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NTU–prenucleoli

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Abstract

We propose a new class of excess functions for coalitional games with non-transferable utility (NTU games) and investigate the resulting class of NTU prenucleoli. We follow the basic ideas of [Kal75] to formulate conditions under which certain functions are believed to measure dissatisfactions of coalitions appropriately. However, we formulate other conditions than [Kal75] which characterize the new excess functions uniquely. The resulting NTU prenucleoli and the well known TU prenucleolus share some important properties like single-valuedness and the characterization by the Kohlberg criterion ([Koh71]). A special member of this class of NTU prenucleoli is introduced which has some additional properties like covariance, consistency (with respect to a new reduced game, which is an extension of the Davis and Maschler reduced (TU) game) etc. Due to this properties we call this member the NTU prenucleolus. Additionally we attack the problem of computing this new solution concept. By demonstrating a closed connection between the NTU prenucleolus and the TU prenucleolus we show how the results concerning the computability of the latter can be used for this.

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1 Introduction

The (pre-) nucleolus for cooperative games with transferable utility (TU games) as introduced for the first time by [Sch69] is a well-known and extensively reasearched solution concept. Some other solution concepts for TU games like the Core or the Shapley value have been successfully extended to a superclass of the class of all TU games, to the class of all cooperative games with *non*-transferable utility (NTU games). For the prenucleolus this task is still unsolved.

One very basic ingredience in the definition of the (TU) prenucleolus is the concept of excess. The excess of a coalition is designed to be a measure of dissatisfaction of the coalition with a given imputation. The prenucleolus minimizes the excesses of all coalitions lexicographically. Thus an intuitive way to extend the prenucleolus to NTU games is to extend the excess to this class of games.

This idea was first considered by E. Kalai in [Kal75]. In that paper a class of excess functions for NTU games was defined by means of some intuitive properties (in the sense that the excess functions for TU games were easily seen to satisfy these properties). The resulting NTU prenucleoli were proven to be nonempty and to be included in the Core, provided the latter is nonempty. However, single-valuedness, continuity, consistency, characterization by balanced collections of coalitions (the Kohlberg criterion, [Koh71]) are properties that these NTU prenucleoli do not satisfy in general.

In this paper we introduce a new class of excess functions (the β-excess functions) and thereby a new class of (NTU) prenucleoli, called β-prenucleoli. We are doing this by providing (and discussing) four properties that uniquely characterize this new class of excess functions, i.e. we axiomatize this class of excess functions. We proof some properties that every β-prenucleolus meets like single-valuedness, validness of the Kohlberg criterion and coincidence with the (TU) prenucleolus on the class of all TU games.

Furthermore, we examine a special member of the class of all β-prenucleoli. For this member we show additional properties like covariance, consistency with respect to a new reduced game and inclusion in the core for a subclass of games. In view of this properties this special β-prenucleolus will be called the (NTU) prenucleolus.

By demonstrating some close connections between the (NTU) prenucleolus and the (TU) prenucleolus of specific TU games, we are able to apply results from a previous paper ([Kla97]) and derive a set-valued dynamical system

that converges to the (NTU) prenucleolus. This result yields a method for computing it. However, the details of this approach are not the subject of the present paper.
In a recent paper, Chang and Chen ([CC02]) consider a class of so-called affine excess functions and its subclass of C-excess functions. The latter is a superclass of the β-excess functions and contains (like the class of Kalai) excess functions that do not necessarily coincide with the TU excess on TU games. They prove single-valuedness and validness of the Kohlberg criterion for the resulting prenucleoli.
Of course, the excess concept is also used to define another well-known solution concept for TU games, the (TU) (pre-) kernel. With the (NTU) excess introduced in this paper, an extension of this solution to the class of all NTU games is also possible. We will not discuss this in the present paper.
This paper is organised as follows. In section 2 we provide the necessary notational conventions and some basic definitions. In section 3 we introduce the (NTU) excess functions introduced by E. Kalai in [Kal75]. Our (NTU) excess functions will be motivated and defined in section 4. In the same section the resulting (NTU) β-prenucleolus will be defined and we prove some results that apply to every member of the class of all β-prenucleoli. In section 5 we investigate in detail a member of this class that we will call the (NTU) prenucleolus since we will show that there enough resemblances to the (TU) prenucleolus to justify this.

2 Definitions and Notations

Let us first agree on some notation. Let U be the (finite or infinite) universe of all players and let $N \subseteq U$ be a finite subset. The subsets $S \subseteq N$ are called coalitions, N is called the grand coalition, 2^N is the set of all coalitions of N. $\mathbb{R}^N$ is the set of all functions from N to $\mathbb{R}$. Every $x \in \mathbb{R}^N$ will be identified with the $|N|$-dimensional vector $(x(i))_{i \in N}$, whose components are indexed by the members of N; we will therefore write x_i instead of $x(i)$ for $x \in \mathbb{R}^N$ and $i \in N$. If $S \in 2^N$ is a coalition and $x \in \mathbb{R}^N$, we denote by x_S the projection of x onto $\mathbb{R}^S := \{x \in \mathbb{R}^N \,|x_i = 0 \quad \forall i \notin S\}$.
If $x, y \in \mathbb{R}^N$, then $x \geq y$ means $x_i \geq y_i$ for every $i \in N$, while $x > y$ denotes the case where $x_i > y_i$ for every $i \in N$. The scalar product of x and y is denoted by $\langle x, y \rangle$, i.e. $\langle x, y \rangle := \sum_{i \in N} x_i y_i, x, y \in \mathbb{R}^N$. If $x \in \mathbb{R}^N$ is a vector and $r \in \mathbb{R}$ is some real number, then $rx := (rx_i)_{i \in N}$. The componentwise multiplication of two vectors $x, y \in \mathbb{R}^N$ is denoted by xy, i.e. $xy := (x_i y_i)_{i \in N}, x, y \in \mathbb{R}^N$. Of course $\frac{x}{y}$ means $\left(\frac{x_i}{y_i}\right)_{i \in N}, x, y \in \mathbb{R}^N, y \neq 0$. For $\lambda \in \mathbb{R}^N$ and $A \subset \mathbb{R}^N$, λA is the set $\{\lambda a \,|a \in A\}$, whereas rA for $r \in \mathbb{R}$ is the set $\{ra \,|a \in A\}$.
The relative interior of A is denoted by $int(A)$ and its boundary by ∂A. If $x \in A$ and $y \leq x$ implies $y \in A$ then A is called comprehensive.

Definition 2.1

A **coalitional game with transferable utility (TU game)** *is a pair (N, v), where $N \subseteq U$ is the set of players and $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$, is the coalitional function that assigns to each coalition $S \in 2^N$ its worth $v(S)$. Let Γ^{TU} be the class of all TU games.*
For every game $(N, v) \in \Gamma^{TU}$ let

$I^*(N, v) = \{x \in \mathbb{R}^N \,|x(N) = v(N)\}$ *be the set of preimputations.*

Definition 2.2

Let $\Gamma \subseteq \Gamma^{TU}$ be a class of games. A **solution concept** *on Γ is a mapping*

$$\begin{aligned} \sigma : \Gamma &\to \bigcup_{(N,v) \in \Gamma} 2^{I^*(N,v)} \\ \sigma(N, v) &\subseteq I^*(N, v), \end{aligned}$$

that assigns to each game $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of the set of preimputations $I^*(N, v)$.

Definition 2.3

A **coalitional game with non-transferable utility (NTU game)** *is a pair (N,V), where $V : 2^N \to 2^{\mathbb{R}^N}, V(S) \subsetneq \mathbb{R}^S$, is the coalitional function that assigns to each coalition $S \in 2^N$ a closed, comprehensive, non-empty proper subset $V(S)$ of outcomes that are attainable to S through cooperation. Here $\mathbb{R}^S := \{x \in \mathbb{R}^N \,|x_i = 0 \quad \forall i \notin S\}$ is the projection of $\mathbb{R}^N$ on the subspace that is spanned by the members of S. Let Γ^{NTU} be the class of all NTU games. If $x \in V(S)$ we say that S is effective for x. We make the assumption that $V(\{i\}) = (-\infty, 0]$ for every player $i \in N$.*

A subclass of Γ^{NTU} is the class of all **hyperplane games**, denoted by Γ^H. In a hyperplane game, every $V(S)$ is a halfspace, given by $V(S) = \{x \in \mathbb{R}^S \,|\langle p_V^S, x\rangle \le c_V^S\}$, where $p_V^S \in \mathbb{R}^S_{++}$ and $c_V^S \in \mathbb{R}_+$. Of course, the representation of $V(S)$ by p_V^S and c_V^S is not unique; in fact, if p_V^S and c_V^S represent $V(S)$, then so does $\frac{1}{r}p_V^S$ and rc_V^S for every $r \in \mathbb{R}_{++}$. If $(N,V) \in \Gamma^H$ is a hyperplane game with $p_V^N = r(1,\ldots,1)$ for some $r \in \mathbb{R}_{++}$, i.e. $\partial V(N)$ is parallel to the boundary of the unit simplex in $\mathbb{R}^N$, then (N,V) is called **simplex game**. Those NTU games, for which $V(N)$ is a halfspace while every other $V(S), S \neq N$, is arbitrary (but satisfies of course the conditions of Definition 2.3), we will call **quasi hyperplane games** and denote by Γ^{qH} the class of all those games. If $(N,V) \in \Gamma^{NTU}$ is an NTU game and $\lambda \in \mathbb{R}^N_{++}$, then we call (N,V) and the game $(N,\lambda V)$ *covariant under a linear transformation of utility.* The game $(N,\lambda V)$ is given by $(\lambda V)(S) = \lambda V(S)$ for every coalition $S \in 2^N$.

Definition 2.4

If $(N,v) \in \Gamma^{TU}$ is a TU game, then denote by $(N,V^v) \in \Gamma^H$ its according NTU game, i.e. (N,V^v) is given by

$$p_{V^v}^S = (1,\ldots,1)\,|_S,$$
$$c_{V^v}^S = v(S)$$

for every $S \in 2^N$. Of course, (N,V^v) is a simplex game.
If $(N,V) \in \Gamma^H$ is a hyperplane game, such that $p_V^S = r_S(1,\ldots,1)\,|_S$ for some $r_S \in \mathbb{R}^S_{++}$ holds true for all $S \in 2^N$, then denote by $(N,v^V) \in \Gamma^{TU}$ its according TU game, i.e.

$$v^V(S) = \frac{1}{r_S}c^S \quad \forall S \in 2^N.$$

Definition 2.5 (monotonic NTU games)

Let $(N, V) \in \Gamma^{NTU}$ be an NTU game. V is **monotonic** *if for all coalitions $S, T \in 2^N$ with $\emptyset \neq S \subset T$ and all $x \in V(S)$, there exists $y \in V(T)$ with $y_S \geq x$.*

An equivalent formulation of Definition 2.5 is to say that V is monotonic if the projection of $V(T)$ on $\mathbb{R}^S$ contains $V(S)$, which means that for every payoff vector that a coalition S is effective for it is possible to assign payoffs to the players in $T \setminus S$ such that the coalition T is effective for the resulting payoff vector. Note that there are no restrictions on the payoffs to players in $T \setminus S$. By introducing such restrictions we get another concept of monotonicity, which is also called "individual superadditivity".

Definition 2.6 (individual superadditive NTU games)

Let $(N, V) \in \Gamma^{NTU}$ be an NTU game. V is **individual superadditive** *if for every player $i \in N$ and every coalition $\emptyset \neq S \subseteq N \setminus \{i\}$ the following holds:*

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}). \tag{1}$$

Since $V(\{i\}) = (-\infty, 0]$ and $V(S)$ is comprehensive for every $S \in 2^N$, equation (1) is equivalent to

$$V(S) \times \{0\} \subset V(S \cup \{i\}). \tag{2}$$

Individual superadditivity requires that feasible outcomes for coalitions must remain feasible in supercoalitions when the "new" players' payoffs are zero. Thus individual superadditivity is a stronger property then monotonicity.

