INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 327

Antagonistic Properties and n-Person Games

by

Jean-Pierre Beaud

September 2001

University of Bielefeld

33501 Bielefeld, Germany

Antagonistic Properties and n-Person Games

Jean-Pierre Beaud[1,2]

[1] Institut für Mathematische Wirtschaftsforschung (IMW), Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany, e-mail: jbeaud@wiwi.uni-bielefeld.de

[2] Laboratoire d'Econométrie, Ecole Polytechnique, 1 rue Descartes, F-75005 Paris, France, email: beaud@poly.polytechnique.fr

September 24, 2001

Abstract In this note, we studied some classes of n-person games possessing properties of two person zero-sum games. We extend the definition of a two-person almost strictly competitive game (Aumann 1961) to the n-person case. We show that the Nash equilibria of a n-person almost strictly competitive game induce the same payoff; and we exhibit the connections between almost strictly competitive games and some classes of n-person games introduced by Kats and Thisse in 1992.

Introduction

In a two-person zero-sum game, the gain of one player is equal to the loss of his opponent. This class of games has some important features: when equilibria exist, they induce a unique payoff, the set of Nash equilibria is convex, the equilibria are interchangeable[1]...

Some classes of two-person non zero-sum games having some of these nice properties have been introduced by different authors. The definitions of these classes are based on different notions of antagonism. Indeed, zero-sum games correspond to the extreme case of competition between two players: what Player 1 wins is equal to what Player 2 loses. By weakening this notion of antagonism, we get some classes of non zero-sum games which satisfy some properties of zero-sum games.

The definitions of some of these classes are also available for games with finitely

[1] Equilibria are interchangeable if for every equilibria (s_1, s_2) and (s'_1, s'_2), (s_1, s'_2) and (s'_1, s_2) are also equilibria (Nash 1951). Note that for the mixed extension of a finite game, if the equilibria are interchangeable, then the set of Nash equilibria is convex. In fact, these two properties are equivalent for the mixed extension of every finite two-person game but it is no longer true in the n-person case when $n > 2$ (Chin, Parthasarathy and Raghavan 1974).

many players. The aim is the same as in the two-person case: to define classes of n-person games which possess some properties of two-person zero-sum games, as for example uniqueness of equilibrium payoff. But the problematic is different: we have to define the notion of antagonism between n players.
In section 1, we recall the definition of n-person game of type A, B and C introduced by Kats and Thisse (1992). In section 2, we define the notions of saddle-point and value of a n-person game. With the help of these definitions, we extend the definitions of games of type I (introduced by Aumann (1961) under the name of almost strictly competitive games(ASC)), II and IV to the n-person case[2]. In section 3, we give some results concerning the connection between these different classes. In section 4, we generalize Aumann's theorem concerning game of type I in extensive form (Aumann 1961) to the n-person case. At last some examples of games are given in section 5.

Notations

We denote by $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ a n-person game where

- $I = \{1, \ldots, n\}$ is the set of players, $n \geq 2$.
- S_i is the set of strategies of Player i.
- u_i is the payoff function of Player i; $u_i : S_1 \times \cdots \times S_n \to \mathbf{R}$ where $\mathbf{R}$ stands for the set of real numbers.

Let $S = \prod_{i \in I} S_i$. For each Player $i \in I$, $-i$ denotes the set $I \setminus \{i\}$ (*i.e.* $-i$ is the set of opponents of Player i). S_A terms the set $\prod_{i \in A} S_i$ ($A \subseteq I$).
From now, we assume the following property:

Hypothesis 1 The sets S_i and the payoff functions u_i are such that the game $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ has a Nash equilibrium.

For example, Hypothesis 1 is satisfied if each set of strategies is a convex compact subset of an Euclidian space and if the payoff function of each player is continuous and quasi-concave in his own action (Glicksberg 1952).
We denote by $NE(G)$ the set of Nash equilibria of G and by $NEP(G)$ the set of its Nash equilibrium payoffs.

1 n-person game of type A, B and C

The antagonism for these three classes of (non zero-sum) games is defined by comparing different n-tuple of strategies according to several evaluation rules (see Figure 1):

[2] Games of type II and IV are generalizations of ASC games (Beaud 1999).

