \in I_m$ and all $\tau^j \in \Theta$. By Lemma 3.2 for each $j \in I_m$ there exists an n-vector x^j such that $\sum_{h_j \in I_n} x^j_{h_j} = M/m$, and

$$H_i(j, \pi^j(i)) + x^j_{\pi^j(i)} \geq H_i(j, h_j) + x^j_{h_j}$$

for all $i \in I_n$, $j \in I_m$ and $h_j \in \mathcal{O}_j$. It follows that

$$\begin{aligned}
\alpha(i,\pi^1(i),\cdots,\pi^m(i)) + \textstyle\sum_{j\in I_m} x^j_{\pi^j(i)} &= \bar{\alpha}(i,\pi^1(i),\cdots,\pi^m(i)) + \textstyle\sum_{j\in I_m} x^j_{\pi^j(i)} \\
&= \textstyle\sum_{j\in I_m} H_i(j,\pi^j(i)) + \sum_{j\in I_m} x^j_{\pi^j(i)} \\
&= \textstyle\sum_{j\in I_m}(H_i(j,\pi^j(i)) + x^j_{\pi^j(i)}) \\
&\geq \textstyle\sum_{j\in I_m}(H_i(j,h_j) + x^j_{h_j}) \\
&= \bar{\alpha}(i,h_1,\cdots,h_m) + \textstyle\sum_{j\in I_m} x^j_{h_j} \\
&\geq \alpha(i,h_1,\cdots,h_m) + \textstyle\sum_{j\in I_m} x^j_{h_j}
\end{aligned}$$

for all $i \in I_n$ and $(h_1,\cdots,h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Thus $(\pi^1,\cdots,\pi^m,\, x^1,\cdots,x^m)$ is a perfectly fair allocation.

Now suppose that $(\pi^1,\cdots,\pi^m,\, x^1,\cdots,x^m)$ is a perfectly fair allocation. Then it holds that

$$\alpha(i,\pi^1(i),\cdots,\pi^m(i)) + \sum_{j\in I_m} x^j_{\pi^j(i)} \geq \alpha(i,h_1,\cdots,h_m) + \sum_{j\in I_m} x^j_{h_j}$$

for all $i \in I_n$ and $(h_1,\cdots,h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. It is readily seen that $(\pi^1,\cdots,\pi^m)$ is an optimal assignment with respect to $(\alpha(i,h_1,\cdots,h_m))$. Let $A_i = \alpha(i,\pi^1(i),\cdots,\pi^m(i)) + \sum_{j\in I_m} x^j_{\pi^j(i)}$ for each $i \in I_n$. Let $d_i(h_1,\cdots,h_m) = A_i - \alpha(i,h_1,\cdots,h_m) - \sum_{j\in I_m} x^j_{h_j}$ for every $(h_1,\cdots,h_m) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_m$. Clearly, $d_i(h_1,\cdots,h_m) \geq 0$. Furthermore,

$$d_i(\pi^1(i),\cdots,\pi^m(i)) = 0$$

for all $i \in I_n$. Let $H_i(j) = -x^j$ for all $j \in I_m \setminus \{m\}$ and $H_i(m) = (A_i - x^m_i,\cdots,A_i - x^m_n)$. Now define $\bar{\alpha}(i,h_1,\cdots,h_m) = \alpha(i,h_1,\cdots,h_m) + d_i(h_1,\cdots,h_m)$. Clearly $\bar{\alpha}(i,h_1,\cdots,h_m) = \sum_{j\in I_m} H_i(j,h_j)$ and $\bar{\alpha}(i,\pi^1(i),\cdots,\pi^m(i)) = \alpha(i,\pi^1(i),\cdots,\pi^m(i))$ for all $i \in I_n$. Thus $(\bar{\alpha}(i,h_1,\cdots,h_m))$ satisfies Condition 5.2. Furthermore, for any $(\tau^1,\cdots,\tau^m) \in \Theta \times \cdots \times \Theta$, we have

$$\begin{aligned}
&\textstyle\sum_{i\in I_n} \bar{\alpha}(i,\pi^1(i),\cdots,\pi^m(i)) \\
&= \textstyle\sum_{i\in I_n} \alpha(i,\pi^1(i),\cdots,\pi^m(i)) \\
&= \textstyle\sum_{i\in I_n}(A_i - \sum_{j\in I_m} x^j_{\pi^j(i)}) \\
&\geq \textstyle\sum_{i\in I_n}(\bar{\alpha}(i,\tau^1(i),\cdots,\tau^m(i)) + \sum_{j\in I_m} x^j_{\tau^j(i)} - \sum_{j\in I_m} x^j_{\pi^j(i)}) \\
&= \textstyle\sum_{i\in I_n} \bar{\alpha}(i,\tau^1(i),\cdots,\tau^m(i)) + \sum_{i\in I_n}\sum_{j\in I_m} x^j_{\tau^j(i)} - \sum_{i\in I_n}\sum_{j\in I_m} x^j_{\pi^j(i)} \\
&= \textstyle\sum_{i\in I_n} \bar{\alpha}(i,\tau^1(i),\cdots,\tau^m(i))
\end{aligned}$$

This means that $(\alpha(i, h_1, \cdots, h_m))$ is an optimality preserved matrix. This completes the proof. □

REFERENCES

Alkan, A., Demange, G., Gale, D., 1991. Fair allocation of indivisible objects and criteria of justice. *Econometrica* 59, 1023-1039.

Aragones, E., 1995. A derivation of the money Rawlsian solution. *Social Choice and Welfare* 12, 267-276.

Foley, D., 1967. Resource allocation and the public sector. *Yale Economic Essays* 7, 45-98.

Haake, C.J, Raith, M.G., and Su, F.E., 2002. Bidding for envy-freeness: a procedural approach to n-player fair division problems. *Social Choice and Welfare* 19, 723-749.

Klijn, F., 2000. An algorithm for envy-free allocations in an economy with indivisible objects and money. *Social Choice and Welfare* 17, 201-216.

Maskin, E., 1987. On the fair allocation of indivisible objects. in: *Arrow and the Foundations of the Theory of Economic Policy*, MacMillan, London, pp. 341-349.

Pazner, E., and Schmeidler, D., 1978. Decentralization, income distribution and the role of money in socialist economies. *Economic Inquiry* **16**, 257-264.

Rawls, J., 1971. *A Theory of Justice,* Harvard University Press, Cambridge, MA.

Shapley, L.S., Shubik, M., 1972. The assignment game, I: the core. *International Journal of Game Theory* 1, 111-130.

Steinhaus, H., 1948. The problem of fair division. *Econometrics* 16, 101-104.

Su, F.E., 1999. Rental harmony: Sperner's lemma in fair division. *American Mathematical Monthly* 106, 930-942.

Sun, N., Yang, Z., 2000. On fair allocations and indivisibilities. Preprint, Faculty of Business Administration, Yokohama National University, Yokohama.

Svensson, L., 1983. Large indivisibilities: an analysis with respect to price equilibrium and fairness. *Econometrica* 51, 939-954.

Tadenuma, K., Thomson, W., 1991. No-envy and consistency in economies with indivisibilities. *Econometrica* 59, 1755-1767.

Varian, H., 1974. Equity, envy, and efficiency. *Journal of Economic Theory* 9, 63-91.

Yang, Z., 2001. An intersection theorem on an unbounded set and its application to the fair allocation problem. *Journal of Optimization Theory and Applications* 110, 429-443.