Definition 2.7

A **solution concept** *on a class $\Gamma \subseteq \Gamma^{NTU}$ is a mapping*

$$\sigma : \Gamma \to \bigcup_{(N,V) \in \Gamma} 2^{V(N)}$$

$$\sigma(N, V) \subseteq V(N),$$

that assigns to each game $(N, V) \in \Gamma$ a subset $\sigma(N, V)$ of the set of outcomes $V(N)$ for which the grand coalition is effective.

The main solution concept in this paper is the (pre-) nucleolus. Therefore we will now introduce the concept of the general nucleolus. Every nucleolus considered in this paper, e.g. the (TU) prenucleolus, the Kalai (NTU) prenucleoli or the (NTU) β-prenucleoli, are special cases of this general concept. Theorems about existence and uniqueness of these solution concepts are more or less simple corollaries of theorems that are valid for the general nucleolus.

Definition 2.8 (General Nucleolus)

Let X be a (finite or infinite) set, let D be a finite set and let $H := \{h_i\}_{i \in D}$, $|D| =: d < \infty, h_i : X \to \mathbb{R} \ \ \forall i \in D$, be a finite family of real-valued functions on X.
Let $\Theta : X \to \mathbb{R}^d$ be defined by

$$\Theta_i(x) := \max \{\min \{h_j(x) \,|j \in S\} \,|S \subseteq D, |S| = i\}, i \in D, x \in X.$$

Thus Θ arranges the components of $(h_i(x))_{i \in D}$ non-increasingly.
The set

$$\mathcal{N}(H, X) := \{x \in X \,|\Theta(x) \leq_{lex} \Theta(y) \quad \forall y \in X\}$$

is the **general nucleolus** *of X w.r.t. H.*

Here $\leq_{\text{lex}}$ denotes the lexicographical ordering of $\mathbb{R}^d$. That means that $x \leq_{\text{lex}} y$ if $x = y$ or there exists a number $k \in D$ with $x_i = y_i$ for all $1 \leq i \leq k - 1$ and $x_k < y_k$.

Theorem 2.9

- *If X is non-empty and compact and h_i is continuous for every $i \in D$, then $\mathcal{N}(H, X) \neq \emptyset$.*
- *If X is convex and h_i is convex and continuous for every $i \in D$, then*
  1. *$\mathcal{N}(H, X)$ is convex and*
  2. *$h_i(x) = h_i(y) \quad \forall x, y \in \mathcal{N}(H, X), i \in D$.*

Proof:

See [Pel88], Theorem 5.1.3. and Theorem 5.1.5. ■

Definition 2.10 (Prenucleolus of TU games)

Let

$$\begin{aligned} \mathcal{PN}: \Gamma^{TU} &\to \bigcup_{(N,v)\in\Gamma^{TU}} 2^{I^*(N,v)} \\ \mathcal{PN}(N,v) &\subseteq I^*(N,v) \quad \forall (N,v) \in \Gamma^{TU} \end{aligned}$$

be defined by

$$\mathcal{PN}(N,v) := \mathcal{N}\left(I^*(N,v), \left\{v(S) - \bullet(S) \,\middle|\, S \in 2^N\right\}\right), (N,v) \in \Gamma^{TU}.$$

Then $\mathcal{PN}$ is called **prenucleolus for TU games**. *Let $e(S,x,v) := v(S) - x(S) \quad \forall (N,v) \in \Gamma^{TU}, x \in \mathbb{R}^N, S \in 2^N$, denote the* **excess** *of* **coalition S at x**.

Theorem 2.11

$|\mathcal{PN}(N,v)| = 1 \quad \forall (N,v) \in \Gamma^{TU}$.

Proof:

Let $(N,v) \in \Gamma^{TU}$ be a game. It is easily checked that $I^*(N,v)$ is non-empty and convex and that the excess function $e(S,x,v) = v(S) - x(S), x \in I^*(N,v), S \in 2^N$, is continuous and convex (even affine linear).

Of course, $I^*(N,v)$ is not compact, thus the first part of Theorem 2.9 does not apply directly to show non-emptiness of $\mathcal{PN}(N,v)$. But let $x \in I^*(N,v)$ and define $t := \max\left\{e(S,x,v) \,\middle|\, S \in 2^N\right\}$. Let $X := \left\{y \in I^*(N,v) \,\middle|\, e(S,x,v) \le t \quad \forall S \in 2^N\right\}$, then X is non-empty ($x \in X$), convex and compact and $\mathcal{PN}(N,v) = \mathcal{N}\left(X, \left\{e(S,\bullet,v) \,\middle|\, S \in 2^N\right\}\right) \neq \emptyset$.

The second part of Theorem 2.9 ensures $e(S,x,v) = e(S,y,v)$ for all $S \in 2^N$ and all $x,y \in \mathcal{PN}(N,v)$. From this $x = y$ follows, thus $|\mathcal{PN}(N,v)| = 1$. ∎

Let $(N,v) \in \Gamma$ be a TU game. Then

$$\mathcal{D}(\alpha,x,v) := \left\{S \in 2^N \,\middle|\, e(S,x,v) \ge \alpha\right\} \quad \forall \alpha \in \mathbb{R}, x \in \mathbb{R}^N,$$

denotes the set of all coalitions with excess greater than α.

A collection of coalitions $\mathcal{S} \subseteq 2^N$ is said to be **balanced**, if there exist balancing coefficients $(\delta_S)_{S \in \mathcal{S}}, \delta_S \in \mathbb{R} \quad \forall S \in \mathcal{S}$, such that

$$\sum_{S \in \mathcal{S}} 1_S \delta_S = 1_N.$$

The well-known Kohlberg characterization of the prenucleolus ([Koh71]) can now be stated as follows. Actually Kohlberg proved a version of this theorem for thè nucleolus.

Theorem 2.12

Let $(N, v) \in \Gamma$ be a TU game and let $x \in I^*(N, v)$ be a preimputation. Then $x = \mathcal{PN}(N, v) \Leftrightarrow \mathcal{D}(\alpha, x, v)$ is balanced for all $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\alpha, x, v) \neq \emptyset$.

Proof:

See [Pel88] (Definition 5.2.5., Theorem 5.2.6. and Theorem 6.1.1.). ■

For shortage of notation we will say that Theorem 2.12 means that for $\mathcal{PN}$ the Kohlberg criterion holds.
For any two players $i, j \in N, i \neq j$, denote by

$$s_{ij}(x) := \max_{S \in 2^N, i \in S, j \notin S} e(S, x, v)$$

the **maximal surplus** of player i against player j at x. The set

$$\mathcal{PK}(N, v) := \left\{ x \in \mathbb{R}^N \mid s_{ij}(x) = s_{ji}(x) \quad \forall i, j \in N, i \neq j \right\}$$

is called the **prekernel** of the game $(N, v) \in \Gamma^{TU}$.

Remark 2.13

A useful property of the prekernel is that it always contains the prenucleolus. Thus the prekernel is always non-empty.

An important concept for solution concepts in cooperative game theory is consistency. Suppose a solution concept Φ on a class Γ of (TU or NTU) games is agreed upon by all players. Then in a game $(N, V) \in \Gamma$ a coalition $S \in 2^N$ might want to analyse "its own game" (S, V^*), called the *reduced game*, where V^* is that coalitional function that reflects in some sense the possible gains of cooperation, when the "outside players" $N \setminus S$ are payed

according to Φ. Whenever the outcomes according to Φ for players in S differ from game V to game V^* some players have an incentive to prefer building coalition S (and "play" the game (S, V^*)) rather then joining in the grand coalition N.
The solution Φ is immune against such sort of instability, if $\Phi(S, V^*) = \Phi(N, V)|_S$, i.e. the payoffs to the players of the "split off" coalition S do not change. Thus there are actually no incentives to depart from the grand coalition. This informally described property of solution concepts is either called "reduced game property" or "consistency". Of course, specifying the coalitional function V^* of the reduced game is crucial to this concept, but by no means canonical.
We present here the definition of a reduced (TU) game that was introduced by Davis and Maschler ([DM65]), because it plays an important role in the theory of the (TU) prenucleolus and we will later define an extension of this reduced game to the class of all NTU games which will be useful in the analysis of the (yet to be defined) NTU prenucleolus.

Definition 2.14

Let $(N, v) \in \Gamma^{TU}$ be a TU game, let $x \in \mathbb{R}^N$ be an arbitrary vector and let $S \in 2^N \setminus \{\emptyset, N\}$ be a coalition. The (TU) **reduced game** (S, v_x^S) *of S w.r.t. x is defined by*

$$\begin{aligned} v_x^S(T) &:= \max_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\}, T \in 2^S \setminus \{\emptyset, S\} \\ v_x^S(S) &:= v(N) - x(N \setminus S) \\ v_x^S(\emptyset) &:= 0. \end{aligned}$$

Definition 2.15

Let Φ be a solution concept on a class $\Gamma \subseteq \Gamma^{TU}$. Φ is **consistent** *(or satisfies the* **reduced game property, RGP***), if for every $x \in \Phi(N, v)$ and every $S \in 2^N \setminus \{\emptyset, N\}$ the following is true: $(S, v_x^S) \in \Gamma$ and $x_S \in \Phi(S, v_x^S)$.*

Lemma 2.16

The prenucleolus $\mathcal{PN}$ and the prekernel $\mathcal{PK}$ are consistent on Γ^{TU}.

For the proof again [Pel88] is referred to[1].

[1]Lemma 5.2.1 (and Corollary 5.2.2) and Theorem 5.2.7 (and Corollary 5.2.8)

Definition 2.17 (TU and NTU Core)

1. *Let* $(N, v) \in \Gamma^{TU}$ *be a TU game. The* **(TU) Core** *of* (N, v) *is defined as*

$$\mathbf{Core}\,(N,v) := \left\{x \in I^*\,(N,v)\,\middle|\,x(S) \geq v(S) \quad \forall S \in 2^N\right\}.$$

2. *Let* $(N, V) \in \Gamma^{NTU}$ *be an NTU game. The* **(NTU) Core** *of* (N, V) *is defined as*

$$\mathbf{Core}\,(N,V) := \left\{x \in \mathbb{R}^N\,\middle|\,x_S \in \mathbb{R}^S \setminus intV(S) \quad \forall S \in 2^N\right\}.$$

3 The NTU nucleoli of Kalai

We now introduce the approach of E. Kalai ([Kal75]) towards an extension of the prenucleolus to the class of NTU games.
Let $\Gamma \subseteq \Gamma^{NTU}$ be the subclass of NTU games that satisfy

$$\forall S \in 2^N : \quad \exists a^S \in \mathbb{R}^S \text{ such that } a^S \geq x \quad \forall x \in V(S).$$

Definition 3.1 (K(alai)-excess function)
Let $(N, V) \in \Gamma$ be a game. The function

$$l^{(N,V)} : 2^N \times \mathbb{R}^N \to \mathbb{R}$$

is called K-excess function, if the following conditions hold for all coalitions $S \in 2^N$:

1. *Independence of other coalitions*
   $[x, y \in \mathbb{R}^N, x_S = y_S] \Rightarrow l^{(N,V)}(S, x) = l^{(N,V)}(S, y).$
2. *Monotonicity*
   $[x, y \in \mathbb{R}^N, x_S < y_S] \Rightarrow l^{(N,V)}(S, x) > l^{(N,V)}(S, y).$
3. *Normalization*
   $[x \in \mathbb{R}^N, x \in \partial V(S)] \Rightarrow l^{(N,V)}(S, x) = 0.$
4. *Continuity in both arguments.*

As we already mentioned earlier, one can easily see that the conditions of Definition 3.1 are satisfied by the (TU) excess function $e(S, x, v) = v(S) - x(S)$ (see Definition 2.10), when they are properly reformulated within the TU environment. But notice that TU games as members of Γ^{NTU} do not belong to the class Γ considered by Kalai.