Type	Couples of strategies	Evaluation rule
A	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, \tilde{s}_{-i})$	$u_i(s) \geq u_i(\tilde{s}) \Leftrightarrow u_j(s) \leq u_j(\tilde{s}), \forall j \in I \setminus \{i\}$
B	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, s_{-i})$	$u_i(s) \geq u_i(\tilde{s}) \Leftrightarrow u_j(s) \leq u_j(\tilde{s}), \forall j \in I \setminus \{i\}$
C	Compare $s = (s_i, s_{-i})$ with $\tilde{s} = (\tilde{s}_i, s_{-i})$	$u_i(s) > u_i(\tilde{s}) \Rightarrow u_j(s) \leq u_j(\tilde{s})$ and $u_i(s) = u_i(\tilde{s}) \Rightarrow u_j(s) = u_j(\tilde{s}), \forall j \in I \setminus \{i\}$

Fig. 1 Definitions of the classes.

This leads to the following definitions:

Definition 1 (Kats-Thisse, 1992) *Let $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ be a n-person game.*

- *G is a game of type* A *if for all $i \in I$, $s, s' \in S$, $u_i(s) \geq u_i(s') \Leftrightarrow u_j(s) \leq u_i(s')$ $\forall j \in I \setminus \{i\}$.*
- *G is a game of type* B *if for each $i \in I$, for all $s_i, s'_i \in S_i$ and for all $s_{-i} \in S_{-i}$, we have*

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \Leftrightarrow u_j(s_i, s_{-i}) \leq u_j(s'_i, s_{-i}) \ \forall j \in I \setminus \{i\} \qquad (1.1)$$

- *G is a game of type* C *if for each $i \in I$, for all $s_i, s'_i \in S_i$ and all $s_{-i} \in S_{-i}$, we have*

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \Rightarrow u_j(s_i, s_{-i}) \leq u_j(s'_i, s_{-i}) \ \forall j \in I \setminus \{i\} \qquad (1.2)$$

and

$$u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i}) \Rightarrow u_j(s_i, s_{-i}) = u_j(s'_i, s_{-i}) \ \forall j \in I \setminus \{i\} \qquad (1.3)$$

Remarks:

1. By definition, every game of type A is of type B and every game of type B is of type C.
2. Two-person games of type A have been introduced under the name of strictly competitive games (Friedman 1983, Moulin 1976). Games of type B and C are also called unilaterally competitive games and weakly unilaterally competitive games (Kats and Thisse 1992).

Kats and Thisse (1992) have shown that every game of type C has a unique equilibrium payoff, and that equilibria of a game of type B are interchangeable under some conditions on the sets of strategies and on the payoff functions.

2 n-person game of type I, II and IV

The classes of games of type A, B and C are defined directly be the preferences of each player without resorting to other concepts. This is no more the case for the classes we introduce now: we compare the strategic behavior of the players with the help of the notion of twisted equilibrium.

2.1 $n = 2$

Let $G = (S_1, S_2, u_1, u_2)$ be a two-person game. We associate to G the game $\bar{G} = (S_1, S_2, -u_2, -u_1)$. $\bar{G}$ is called the twisted game.
$s \in S$ is a twisted equilibrium of G if s is a Nash equilibrium of $\bar{G}$ (Aumann 1961).
$\mathbf{e} \in \mathbb{R}^2$ is a twisted equilibrium payoff of G if there exists a twisted equilibrium s such that $u_i(s) = e_i$ for each $i = 1, 2$.
Aumann gives the following definition of an almost strictly competitive game when $n = 2$ (Aumann 1961).

Definition 2 *G is an almost strictly competitive (ASC) game if*

(i) there exists $s \in S$ which is a Nash and a twisted equilibrium;
(ii) the set of Nash equilibrium payoffs is equal to the set of twisted equilibrium payoffs.

Condition (i) of Definition 2 may be defined using the notion of a saddle-point of a two-person game (Beaud 1999):

Definition 3 *$\bar{s} \in S$ is a saddle-point of the game G if for all $s \in S$, $i \in I$,*

$$u_i(s_i, \bar{s}_{-i}) \leq u_i(\bar{s}) \leq u_i(\bar{s}_i, s_{-i})$$

It is shown that the set of saddle-points of G, denoted by $S(G)$, is equal to the intersection of the sets of Nash and twisted equilibria of G. Hence, condition (i) of Definition 2 is equivalent to: $S(G) \neq \emptyset$.
Aumann has shown that every almost strictly competitive game has a unique Nash equilibrium payoff.