Definition 3.2
Let $(N, V) \in \Gamma$ be a game and let $l^{(N,V)}$ be a K-excess function. Define the **K(alai)-nucleolus** *of (N, V) w.r.t. $l^{(N,V)}$ as*

$$K - \mathcal{P}N^l(N, V) := Nu\left(l^{(N,V)}, V(N)\right).$$

Before we proceed, we will give some examples for K-excess functions. These examples are visualized in Figure 1.

Example 3.3

1. *Let $\mu \in \mathbb{R}^N_{++}$ be a vector and $(N, V) \in \Gamma$ be a game. Define*

$$\begin{aligned} f_\mu(S,x) &:= f_\mu^{(N,V)}(S,x) \\ &:= \sup\{t \in \mathbb{R} \,|\, x_S + t\mu_S \in V(S)\} \\ &\forall \quad S \in 2^N, \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

   When the vector μ is thought of as a direction in which coalitions are able to "move" from a starting point $x \in \mathbb{R}^N$ then $f_\mu(S,x)$ is the maximal distance the coalition S can "move in direction μ" without leaving $V(S)$ or the minimal distance S has to move when x is not a member of $V(S)$ and S is "forced back" to $V(S)$.

   It is easily checked that f_μ is indeed a K-excess function, i.e. meets the conditions 1 to 4 of definition 3.1.

2. *A special case of $f^{(N,V)}$ is given by*

$$g^{(N,V)}(S,x) := f_{\frac{\bar{\mu}}{|S|}}^{(N,V)}(S,x),$$

   where $\bar{\mu} := (1, \ldots, 1) \in \mathbb{R}^N$.

   Here the direction in which to move is the "egalitarian" one.

3. *The sum of "individual excesses" is another possibility.*

$$\begin{aligned} &h^{(N,V)} : 2^N \times \mathbb{R}^N \to \mathbb{R} \\ &h^{(N,V)}(S,x) = \sum_{i \in S} h_i^{(N,V)}(S,x), \end{aligned}$$

   where

$$h_i^{(N,V)}(S,x) := \max\{t \in \mathbb{R} \,|\, x_i + te_i \in V(S)\}.$$

Without going into further details, we briefly list some of the properties that K-nucleoli have or do not have in the next two remarks.

Remark 3.4

The following two important properties of the K-nucleoli are proven in [Kal75].

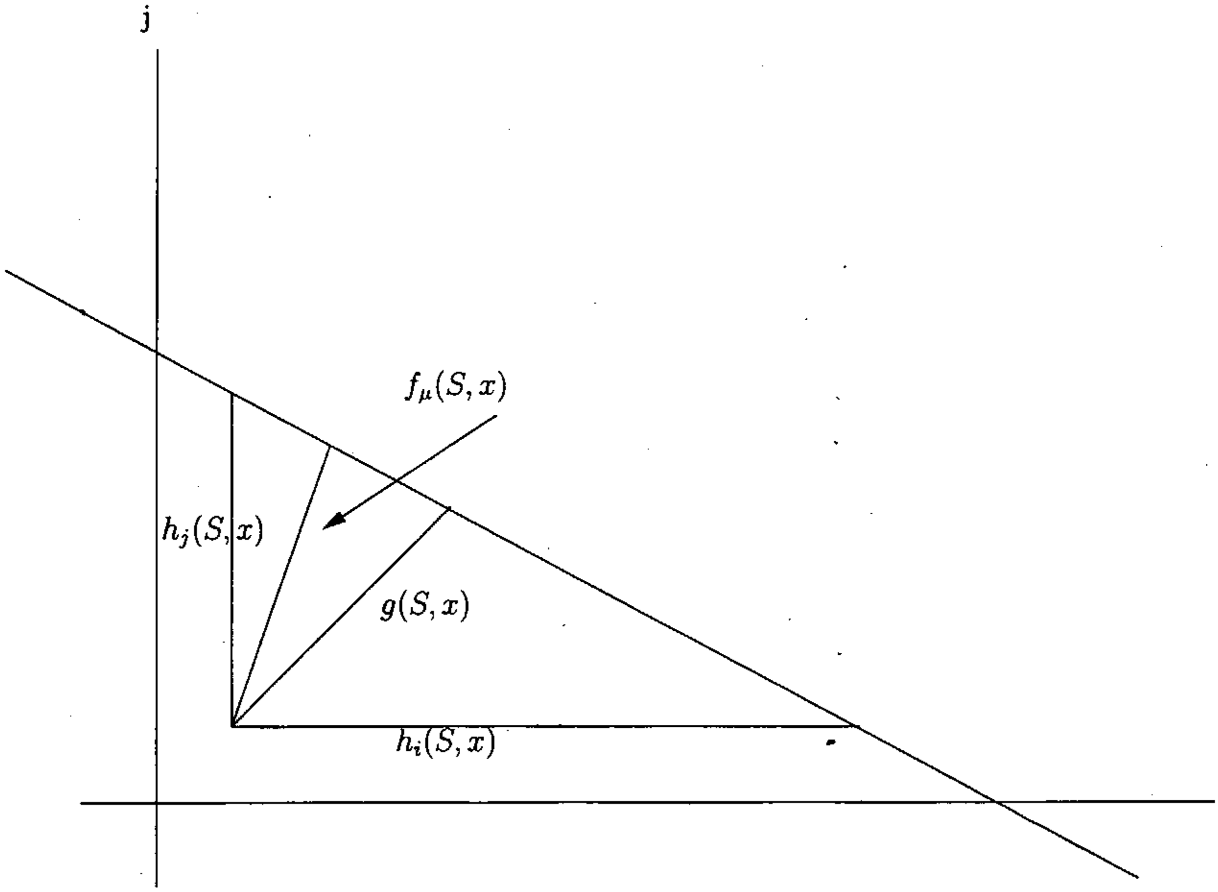

Figure 1: Three K-excess functions

- *$K\mathcal{P}N^l(N,V) \neq \emptyset$ for every game $(N,V) \in \Gamma$ and for every K-excess function l.*
- *$K\mathcal{P}N^l(N,V) \in$ **Core** (N,V) for every game $(N,V) \in \Gamma$, such that **Core** $(N,V) \neq \emptyset$, and for every K-excess function l.*

Remark 3.5

1. *The results on single-valuedness of the general nucleolus can not be applied to state single-valuedness of $K\mathcal{P}N^l$ for every choice of a K-excess function l, because in general K-excess functions are not convex. Look at $g^{(N,V)}$ of example 3.3, which might even be concave. There is, however, a "generic uniqueness" result in*

[Kal75], that holds under some additional restrictions to the K-excess functions. But also an example of a K-nucleoli that consists of three distinct points is given is that paper.

2. *The Kohlberg criterion does not hold in general. A look at the proof of Theorem 2.12 reveals that the fact that the K-excess functions are in general not affine linear might be the reason for this. Also see example 3.6 below for a counterexample.*

3. *According to a theorem in [Yan97], there is no K-excess function l such such $K\mathcal{P}N^l$ is consistent. Of course, we did not yet specify any reduced (NTU) game. We postpone this until the discussion of the consistency of the (NTU) prenucleolus in section 5. We only mention that the (TU) reduced game (Defintion 2.14) has a direct analogon for NTU games, which is used in the stated theorem.*

4. *$K\mathcal{P}N^l$ does not necessarily coincide with the prenucleolus on the class of TU games considered as a subclasss of Γ^{NTU}. As mentioned earlier, this subclass does not belong to the class Γ used by Kalai. It seems though like his results can be formulated and proven for hyperplane games such that this question of coincidence is valid.*

Example 3.6

Let $N = \{1,2,3\}$ and let V_0 the hyperplane game where $p^S = (1,1,1)|_S \quad \forall S \in 2^N$ and

$$c^S = \begin{cases} 1 & , if \quad |S| \geq 2 \\ 0 & , if \quad |S| < 2 \end{cases} \qquad \forall S \in 2^N.$$

Then $\nu_0 := \mathcal{P}N^h(N, V_0) = \frac{1}{3}(1,1,1)$ and the K-excesses are given in Table 1 (we use $h^{(N,V)}$ as K-excess function, see 3. in example 3.3). When we change the game by decreasing $p_1^{\{1,2\}}$ and $p_1^{\{1,3\}}$, see Figure 2, then, as $p^{\{12\}}$ and $p^{\{13\}}$ approach $\left(\frac{1}{2}, 1\right)$, the K-nucleolus of the so derived game V_1 approaches $x := \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$ with the K-excesses as given in Table 2.

From the view of the Kohlberg criterion this looks right, i.e. the collection of coalitions that attain maximal K-excess at x is balanced and so is every other collection attaining at least the second highest excess

S	$e^h(S, \nu_0, V_0)$
$\{1,2\}$	$\frac{2}{3}$
$\{1,3\}$	$\frac{2}{3}$
$\{2,3\}$	$\frac{2}{3}$
N	0
$\{1\}, \{2\}, \{3\}$	$-\frac{1}{3}$

Table 1: K-excesses w.r.t. the K-nucleolus of V_0

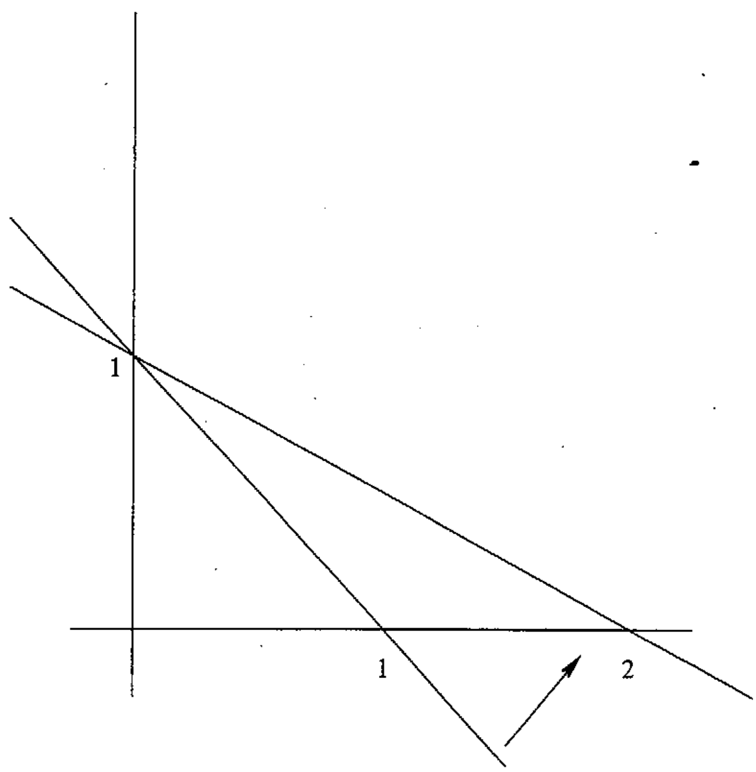

Figure 2: The game V_1

S	$e^h(S, x, V_1)$
$\{1,2\}$	$\frac{3}{2}$
$\{1,3\}$	$\frac{3}{2}$
$\{2,3\}$	$\frac{3}{2}$
N	0
$\{2\},\{3\}$	$-\frac{1}{8}$
$\{1\}$	$-\frac{3}{4}$

Table 2: K-excesses w.r.t. x in game V_1

etc. But as we already mentioned, the Kohlberg criterion does not necessarily hold for the K-nucleoli and indeed we can show that x is not the K-nucleolus of the game (N, V_1). Let therefore $\epsilon > 0$ and define x^ϵ by

$$\begin{aligned} x_1^\epsilon &:= x_1 - \epsilon \\ x_2^\epsilon &:= x_2 + \frac{\epsilon}{2} \\ x_3^\epsilon &:= x_3 + \frac{\epsilon}{2}. \end{aligned}$$

Then we have $x^\epsilon \in \partial V(N)$. Since $x^\epsilon|_{\{1,2\}}$ and $x^\epsilon|_{\{1,3\}}$ lie on a line that is parallel to $\partial V(\{1,2\})$ and $\partial V(\{1,3\})$, respectively, the K-excesses of the coalitions $\{1,2\}$ and $\{1,3\}$ are the same with respect to x and to x^ϵ. But we have

$$e^h(\{2,3\}, x^\epsilon, V_1) < e^h(\{2,3\}, x, V_1)$$

from which it follows that x is not the K-nucleolus of the game (N, V_1). Actually, it is $y := \left(0, \frac{1}{2}, \frac{1}{2}\right) = x^{\frac{3}{4}}$. The K-excesses with respect to y are given in Table 3.

This example shows the non-validness of the Kohlberg criterion and some form of discontinuity of the K-nucleolus. This unwanted behavior can also be observed for other K-excess functions.

S	$e^h(S, y, V_1)$
$\{1, 2\}$	$\frac{3}{2}$
$\{1, 3\}$	$\frac{3}{2}$
$\{2, 3\}$	0
$\{1\}, N$	0
$\{2\}, \{3\}$	$-\frac{1}{2}$

Table 3: K-excesses w.r.t. y in game V_1

4 The NTU β-Nucleoli for quasi hyperplane games

The previous section showed that Kalai's excess functions, although based on rather intuitive axioms, did not exhaustively establish a theory of nucleoli-like solution concepts for NTU games.
In this section we will develope a new class of excess functions and investigate (in Chapter 5) in detail a member of this class with yields an (NTU) prenucleolus with some nice properties.
For the remainder of this section the class of NTU games under consideration is Γ^{qH}, the class of all quasi hyperplane games, thus for every game $(N, V) \in \Gamma^{qH}$ we have $p^N \in \mathbb{R}^N$ and $c^N \in \mathbb{R}$ such that $V(N) = \{x \in \mathbb{R}^N \,|\langle p^N, x\rangle \leq c^N\}$ and every $V(S), S \in 2^N \setminus \{N\}$, merely satisfies the conditions formulated in Definition 2.3. We will develop all the necessary theory for this class of NTU games and propose an extension to general NTU games in a subsequent paper.
In the sequel we will have to deal with some form of monotonicity for hyperplane games, but we encounter a problem with the known concepts as described in section 2.

Lemma 4.1

1. *Hyperplane games are always monotonic (Definition 2.5).*

2. *If a hyperplane game $(N, V) \in \Gamma^H$ is individual superadditive (Definition 2.6), then (N, V) is covariant (under a linear transformation of utility) to a simplex game (N, V^v) belonging to some TU game (N, v).*

Proof:

1. *Since $p_V^S \in \mathbb{R}^N_{++}$ $\forall S \in 2^N$, it is immediately clear that the projection of any $V(T)$ onto a lower dimensional $\mathbb{R}^S$ $(S \subset T)$ is always the entire $\mathbb{R}^S$ itself, thus it contains $V(S)$.*

2. *If (N, V) is individually superadditive, then $p^S = p^T|_S$ must hold for all $S \subset T, S, T \in 2^N$. Thus $p^S = p^N|_S$ $\forall S \in 2^N$.* ■

We will use another concept of monotonicity, which we call *weak individual superadditivity.*

Definition 4.2

Let $(N,V) \in \Gamma^{NTU}$ be a game. (N,V) is called **weak individual superadditive,** *if for every player $i \in N$ and every coalition $\emptyset \neq S \in 2^{N\setminus\{i\}}$ the following holds:*

$$\left(V(S) \cap \mathbb{R}^S_+\right) \times V(\{i\}) \subset V(S \cup \{i\}). \tag{3}$$

Definition 4.2 requires that at least the individual rational outcomes in $V(S), S \in 2^N$, are also obtainable in supercoalitions $T \supset S, T \in 2^N$, by assigning zero payoff to players in $T - S$.

4.1 Definitions and general results

We will now introduce the key concept of the new (NTU) excess functions.

Definition 4.3

For every $(N,V) \in \Gamma^{qH}$ let $\beta^{(N,V)} : 2^N \to \mathbb{R}^N$, s.t. $\beta^{(N,V)}(S) \in \mathbb{R}^S \quad \forall S \in 2^N$, be a function that assigns to every coalition $S \in 2^N$ its **reference point** *$\beta^{(N,V)}(S) \in \mathbb{R}^S$. Call every such $\beta^{(N,V)}$ a* **reference function** *and denote by*

$$\mathcal{B} := \left\{\beta^{(N,V)} \,\middle|\, (N,V) \in \Gamma^{qH}, \beta^{(N,V)} : 2^N \to \mathbb{R}^N, \beta^{(N,V)}(S) \in \mathbb{R}^S \quad \forall S \in 2^N\right\}$$

the set of all reference functions.
Instead of $\beta^{(N,V)}(S)$ we will write $\beta^V(S)$ or $\beta(S)$ whenever there is no danger of confusion.

The purpose of introducing the concept of reference points and reference functions is to identify a "point of indifference" for every coalition, that means a point $x \in \mathbb{R}^S$ yielding an excess of zero to coalition S, i.e. a point at which the coalition is neither satisfied nor dissatisfied[2]. Although the concept of a reference function resembles somehow a solution concept itself, i.e. it could be interpreted as assigning a payoff configuration to any NTU game, it is meant as a mere auxiliary concept. Note that so far we did not impose any conditions on β such as $\beta(S) \in V(S)$ or the like.
Once a point of indifference is chosen for every coalition (we will later discuss the way this can or should be done), there are of course other points in

[2]or, in other words, is indifferent between satisfaction and dissatisfaction.

$\mathbb{R}^S$ yielding equal excess (of zero at the moment) to the coalition. Another important feature of the excess function we are about to introduce is the way those points are characterized. This characterization is based on the following considerations. Technically the domain of any excess function for coalition $S \in 2^N$ is $\mathbb{R}^S$, hence the (dis-) satisfaction of the coalition only depends on the outcome for this coalition and ignores the payments to the complementary coalition (compare Axiom 1 of Kalai in Definition 3.1). But since we are using excess functions in order to determine a solution concept for the grand coalition via minimization of dissatisfactions, this domain should be interpreted as the projection of $\partial V(N)$ onto $\mathbb{R}^S$. Note that this projection is indeed the whole $\mathbb{R}^S$ since we are dealing with quasi hyperplane games.

Now suppose there is an imputation $x \in \partial V(N)$ such that x_S is a point of indifference for the coalition $S \in 2^N$. The coalition might consider a redistribution of its share x_S according to those transfer rates that are relevant to them in the grand coalition, namely p_S^N. Since the imputation x has been made possible through cooperation of all players and coalition S might well be not effective for x_S these transfer rates are surely the only possible basis for any such redistribution. Of course, these are only virtual redistributions: The imputation $x \in \partial V(N)$ has not been allocated to the players yet. It is only a proposal which is to be checked wether or not is minimizes dissatisfaction. We are still within the process of determining the difference between the status quo x_S and what might be, i.e. we are "calculating dissatisfaction". Due to the principle of independence of other players (again compare Axiom 1 in Definition 3.1) such a redistribution should not effect the excess of any coalition outside of S. We argue that it should not change the excess of coalition S either.

Otherwise, i.e. when the coalition should be able to change its excess by redistributing x_S according to p_S^N, then the nucleolus defined by such an excess function would not really be a lexicographical minimizer of dissatisfaction/excesses because it is in this sense not well defined what the dissatisfaction of a coalition actually is. This we want to avoid. Therefore we will impose another property on the new excess functions which might be informally described as "invariance under changes according to p_S^N". Since the motivation we gave for this property of course also holds for imputations $x \in \partial V(N)$ such that x_S is not a point of indifference for S, we might also say that the excess function for S should have contour sets that are hyperplanes with a normal vector proportional to p_S^N.

In the case that a TU game $(N, v) \in \Gamma^{TU}$ is under consideration, we already know which points $x \in \mathbb{R}^S, S \in 2^N$, are candidates for being a "point of indifference" by looking at the TU excess function $e(S, x) = v(S) - x(S)$. In other words, when considering (N, v) as a member of Γ^{NTU}, i.e. as (N, V^v), then the points of indifference of S lie on the boundary of $V^v(S)$. This tells us that $\beta(S) \in \partial V^v(S)$ should be satisfied, or in other words $\sum_{i \in S} \beta_i(S) = v(S)$, if we want to make the new excess function compatible to the TU excess. This motivates the next definition.

Definition 4.4

Let $\overline{\mathcal{B}} \subset \mathcal{B}$ be the set of all reference functions β that satisfy

$$\sum_{i \in S} \beta_i^{(N,V^v)}(S) = v(S) \quad \forall S \in 2^N,$$

for every TU game $(N, v) \in \Gamma^{TU}$.

The following theorem states that if an excess function for NTU games should satisfy the two properties just discussed, i.e. vanishing on $\beta(S)$ for $\beta \in \overline{\mathcal{B}}$ and having contour sets that are hyperplanes with normal vectors proportional to p_S^N, and if it is furthermore an affine linear function that coincides with the TU excess function on Γ^{TU}, then it is uniquely defined.

Theorem 4.5

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game. Let

$$e := e^V : 2^N \times \mathbb{R}^N \times \overline{\mathcal{B}} \to \mathbb{R}$$

be a function that satisfies

1. $\forall x, y \in \mathbb{R}^N : \langle x, p_V^N|_S \rangle = \langle y, p_V^N|_S \rangle \Rightarrow e(S, x, \beta) = e(S, y, \beta) \ \forall S \in 2^N, \beta \in \overline{\mathcal{B}}$,
2. $e(S, \beta(S), \beta) = 0 \ \forall S \in 2^N, \beta \in \overline{\mathcal{B}}$,
3. $e(S, x, \beta) = \langle x, r_S \rangle + c_S, r_S \in \mathbb{R}^S, c_S \in \mathbb{R} \ \forall S \in 2^N, x \in \mathbb{R}^N, \beta \in \overline{\mathcal{B}}$, *and*
4. *If* $(N, V) \in \Gamma^{TU}$, *then* $e(S, x, \beta) = v^V(S) - x(S) \ \forall S \in 2^N, x \in \mathbb{R}^N, \beta \in \overline{\mathcal{B}}$ *(coincidence with the TU excess function on Γ^{TU}).*

Then

$$\begin{aligned} e(S,x,\beta) &= \sum_{i\in S} (\beta_i(S) - x_i)\, p^N_{V,i} \\ &= \left\langle \beta(S) - x_S, p^N_V \right\rangle . \end{aligned}$$

Proof:

Let $S \in 2^N$ be a coalition.

Claim 1

Axioms 1 and 3 imply $r_S = \gamma_{r_S} p^N_V|_S$ for some $\gamma_{r_S} \in \mathbb{R}$.

Claim 2

The value γ_{r_S} in Claim 1 is negative.

Proof of Claim 2

Let $(N,V) \in \Gamma^{TU}$ be a TU game. Then the Axioms 3 and 4 imply (together with Claim 1):

$$\gamma_{r_S} \left\langle x, p^N_V|_S \right\rangle + c_S = \langle x, -(1,\ldots,1)\rangle + v^V(S), \tag{4}$$

which at once yields $\gamma_{r_S} < 0$.

Claim 3

Claim 1 and Axiom 2 imply

$$e(S,x,\beta) = \alpha_{r_S} \left\langle \beta(S) - x, p^N_V|_S \right\rangle , \tag{5}$$

with $\alpha_{r_S} \in \mathbb{R}_{++}$.