2.2 $n \geq 3$: *saddle-point and value of a n-person game*

The definition of a twisted game does not extend when the number of players is greater than 2. In this latter case, how can we generalize the notion of a twisted equilibrium? Kats and Thisse suggest the following definition of a twisted equilibrium (Kats and Thisse 1992):

Definition 4 *$\bar{s} \in S$ is a twisted equilibrium of a game G if $u_j(\bar{s}) \leq u_j(s_i, \bar{s}_{-i})$ for all $i \in I$, $s_i \in S_i$ and for all $j \in I \setminus \{i\}$.*

By using this definition of a twisted equilibrium, the definition of an almost strictly competitive game can be extended to n-person games.
Unfortunately, Kats and Thisse's definition is not satisfactory: we give now an example of a three-person almost strictly competitive game having two different Nash equilibrium payoffs.
Example 1. $n = 3$, $S_i = \{A_i, B_i\}$ for each $i \in I$.
If $s_3 = A_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,3,5 & 1,3,5 \\ 1,4,5 & 1,4,5 \end{pmatrix} \end{array}$$

If $s_3 = B_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,4,5 & 1,4,5 \\ 1,4,5 & 1,4,5 \end{pmatrix} \end{array}$$

There are two Nash equilibria: (A_1, B_2, A_3) and (B_1, B_2, B_3). Hence $NEP = \{(1,3,5), (1,4,5)\}$.
(A_1, B_2, A_3) and (B_1, B_2, B_3) are also the only twisted equilibria. Hence, $TEP = \{(1,3,5), (1,4,5)\}$.
So, there exists a profile of strategies which is a twisted and a Nash equilibria, and the sets of Nash equilibrium payoffs and twisted equilibrium payoffs coincide: the game is almost strictly competitive and have two distinct Nash equilibrium payoffs contrary to the two-person case.
Let us first generalize the notion of saddle-point to n-person games.

Definition 5 *$\bar{s} \in S$ is a saddle-point of the game $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ if for every $i \in I$, for every $s \in S$,*

$$u_i(s_i, \bar{s}_{-i}) \leq u_i(\bar{s}) \leq u_i(\bar{s}_i, s_{-i}) \tag{2.4}$$

We denote by $S(G)$ the set of saddle-points of G.

Equation (2.4) means that for every $i \in I$, $\bar{s}$ is a saddle-point of the function u_i with respect to maximizing in s_i and minimizing in s_{-i} (Rockafellar 1970).
This leads to the following definition:

Definition 6 *$\bar{s} \in S$ is a strong twisted equilibrium of a n-person game G if:*

$$\forall i \in I,\ \forall s_{-i} \in S_{-i},\ u_i(\bar{s}) \leq u_i(\bar{s}_i, s_{-i}) \tag{2.5}$$

We denote by $STE(G)$ (resp. $STEP(G)$) the set of strong twisted equilibria (resp. the set of the payoffs induced by the strong twisted equilibria).

Remarks:

1. In definition 4, any (unilateral) deviation of Player i induces a gain for all the other players whereas in definition 6, any deviation of (part of) the other players induces a gain for Player i.
2. Definition 6 is the same as the definition of a twisted equilibrium when $n = 2$.
3. In the above example, $(1,4,5)$ is not a strong twisted equilibrium payoff.

Definition 7 *$G = (I,(S_i)_{i\in I},(u_i)_{i\in I})$ is a game of:*

- *type* I *if*
  - *a) there exists a profile of strategies which is a Nash and a strong twisted equilibrium,*
  - *b) the set of Nash equilibrium payoffs is equal to the set of strong twisted equilibrium payoffs;*
- *type* II *if*
  - *a) there exists a profile of strategies which is a Nash and a strong twisted equilibrium,*
  - *b') the intersection between the set of Nash equilibrium payoffs and the set of strong twisted equilibrium payoffs is non empty;*
- *type* IV *if*
  - *b') the intersection between the set of Nash equilibrium payoffs and the set of strong twisted equilibrium payoffs is non empty.*