Proof of Claim 3

By Claim 1 we have $r_S = \gamma_{r_S} p^N_V|_S$ for some $\gamma_{r_S} \in \mathbb{R}, \gamma_{r_S} < 0$. By axiom 2 we have

$$\begin{aligned} e\left(S, \beta(S), \beta\right) &= \langle \beta(S), r_S \rangle + c_S = 0 \\ \Leftrightarrow c_S &= -\gamma_{r_S} \left\langle \beta(S), p^N_V|_S \right\rangle , \end{aligned}$$

which yields

$$\begin{aligned} e(S,x,\beta) &= \langle x, r_S \rangle + c_S \\ &= \gamma_{r_S} \left\langle x, p^N_V|_S \right\rangle - \gamma_{r_S} \left\langle \beta(S), p^N_V|_S \right\rangle \\ &= \gamma_{r_S} \left\langle x - \beta(S), p^N_V|_S \right\rangle \\ &= \alpha_{r_S} \left\langle \beta(S) - x, p^N_V|_S \right\rangle \end{aligned}$$

with $\alpha_{r_S} := -\gamma_{r_S} > 0$ *and the proof of Claim 3 is complete.*
Now Claim 3 together with axiom 4 yield (for any TU game $(N,V) \in \Gamma^{TU}$*)*

$$e(S,x,\beta) = \alpha_{r_S}\left\langle \beta(S) - x, p_V^N|_S \right\rangle = v^V(S) - x(S), x \in \mathbb{R}^S,$$

which is equivalent to

$$\alpha_{r_S}\left(\sum_{i\in S}\beta_i(S) - x(S)\right) = v^V(S) - x(S), x \in \mathbb{R}^S.$$

Since $\beta \in \overline{\mathcal{B}}$*, i.e.* $\sum_{i\in S}\beta_i(S) = v^V(S)$*, this in turn yields* $\alpha_{r_S} = 1$.

■

Theorem 4.5 has anticipated the next definition, which is reformulated now for the sake of clarity.

Definition 4.6 (*β*-excess function)
Let $(N,V) \in \Gamma^{qH}$ *be a (quasi hyperplane) game. The function*

$$\begin{aligned} e^{(N,V)} &: 2^N \times \mathbb{R}^N \times \overline{\mathcal{B}} \to \mathbb{R} \\ e^{(N,V)}(S,x,\beta) &:= \sum_{i\in S}\left(\beta_i^{(N,V)}(S) - x_i\right)p_i^N \\ &= \left\langle \beta^{(N,V)}(S) - x_S, p_V^N \right\rangle \end{aligned}$$

is called ***β*-excess function**.

The previous theorem 4.5 proved that the axioms 1 – 4 uniquely determine an excess function for NTU quasi-hyperplane games. The next lemma will answer the question affirmatively if these axioms are logically indepent, i.e. no axiom is an implication of the others.

Lemma 4.7
The axioms of Theorem 4.5 that characterize the β-excess functions, are logically independent.

Proof:
Let $\beta \in \overline{\mathcal{B}}$ *be a reference function. We show the independence of the axioms by giving an example of an excess function for every axiom, respectively, that satisfies the other axioms and is different from the β-excess function.*

1. Let

$$e^1 := \langle \beta - x, (1, \ldots, 1) \rangle .$$

Then e^1 satisfies the axioms 2, 3 and 4.

2. Let

$$e^2(x) := c_V^S - \left\langle x, p_V^N \right\rangle .$$

Then e^2 satisfies the axioms 1, 3 and 4.

3. Let

$$e^3(x) := \left(\left\langle \beta - x, p_V^N \right\rangle \right)^{\frac{p_{V,1}^N}{r_N}} .$$

Then e^3 satisfies the axioms 1, 2 and 4.

4. Let

$$e^4(x) := \alpha \left\langle \beta - x, p_V^N \right\rangle$$

with $\alpha > 1$. Then e^4 satisfies the axioms 1, 2 and 3. ■

By Theorem 4.5 and Lemma 4.7 we have shown that the axioms (1) - (4) constitute an axiomatization of the β-excess function.

Remark 4.8

A look at the axioms 2 and 4 of Theorem 4.5 reveals that we can not relax the condition $\beta \in \overline{\mathcal{B}}$ to $\beta \in \mathcal{B}$ because these axioms would then be incompatible.

Apart from the axioms that axiomatize e^β, the β-excess functions also satisfy those properties as stated by the next lemma. These properties are in fact simple corollaries of Definition 4.6, thus the proofs are omitted.

Lemma 4.9

Let $(N, V) \in \Gamma^{qH}$ be a game. For every reference function $\beta := \beta^{(N,V)} \in \mathcal{B}$, the β-excess function e^β has the following properties (compare Definition 3.1):

1. *Independence of other players*
   $\left[x, y \in \mathbb{R}^N, x_S = y_S\right] \Rightarrow e^\beta(S, x) = e^\beta(S, y)$.
2. *Monotonicity*
   $\left[x, y \in \mathbb{R}^N, x_S < y_S\right] \Rightarrow e^\beta(S, x) > e^\beta(S, y)$.
3. *Normalization*
   $x \in \mathbb{R}^N, \left\langle x_S, p^N_{V,S}\right\rangle = \left\langle \beta(S), p^N_{V,S}\right\rangle \Rightarrow e^\beta(S, x) = 0$.
4. *Continuity in x.*

Note that the β-excess functions meet three of the four properties that define the Kalai-excess functions, i.e. independence of other players, monotonicity and continuity.

Definition 4.10 (β-prenucleolus for quasi hyperplane games)
Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game. The set

$$\mathcal{P}N^\beta(N, V) := Nu\left(e^{\beta,(N,V)}(S, \bullet), V(N)\right).$$

is called **β-prenucleolus for quasi hyperplane games** *or* **(NTU) β-prenucleolus** *for short.*

Example 4.11
Let us look at an example of a reference function and the resulting (NTU) β-prenucleolus.
Let $(N, V) \in \Gamma^H$ be a hyperplane game and define $\beta^{(N,V)}(S) \in \mathbb{R}^S$ by

$$\beta_i^{(N,V)}(S) := \begin{cases} \frac{c^S}{|S| p_i^S} & , \; i \in S \\ 0 & , \; i \notin S \end{cases} \quad \left(S \in 2^N\right). \tag{6}$$

The number $\frac{c^S}{p_k^S}$ $(k \in S)$ is the maximal amount that player $k \in S$ can achieve under an imputation for coalition S that is individual rational for players in S, i.e. is contained in $V(S) \cap \mathbb{R}^S_+$. $\beta^{(N,V)}(S), S \in 2^N$, is the mean value of these extreme points, see Figure 3.
Let us compute the β-prenucleolus of the game (N, V_1) of example 3.6, i.e. (N, V_1) consists of $N := \{1, 2, 3\}$ and the coalitional function V_1, which is given by Table 4.
The computation yields $\mathcal{P}N^\beta = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$. The resulting β-excesses are given in Table 5. Note that for this β-excess function and this hyperplane game the Kohlberg criterion holds.

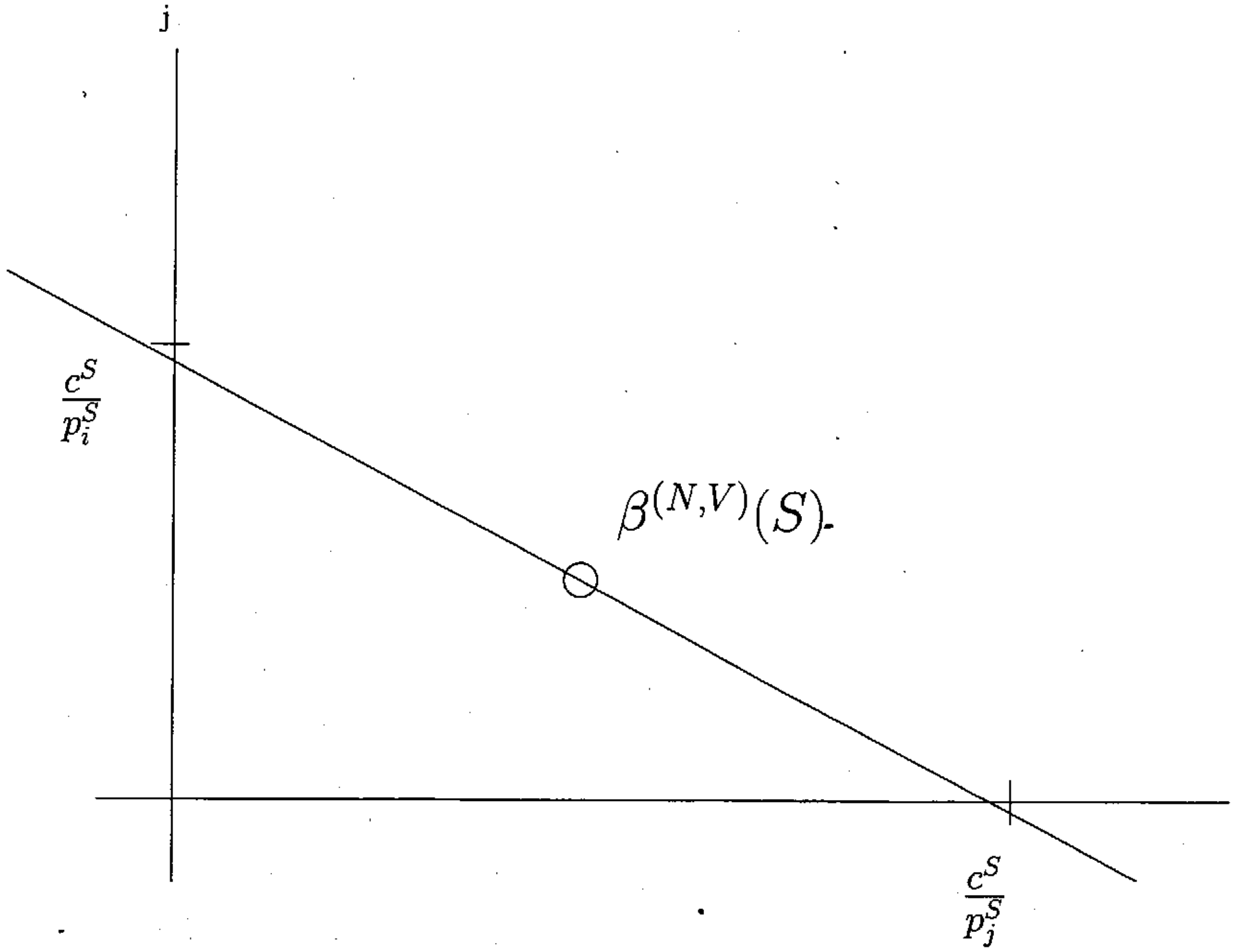

Figure 3: A reference function

S	$p^S_{V_1}$	$c^S_{V_1}$
$\{1\}$	$(1,0,0)$	0
$\{2\}$	$(0,1,0)$	0
$\{3\}$	$(0,0,1)$	0
$\{1,2\}$	$(\frac{1}{2},1,0)$	1
$\{1,3\}$	$(\frac{1}{2},0,1)$	1
$\{2,3\}$	$(0,1,1)$	1
N	$(1,1,1)$	1

Table 4: The game V_1 of example 3.6

We now turn to the analysis of some basic properties of the (NTU) β-prenucleolus. The first result on existence and uniqueness of $\mathcal{P}N^\beta$ follows as directly as those about the TU prenucleolus from Theorem 2.9 about the general nucleolus.

Theorem 4.12

$\left|\mathcal{P}N^\beta(N,V)\right| = 1 \quad \forall (N,V) \in \Gamma^{qH}, \forall \beta \in \mathcal{B}.$

Proof:

See the references to [Pel88] in the proof of Theorem 2.9. ■

A direct advantage of $\mathcal{P}N^\beta$ over $PKNu^l$ is that the both technical and interpretational meaningful Kohlberg criterion holds as the next Theorem states. Again, the proof is a more or less simple reformulation of the proof of Theorem 2.12, so it is omitted here.

Theorem 4.13

Let $(N,V) \in \Gamma^{qH}$ be a game and let $\beta^{(N,V)} \in \mathcal{B}$ be a reference function. Let $x \in V(N)$ be an imputation. Then $x = \mathcal{P}N^\beta(N,V) \Leftrightarrow \mathcal{D}(\alpha,x,V)$ is balanced for all $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\alpha,x,V) \neq \emptyset$.

Note 4.14

So far no restrictions to the choice of the reference function $\beta^{(N,V)}$ were made, i.e. Theorem 4.12 and Theorem 4.13 are true for every

S	$e^{\beta}(S,x)$
$\{1,2\}$	$\left(\frac{1}{2\cdot\frac{1}{2}}-\frac{2}{3}\right)\cdot 1+\left(\frac{1}{2\cdot 1}-\frac{1}{6}\right)\cdot 1=\frac{2}{3}$
$\{1,3\}$	$\frac{2}{3}$
$\{2,3\}$	$\left(\frac{1}{2\cdot 1}-\frac{1}{6}\right)\cdot 1+\left(\frac{1}{2\cdot 1}-\frac{1}{6}\right)\cdot 1=\frac{2}{3}$
N	0
$\{2\},\{3\}$	$-\frac{1}{6}$
$\{1\}$	$-\frac{2}{3}$

Table 5: β-excess for the β-prenucleolus of example 3.6

choice of $\beta^{(N,V)}$. This fact establishes a quite comfortable basis for the following investigations, since no matter what subset of $\mathcal{B}$ is under current consideration, the β-prenucleolus exists, is even single-valued and the validness of the Kohlberg criterion makes computations of β-prenucleoli much more easier.

However, for many of the remaining properties of solution concepts that will be investigated in connection with $\mathcal{P}N^{\beta}$, restrictions to appropriate subsets of $\mathcal{B}$ will be necessary.
A question that arises in connection with every solution concept for NTU games is its behavior on a special subclass of Γ^{NTU}, namely on the class of TU games Γ^{TU}. Often solution concepts for NTU games are extensions from Γ^{TU} to its superclass Γ^{NTU}. Those (NTU) solutions have to coincide on Γ^{TU} with the (TU) solution they stem from, otherwise they would not be an "extension". As we plan to extend the TU prenucleolus to Γ^{NTU}, we have to examine the behavior of $\mathcal{P}N^{\beta}$ on Γ^{TU}.
As we pointed out in section 3 the coincidence of $K\mathcal{P}N^{l}$ with the prenucleolus on TU games is not independent of the choice of a K-excess function. For the β-prenucleolus this coincidence is valid for every β-excess function, for which the axiomatization (Theorem 4.5) is valid, i.e. for $\beta \in \overline{\mathcal{B}}$.

Lemma 4.15

Let $(N,v) \in \Gamma^{TU}$ be a TU game and let $(N,V^{v}) \in \Gamma^{H}$ its according NTU (simplex) game. Then $\mathcal{P}N(N,v) = \mathcal{P}N^{\beta}(N,V^{v})$ holds true for every $\beta^{(N,V^{v})} \in \bar{\mathcal{B}}$.

Definition 4.16 (covariant reference functions)

Let the set of all covariant reference functions $\mathcal{B}^{cov} \subset \mathcal{B}$ be defined by $\mathcal{B}^{cov} := \left\{ \beta^{(N,V)} \,\middle|\, \beta^{(N,V)} \in \mathcal{B}, \beta^{(N,\lambda V)}(S) = \lambda \beta^{(N,V)}(S) \quad \forall S \in 2^N, \forall \lambda \in \mathbb{R}^N_{++} \right\}$.

Lemma 4.17

$\beta \in \mathcal{B}^{cov} \Rightarrow \mathcal{P}N^\beta$ *is covariant.*

Proof:

Let $(N,V) \in \Gamma^{qH}$ be a game and let $\beta \in \mathcal{B}^{cov}$ be a reference function. Then

$$\begin{aligned} e^{\beta,(N,\lambda V)}(S,\lambda x) &= \sum_{i \in S} \left(\lambda_i \beta^{(N,V)}(S)_i - \lambda_i x_i \right) \cdot \frac{p_i^N}{\lambda_i} \\ &= e^{\beta,(N,V)}(S,x) \\ \forall &\quad S \in 2^N, x \in \mathbb{R}^N. \end{aligned}$$

Hence $\mathcal{P}N^\beta(N,\lambda V) = \lambda \mathcal{P}N^\beta(N,V) \quad \forall \lambda \in \mathbb{R}^N_{++}$. ■

Remark 4.18

The reference function of example 4.11 is covariant.

Let $(N,V) \in \Gamma^{qH}$ be a simplex game and let $\beta^{(N,V)} \in \mathcal{B}$ be a reference function. The excess of a coalition $S \in 2^N$ at $x \in \mathbb{R}^N$ computes as

$$\begin{aligned} e^\beta(S,x,V) &= \sum_{i \in S} (\beta_i(S) - x_i)\, p_i^N \\ &= \sum_{i \in S} \beta_i(S) - x(S). \end{aligned}$$

Thus the reference function β induces a TU game via

$$v_V^\beta(S) := \sum_{i \in S} \beta_i(S) \quad \forall S \in 2^N \setminus \{N\} \tag{7}$$

$$v_V^\beta(N) := c_V^N \tag{8}$$

with the property that for every coalition $S \in 2^N$ and every imputation $x \in V(N)$ the (TU) and (NTU) excesses are equal:

$$e^\beta(S,x,V) = e\left(S,x,v_V^\beta\right)$$

and thus the β-prenucleolus and the (TU) prenucleolus coincide:

$$\mathcal{P}N^{\beta}(N,V)=\mathcal{P}N\left(N,v_V^{\beta}\right).$$

This fact will not only help in computing PNU^{β} (see below) but also yields conclusions about $\mathcal{P}N^{\beta}$.

Definition 4.19

Let $(N,V)\in\Gamma^{qH}$ be a simplex game and let $\beta^{(N,V)}\in\mathcal{B}$ be a reference function. The TU game $\left(N,v_V^{\beta}\right)$ as defined by (7) and (8) is called the (TU) β-game *of the NTU game (N,V).*

Lemma 4.20

Let $(N,V)\in\Gamma^{qH}$ be a game and let $\beta^{(N,V)}\in\mathcal{B}$ be a reference function. Let $(N,W)\in\Gamma^{qH}$ be a simplex game that is derived from (N,V) by a linear transformation. If v_W^{β} is individual superadditive, i.e. $v_W^{\beta}(S)+v_W^{\beta}(\{i\})\leq v_W^{\beta}(S\cup i)\quad\forall S\in 2^{N\setminus\{i\}},i\in N$, then $\mathcal{P}N^{\beta}(N,V)$ is individual rational.

Proof:

In view of the comments about the (TU) β-game of an NTU game above, we show that $\mathcal{P}N\left(v_W^{\beta}\right)$ is individual rational. To this end, let $x:=\mathcal{P}N\left(v_W^{\beta}\right)$ and suppose there exists a player $i\in N$ such that $x_i<v_W^{\beta}(\{i\})$. For every coalition $S\in 2^{N\setminus\{i\}}$ that does not contain i the following is true:

$$\begin{aligned}
e\left(S,x,v_W^{\beta}\right) &= v_W^{\beta}(S)-x_S\\
&< v_W^{\beta}(S)+v_W^{\beta}(\{i\})-x(S)-x_i\\
&\leq v_W^{\beta}(S\cup\{i\})-x(S\cup\{i\})\\
&= e\left(S\cup\{i\},x,v_W^{\beta}\right).
\end{aligned}$$

Thus coalitions that attain maximal excess under the imputation x must all contain player i. Since $e\left(N,x,v_W^{\beta}\right)=0$ and $e\left(\{i\},x,v_W^{\beta}\right)>0$, N is not a coalition with maximal excess. It follows that the collection of coalitions with maximal excess is not balanced, contrary to our assumption that x is the prenucleolus of v_W^{β}. ■

Corollary 4.21

If a TU game (N, v) is individual superadditive, then $\mathcal{P}N(N, v)$ is individual rational.

If $(N, V) \in \Gamma^{qH}$ is not a simplex game but $\beta \in \mathcal{B}$ is covariant, then we can also describe the (NTU) β-prenucleolus of (N, V) by the (TU) prenucleolus of a suitably chosen TU game. Since the game $\left(N, p_V^N V\right)$ is a simplex game, we have

$$\mathcal{P}N^{\beta}\left(N, p_V^N V\right) = \mathcal{P}N\left(N, v^{\beta}_{p_V^N V}\right).$$

Now the covariance of β and Lemma 4.17 yield

$$\mathcal{P}N^{\beta}(N, V) = \frac{1}{p_V^N}\mathcal{P}N\left(N, v^{\beta}_{p_V^N}\right).$$

The Core is a very well established solution concept both for TU and for NTU games. Therefore it is considered a major advantage of the (TU) prenucleolus that it is always a member of the core whenever the latter is non-empty. This is important in situations where there is demand for (in the core-sense) stable but *single-valued* solutions for TU games. Sometimes the (TU) prenucleolus is therefore said to be a core-selector.

As noticed in section 3, the Kalai prenucleoli for NTU games are also contained in the core, when the core exists. We will now investigate this property in connection with the class of β-prenucleoli.

The simplicity of the proofs that both the (TU) prenucleolus and the Kalai (NTU) prenucleoli are core-selectors is due to the fact that the respective cores can be defined as those imputations yielding non-positive excesses. This is not true for the β-prenucleoli and, moreover, core-inclusion will turn out to be a property of some reference functions on some subclass of games.

Example 4.22

Let $N = \{1, 2, 3\}$ and consider the hyperplane game as given by Table 6:

The core of this game is a singleton:

$$\textbf{Core}(N, V) = \{(0; 0.7; 1.7)\}.$$

Take, for example, the reference function β as defined in example 4.11, then

$$\mathcal{P}N^{\beta}(N, V) = (0.8; -0.61; 1.05),$$

S	p_V^S	c_V^S
1	1 0 0	0
2	0 1 0	0
3	0 0 1	0
12	5 10 0	7
13	8 0 10	17
23	0 3 7	11
123	9 1 9	16

Table 6: The game of example 4.22

thus $\mathcal{PN}^\beta(N,V) \notin$ **Core** (N,V). *Furthermore, with Lemma 4.20 in mind we can imply that the β-game of* (N,V) *is not individual super-additive.*

Example 4.22 also serves to proof the next lemma, which states an impossibility.

Lemma 4.23

There exists a game $(N,V) \in \Gamma^H$ *with* $|N| = 3$, *such that* **Core** $(N,V) \neq \emptyset$ *and* $\mathcal{PN}^\beta(N,V) \notin$ **Core** (N,V) *for every* $\beta^{(N,V)} \in \mathcal{B}$ *with* $\beta^{(N,V)}(S) \in \partial V(S) \cap \mathbb{R}^S_{++} \quad \forall S \in 2^N$.

Proof:

Consider the game of example 4.22 and denote by $x := (0; 0.7; 1.7)$ *its unique core-element. Let* $\beta^{(N,V)} \in \mathcal{B}$ *be a reference function, such that* $\beta^{(N,V)}(S) \in \partial V(S) \cap \mathbb{R}^S_{++}$. *Then* $e^\beta(\{12\}, x, V) > 0$ *and* $e^\beta(\{13\}, x, V) > 0$. *Since the β-excess for the coalitions* $\{1\}$, N *and* $\emptyset$ *are zero and the β-excess for the coalitions* $\{2\}$ *and* $\{3\}$ *are negative, the β-excess for the coalition* $\{23\}$ *must be positive - and furthermore equal to* $e^\beta(\{12\}, x, V)$ *and* $e^\beta(\{13\}, x, V)$ *- for the collection of coalitions with highest excesses to be balanced.*

For $0 < \lambda < 1$ *let* $\beta^\lambda := \lambda\left(0, \frac{11}{7}\right) + (1-\lambda)\left(\frac{11}{3}, 0\right)$ *be any point on the* *open line segment* $\partial V(\{23\}) \cap \mathbb{R}^{\{23\}}_{++}$. *Then*

$$e^{\beta}\left(\{23\}, \beta^{\lambda}, V\right) = \left((1-\lambda)\frac{11}{3} - \frac{7}{10}\right) + \left(\lambda\frac{11}{7} - \frac{17}{10}\right)\cdot 9$$
$$= \lambda\frac{220}{21} - \frac{37}{3}$$

Thus $e^{\beta}\left(\{23\}, \beta^{\lambda}, V\right) > 0$ *if and only if* $\lambda > \frac{259}{220} > 1$. ∎

5 The NTU prenucleolus

In chapter 4 we have introduced the new class of β-excess functions for NTU games and the according class of NTU βprenucleoli. Of course among the members of this class there exist prenucleoli which fail to satisfy some essential conditions such as covariance, coincidence with the TU prenucleolus on the class of all TU games and the like. Thus not much more than the definition of these solution concepts seem to justify the name prenucleolus.
In this chapter we will examine more detailed a subclass of some NTU β-excess functions which all yield the same NTU β-prenucleolus. This NTU β-prenucleolus is covariant, symmetric, single-valued and consistent with respect to a new reduced game, thus satisfies all the axioms that in the TU case characterize the TU prenucleolus uniquely. Although so far we do not know whether or not these four axioms consitute an axiomatization in the NTU case, we feel that they are certainly enough reason to justify the name 'NTU prenucleolus'.

Definition 5.1

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game. For every coalition $S \in 2^N$ define

$$m_V^S := \max_{x \in p_V^N V(S)+} \langle x, p_V^N \rangle$$

and denote the maximizers by M_V^S, i.e.

$$M_V^S := arg \max_{x \in p_V^N V(S)+} \langle x, p_V^N \rangle .$$

Remark 5.2

1. *If $(N, V) \in \Gamma^H$, i.e. (N, V) is a hyperplane game, then an alternative formulation of Definition 5.1 is*

$$m_V^S := \frac{c_V^S}{\min_{i \in S} \frac{p_{V,i}^S}{p_{V,i}^N}} \quad (S \in 2^N).$$

2. *The value m_V^S is invariant under positive linear transformations, i.e. for every $\lambda \in \mathbb{R}_{++}^N$:*

$$m_{\lambda V}^S = m_V^S \quad \forall (N, V) \in \Gamma^{qH}, \forall S \in 2^N.$$

Definition 5.3

Let $(N,V) \in \Gamma^{qh}$ be a quasi hyperplane game. Let $\beta^{(N,V)} \in \mathcal{B}$ be a reference function. $\beta^{(N,V)}$ is called **maximal feasible**, *if*

$$\sum_{i \in S} \beta_i^{(N,V)}(S) \cdot p_i^N = m_V^S$$

holds true for every coalition $S \in 2^N$.

Lemma 5.4

If $\beta \in \mathcal{B}$ is maximal feasible, then $\beta \in \overline{\mathcal{B}}$.

Proof:

Let $(N,v) \in \Gamma^{TU}$ be a TU game and let $(N,V^v) \in \Gamma^H$ be its according simplex game. Then

$$\begin{aligned} \sum_{i \in S} \beta_i^{(N,V^v)}(S) &= m_{V^v}^S \\ &= c_{V^v}^S \\ &= v(S) \end{aligned}$$

holds true, hence $\beta \in \overline{\mathcal{B}}$. ■

Lemma 5.4 tells us that if we consider maximal feasible reference functions, we are within the framework of Theorem 4.5.

Assume that $(N,V) \in \Gamma^H$ and $p^N = (1,\ldots,1)$, then Figure 4 illustrates definitions 5.1 and 5.3.

The number m_V^S is the maximal amount a member of coalition S can get under an individual rational imputation. Every maximal feasible reference function assigns to a coalition S a redistribution of m_V^S according to the transfer rates p_S^N that are relevant to the grand coalition.

For every two maximal feasible reference functions β and β' the excess functions e^β and $e^{\beta'}$ coincide:

$$\begin{aligned} e^\beta(S,x,V) &= \sum_{i \in S} \left(\beta_i^{(N,V)}(S) - x_i\right) p_{V,i}^N \\ &= m_V^S - \sum_{i \in S} x_i p_{V,i}^N \\ &= \sum_{i \in S} \left(\beta_i'^{(N,V)}(S) - x_i\right) p_{V,i}^N \\ &= e^{\beta'}(S,x,V), \end{aligned}$$

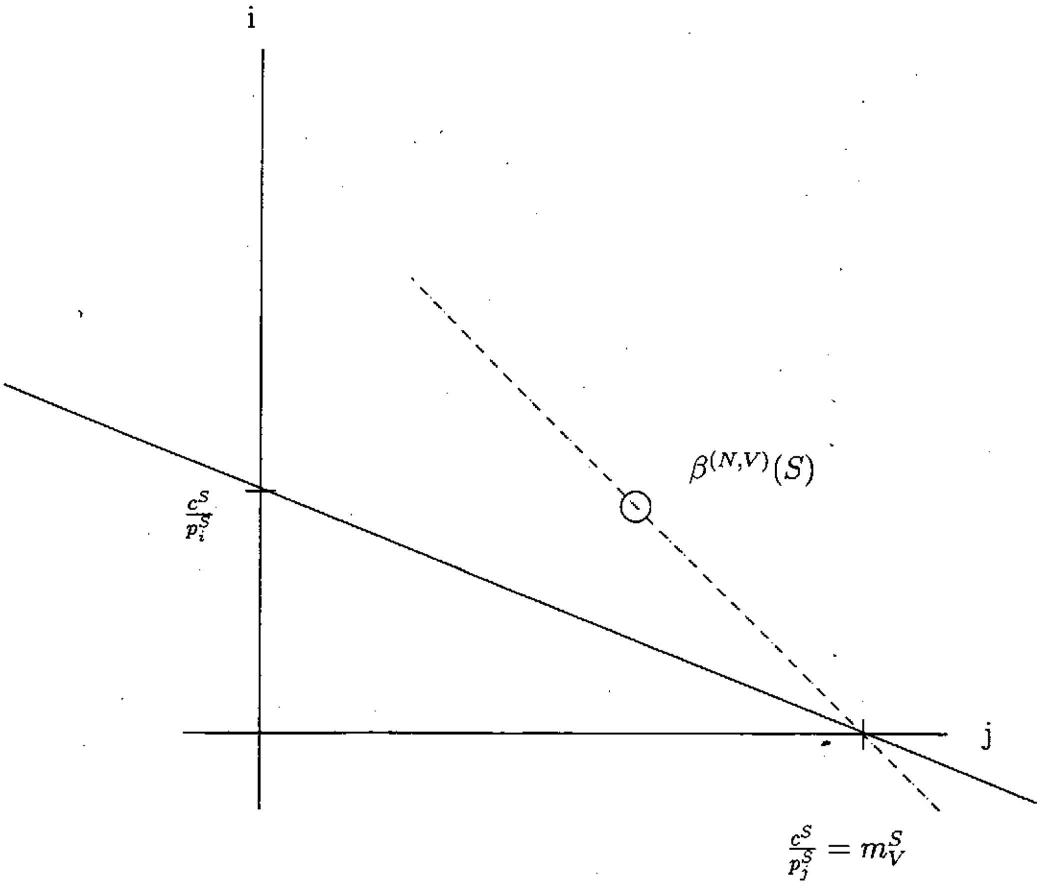

Figure 4: The value m_V^S in a special case

and hence also $\mathcal{P}N^{\beta}$ and $\mathcal{P}N^{\beta'}$ coincide. We will therefore shorten the notation by defining $\mathcal{P}N(N,V) := \mathcal{P}N^{\beta}(N,V)$ whenever β is maximal feasible and call $\mathcal{P}N$ **the (NTU) prenucleolus for quasi hyperplane games.**

Lemma 5.5

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. If $\beta^V \in \mathcal{B}$ is maximal feasible ("for V"), then $\lambda\beta^V$ is maximal feasible ("for λV") for every $\lambda \in \mathbb{R}^N_{++}$.

Proof:

Let $S \in 2^N$ be a coalition and let $\lambda \in \mathbb{R}^N_{++}$. From

$$\begin{aligned}\sum_{i\in S} \left(\lambda\beta^V\right)_i (S) p^N_{\lambda V}, i &= \sum_{i\in S} \lambda_i \beta^V_i (S) \frac{1}{\lambda_i} p^N_{V,i} \\ &= \sum_{i\in S} \beta^V_i (S) p^N_{V,i} \\ &= m^S_V \\ &= m^S_{\lambda V}\end{aligned}$$

we see at once that $\lambda\beta^V$ is maximal feasible. ■

Thus the property of maximal feasibility of a reference function is invariant under a positive linear transformation of utility. This directly yields the covariance of the (NTU) prenucleolus.

Corollary 5.6

The NTU prenucleolus $\mathcal{PN}$ is covariant.

Proof:

Let $(N, V) \in \Gamma^{qH}$ be a game und let $\lambda \in \mathbb{R}^N_{++}$. Let $\beta^{(N,V)} \in \mathcal{B}$ be maximal feasible. Lemma 5.5 yields that $\beta^{(N,\lambda V)} := \lambda\beta^{(N,V)}$ is maximal feasible for $(N, \lambda V)$. Then

$$\begin{aligned}e^{\beta(N,\lambda V)} &= \sum_{i\in S} \left(\beta^{(N,\lambda V)}_i (S) - \lambda_i x_i\right) p^{\lambda V}_{V,i} \\ &= \sum_{i\in S} \left(\lambda_i \beta^{(N,V)}_i (S) - \lambda_i x_i\right) \frac{1}{\lambda_i} p^N_{V,i} \\ &= e^{\beta^{(N,V)}} (S, x, V),\end{aligned}$$

which yields $\mathcal{PN}(N, \lambda V) = \lambda \mathcal{PN}(N, V)$. ■

As the next result concerning the NTU prenucleolus we will present a Theorem similar to Lemma 4.20, which states that for weak individual superadditive quasi hyperplane games the NTU prenucleolus is individual rational.

Theorem 5.7

If $(N, V) \in \Gamma^{qH}$ is weak individual superadditive (see Definition 4.2), then the NTU prenucleolus $\mathcal{PN}(N, V)$ is individual rational.

Proof:

Let $(N,V) \in \Gamma^{qh}$ be weak individual superadditive, and let $\beta \in \mathcal{B}$ be a maximal feasible reference function.

We know that $\mathcal{P}N(N,V) = \mathcal{P}N\left(N, v_V^\beta\right)$, thus all we have to show is that v_V^β is individual superadditive, i.e. v_V^β satisfies

$$v_V^\beta(S) + v_V^\beta(\{i\}) \leq v_V^\beta(S \cup \{i\}) \quad \forall S \in 2^{N \setminus \{i\}}, i \in N,$$

since for individual superadditive TU games the TU prenucleolus is individual rational (Corollary 4.21).

Now suppose that v_V^β is not individual superadditive, thus there is a player $i \in N$ and a coalition $S \in 2^{N\setminus\{i\}}$ such that

$$v_V^\beta(S) + v_V^\beta(\{i\}) > v_v^\beta(S \cup \{i\}).$$

Since $v_V^\beta(\{i\}) = 0$ and $v_V^\beta(S) = m_V^S$ we have

$$m_V^S > m_V^{S\cup\{i\}}.$$

Let $\bar{x}_V^S$ be "a maximizer for m^S, i.e.

$$\bar{x} \in M_V^S = \arg \max_{x \in V(S)^+} \left\langle x, p_V^N|_S \right\rangle.$$

Since $\bar{x}_V^S \in V(S)^+$ and the game is weak individual superadditive, $\left(\bar{x}_V^S, 0\right) \in V(S \cup \{i\})$ must hold.

But

$$\begin{aligned} \left\langle \left(\bar{x}_V^S, 0\right), p_V^N|_{S\cup\{i\}} \right\rangle &= \left\langle \bar{x}_V^S, p_V^N|_S \right\rangle \\ &= m_V^S \\ &> m_V^{S\cup\{i\}} \end{aligned}$$

holds, thus $\left(\bar{x}_V^S, 0\right) \notin V(S \cup \{i\})^+$. By $\left(\bar{x}_V^S, 0\right) \in \mathbb{R}_+^{S\cup\{i\}}$ also $\left(\bar{x}_V^S, 0\right) \notin V(S)$ holds, a contradiction to the assumption of weak individual superadditivity of (N,V). ■

5.1 Computation

In chapter 4 we showed how general β-prenucleoli can be expressed via (TU) prenucleoli of suitably chosen TU games. In the case of maximal feasible

reference functions, i.e. for the NTU prenucleolus, things are even simpler because of the invariance of the values $m_V^S, S \in 2^N$.
To be more precise, let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game, not necessarily simplex. The game $\left(N, p_V^N V\right)$ is simplex and, because of the maximal feasibility of β,

$$\begin{aligned} v^{\beta}_{p_V^N V}(S) &= \sum_{i \in S} \beta_i^{p_V^N V}(S) \\ &= \sum_{i \in S} \beta_i^{p_V^N V}(S) \cdot p^N_{p_V^N V, i} \\ &= m^S_{p_V^N V} \\ &= m_V^S \quad \forall S \in 2^N, S \neq N \end{aligned}$$

holds true.
Again by the covariance of the NTU prenucleolus we have

$$\mathcal{PN}(N, V) = \frac{1}{p_V^N} \mathcal{PN}\left(N, v^{\beta}_{p_V^N V}\right).$$

It follows that the computation of the NTU prenucleolus for a given quasi hyperplane game (N, V) consists of determining the values $m_V^S, S \in 2^N$, and computing the (TU) prenucleolus of the TU game $\left(N, v^{\beta}_{p_V^N V}\right)$.
Having this in mind, we are now able to apply the results of [Kla97] which will yield a set-valued dynamical system that converges to the (NTU) prenculeolus[3]. As it was shown in [Kla97] these results can be used to develop a computer program to compute the (NTU) prenucleolus. A detailed discussion of this will be the subject of a subsequent paper.

5.2 Inclusion in the Core

Definition 5.8

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game. If for all balanced collections $\mathcal{S} \subseteq 2^N$ with balancing coefficients $(\delta_S)_{S \in \mathcal{S}}$ we have

$$c^N \geq \sum_{S \in \mathcal{S}} \delta_S m_V^S,$$

then (N, V) is called **m-balanced**.

[3] See also Justman ([Jus77]) for a different set-valued dynamical system.

Theorem 5.9

Let $(N,V) \in \Gamma^{qH}$ be a quasi hyperplane game. If (N,V) is m-balanced, then $\mathcal{P}N\,(N,V) \in \mathbf{Core}\,(N,V)$.

Proof:

Without loss of generality, assume that (N,V) is a simplex game. Let β be a maximal feasible reference function. Let $\left(N, v_V^\beta\right)$ be the TU game derived from β and V, i.e.

$$\begin{aligned} v_V^\beta(S) &= \sum_{i \in S} \beta_i^{(N,V)}(S), \quad S \in 2^N \setminus \{N\} \\ v_V^\beta(N) &= c^N \end{aligned}$$

Since β is maximal feasible, we have

$$v_V^\beta(S) = m_V^S \quad \forall S \in 2^N.$$

Thus the m-balancedness of (N,V) is equivalent to

$$v_V^\beta(N) \geq \sum_{S \in \mathcal{S}} \delta_S v_V^\beta(S) \tag{9}$$

for all balanced collections $\mathcal{S} \subseteq 2^N$ with balancing coefficients $(\delta_S)_{S \in \mathcal{S}}$, which yields $\mathbf{Core}\left(N, v_V^\beta\right) \neq \emptyset$ by Scarf's Theorem ([Sca67], equation (9) tells that v_V^β is balanced).

Let $\nu := \mathcal{P}N\left(N, v_V^\beta\right)$ be the (TU) prenucleolus of $\left(N, v_V^\beta\right)$, then $\nu \in \mathbf{Core}\left(N, v_V^\beta\right)$, in other words:

$$\sum_{i \in S} \nu_i \geq v_V^\beta(S) \quad \forall S \in 2^N.$$

Thus ν satisfies

$$\sum_{i \in S} \nu_i \geq m_V^S \quad \forall S \in 2^N,$$

and, since $\nu \in \mathbf{Core}\left(N, v_V^\beta\right)$, ν is individually rational:

$$\nu_i \geq 0 \quad \forall i \in N.$$

Thus we have

$$p_V^S \cdot \nu \geq c_V^S \quad \forall S \in 2^N$$

$$\Rightarrow \nu \in \mathbf{Core}\,(N, V)\,.$$

The observation $\nu = \mathcal{P}N\,(N, V)$ (see the considerations on page 29) now completes the proof. ■

We immediately see that m-balancedness of a quasi hyperplane game is a sufficient condition for the core of this game to be non-empty.

Corollary 5.10
If $(N, V) \in \Gamma^{qH}$ is m-balanced, then $\mathbf{Core}\,(N, V) \neq \emptyset$.

5.3 Consistency

For the class Γ^{TU} of TU games two different versions of reduced games are used in axiomatizations of solution concepts. The reduced game defined in Chapter 2 is due to Davis and Maschler ([DM65]) and is used to axiomatize the (TU) prenucleolus and the (TU) prekernel. The other reduced game, due to Hart and Mas-Colell ([HMC89]), yields an axiomatization of the Shapley value via the same axioms used in the axiomatization of the prenucleolus just by exchanging the two reduced games in the definition of consistency.
These two (TU) reduced games can easily be generalized to the class of NTU games and the question arises wether or not the (NTU) prenucleolus is consistent w.r.t. one of these (NTU) reduced games. But, as [MO89] have shown, there does not exist a solution concept for hyperplane games that is

- efficient,
- symmetric,
- covariant and
- consistent w.r.t. to the reduced games of Davis and Maschler or Hart and Mas-Colell.

Since $\mathcal{PN}$ satisfies the first three properties, we can imply that $\mathcal{PN}$ is not consistent, although one should be aware that the covariance that [HMC89] used contains also additive transformations of utility.
To overcome this problem, two different ways are possible. The first way, as undertaken is [MO89], is to keep the definition of the reduced game and to modify the definition of consistency. By this means they axiomatized their new solution concept, called the *consistent NTU shapley value*, by efficiency, symmetry, covariance and the so-called bilateral consistency.
However, we will take the other possible way and keep the definition of consistency but use a new reduced game in order to establish consistency of $\mathcal{PN}$. This new reduced game coincides with that of Davis and Maschler on the class of all TU games. As yet we do not know if the properties of $\mathcal{PN}$ together with consistency constitute an axiomatization.

Definition 5.11

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game and let $S \in 2^N$ be a coalition. Denote by $\bar{M}_V^S \in V(S)^+$ the maximizing member of $V(S)$ in the definition of m_V^S. If there are more than one maximizer, choose one arbitrarily.
Let $C_S := \left\{ i \in S \,\middle|\, \bar{M}_{V,i}^S > 0 \right\}$ be the set of those players of S whose outcome under $\bar{M}_V^S$ is strictly positive.

Definition 5.12

Let $(N, V) \in \Gamma^{qH}$ be a quasi hyperplane game, let $x \in \mathbb{R}^N$ be an imputation and let $S \in 2^N \setminus \{\emptyset, N\}$ be a coalition. The **(NTU) reduced game** $\left(S, V_x^S\right)$ *of S w.r.t. x is defined by*

$$\begin{aligned} V_x^S(T) &:= V(T) + s_T \quad (T \subset S) \\ V_x^S(S) &:= \left\{ x_S \in \mathbb{R}^S \,\middle|\, \langle p_V^N, x_S \rangle \leq c^N - \langle p_V^N, x_{N \setminus S} \rangle \right\}. \end{aligned}$$

Here $s_T \in \mathbb{R}^T$ is defined by

$$s_i := \begin{cases} \frac{\max_{Q \subseteq N \setminus S} \left\{ m_V^{T \cup Q} - x(Q) \right\} - m_V^T}{|C_T| p_{V,i}^N} & : \; i \in C_T \\ 0 & : \; i \notin C_T \end{cases}$$

Remark 5.13

In the case when (N, V) is a hyperplane game, then Definition 5.12 can as well be formulated as

$$V_x^S(T) = \left\{ x_T \in \mathbb{R}^T \,\middle|\, \langle p_V^T, x_T \rangle \leq c_{V_x^S}^T \right\} \quad (T \subset S)$$

with

$$c^T_{V^S_x} := \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q}_V - x(Q) \right\} \min_{i \in T} \frac{p^T_{V,i}}{p^N_{V,i}}.$$

Remark 5.14

1. *The (NTU) reduced game of a simplex game is itself a simplex game.*
2. *Remark 5.13 shows that the (NTU) reduced game coincides with the (TU) reduced game of Davis and Maschler when the original game is a TU game. For this we only need to notice that in this case* $p^S_V = (1, \ldots, 1)$ *and ,* $m^S_V = v(S)$ *holds true for all* $S \in 2^N$.

Definition 5.15

Let Φ *be a solution concept on* Γ^{qH}. Φ *satisfies the* **reduced game property (RGP)** *or is called* **consistent**, *if the following is true for every game* $(N, V) \in \Gamma^{qH}$:

$$x \in \Phi(N, V) \Rightarrow x_S \in \Phi\left(S, V^S_x\right) \quad \forall S \in 2^N.$$

Theorem 5.16

The NTU prenucleolus is consistent.

Proof:

Essential for this proof is the (TU) consistency of the (TU) prenucleolus (Lemma 2.16).
Let $(N, V) \in \Gamma^{qH}$ *be a quasi hyperplane game and let* $\beta^{(N,V)} \in \mathcal{B}$ *be a maximal feasible reference function. Assume w.l.o.g. that* (N, V) *is a simplex game, hence* $p^N_V = (1, \ldots, 1)$.
Denote by $\nu := \mathcal{P}N(N, V)$ *the (NTU) prenucleolus of* (N, V) *and by* $\left(N, v^\beta_V\right)$ *the (TU)* β*-game of* V, *i.e.* $v^\beta_V(S) = \sum_{i \in S} \beta^{(N,V)}_i(S) = m^S_V$, *since* β *is maximal feasible.*
Let $S \in 2^N, S \neq N,$ *be a coalition. We have to show that*

$$\nu_S = \mathcal{P}N\left(S, V^S_\nu\right),$$

or equivalently

$$\nu_S = \mathcal{P}N\left(S, v^\beta_{V^S_x}\right).$$

Since $\nu = \mathcal{P}N\left(N, v_V^\beta\right)$ and the (TU) prenucleolus is (TU) consistent, it follows that

$$\nu_S = \mathcal{P}N\left(S, \left(v_V^\beta\right)_\nu^S\right).$$

Thus all we have to show is

$$\left(v_V^\beta\right)_\nu^S = v_{V_\nu^S}^\beta,$$

i.e., that the (TU) reduced game of v_V^β w.r.t. S and ν is equal to the (TU) β-game of the (NTU) reduced game of V w.r.t. S and ν.
Note that the definition of s_T implies that if M_V^T is a maximizer for m_V^T, then $M_V^T + s_T$ is a maximizer for $m_{V_x^S}^T$. Thus we have for all $T \neq S$:

$$\begin{aligned} m_{V_x^S}^T &= \left\langle M_V^T + s_T, p_V^N \right\rangle \\ &= m_V^T + \left\langle s_T, p_V^N \right\rangle \\ &= m_V^T + \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q} - x(Q) \right\} - m_V^T \\ &= \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q} - x(Q) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} v_{V_\nu^S}^\beta(T) &= \sum_{i \in T} \beta_i^{(S, V_\nu^S)}(T) \\ &= m_{V_\nu^S}^T \\ &= \max_{Q \subseteq N \setminus S} \left\{ m^{T \cup Q} - \nu(Q) \right\} \\ &= \max_{Q \subseteq N \setminus S} \left\{ v_V^\beta (T \cup Q) - \nu(Q) \right\} \\ &= \left(v_V^\beta\right)_\nu^S (T). \end{aligned}$$

It remains the case $T = S$:

$$\begin{aligned}
\left(v^{\beta}_{V}\right)^{S}_{\nu}(S) &= v^{\beta}_{N}(N) - \nu\left(N \setminus S\right) \\
&= c^{N} - \nu\left(N \setminus S\right) \\
&= \nu(S) \\
v^{\beta}_{V^{S}_{\nu}}(S) &= \sum_{i \in S} \beta_i^{\left(S, V^{S}_{\nu}\right)}(S) \\
&= m^{S}_{V^{S}_{\nu}} \\
&= c^{N} - \nu\left(N \setminus S\right) \\
&= \nu(S)
\end{aligned}$$

■

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