Example 2. Let $n = 3$, $S_i = \{A_i, B_i\}$ for each $i \in I$.
If $s_3 = A_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 1,0,0 & 0,1,0 \\ 0,0,1 & 0,0,0 \end{pmatrix} \end{array}$$

If $s_3 = B_3$:

$$\begin{array}{cc} & \begin{array}{cc} A_2 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} 0,0,0 & 0,0,1 \\ 0,1,0 & 1,0,0 \end{pmatrix} \end{array}$$

This game has two Nash equilibria, (B_1,B_2,A_3) and (A_1,A_2,B_3), which induce a payoff equal to $(0,0,0)$. Indeed (B_1,B_2,A_3) and (A_1,A_2,B_3) are saddle-points.
By definition of a Nash and of a strong twisted equilibrium, we get the following property:

Property 1 *For every n-person game G, $S(G) = NE(G) \cap STE(G)$.*

When $n = 2$, saddle-points are interchangeable (Beaud 1999). This is no more the case when $n > 2$. In the example above, (B_1,B_2,A_3) and (A_1,A_2,B_3) are saddle-points, but not (A_1,A_2,A_3).

2.2.1 Value of a n-person game
We can associate to each Player i two quantities:

1. The *max-min* of Player i: $\underline{v}_i = \max_{S_i} \min_{S_{-i}} u_i(\cdot,\cdot)$.
2. The *min-max* of Player i: $\bar{v}_i = \min_{S_{-i}} \max_{S_i} u_i(\cdot,\cdot)$.

Note that $\bar{v}_i \geq \underline{v}_i$ for all $i \in I$.

Definition 8 *The n-person game $G = (I,(S_i)_{i\in I},(u_i)_{i\in I})$ has a vector value $v \in \boldsymbol{R}^n$ if $\bar{v}_i = \underline{v}_i = v_i$ for all $i \in I$.*

For example, it is well known that every two-person zero-sum game has a value (recall Hypothesis 1).
De Wolf (1999) generalizes this result to n-person games of type C. In fact, we have this stronger result (see Section 3):

Property 2 *Every n-person game of type* IV *has a value, and this value is the unique Nash equilibrium payoff.*

Proof:
Let $i \in I$ and $\mathbf{e} \in NEP(G) \cap STEP(G)$.
Consider $s^\star \in NE(G)$ and $\bar{s} \in STE(G)$ such that $u_i(\bar{s}) = u_i(s^\star) = e_i$. We have

$$e_i = u_i(s^\star) \geq \max_{S_i} u_i(s_i, s^\star_{-i}) \geq \min_{S_{-i}} \max_{S_i} u_i(s_i, s_{-i})$$

$$e_i = u_i(\bar{s}) \leq \min_{S_{-i}} u_i(\bar{s}_i, s_{-i}) \leq \max_{S_i} \min_{S_{-i}} u_i(s_i, s_{-i})$$

Hence $\bar{v}_i \leq e_i \leq \underline{v}_i$.
So $e_i = \underline{v}_i = \bar{v}_i = v_i$. □

3 Connection between the different classes

The definitions of the different classes of games imply that every game of type A (respectively B, I, II) is a game of type B (resp. C, II, IV).
When $n = 2$, it is known that every game of type C is of type II (Beaud 1999).
When $n > 2$, this is still the case. De Wolf has proved that for a game of type C, for each player $i \in I$, if *any* players $-i$ deviate from their equilibrium strategy, then Player i's payoff increases (De Wolf 1999). This implies that for every game of type C, $NE(G)$ is a subset of $STE(G)$. Hence:

Property 3 *Every n-person game of type* C *is a game of type* II.

Remark: Example 2 is an example of a game of type I but not of type C: when $s_2 = B_2$ and $s_3 = A_3$, Player 1 is indifferent between A_1 and B_1, but not Player 2.
There exists also game of type C but not of type I (Beaud 1999, Example 2.3) [3].

[3] A figure showing the connections between the different classes is placed at the end of this paper.

4 Extensive form game

4.1 n-person game of type I

The aim of this section is to generalize Theorem D of Aumann (1961) to the n-person case. We refer to Owen (1995) for the definition of an extensive game and its properties.

Theorem 1 *Let G be a n-person extensive game which decomposes at a move X and G^X be of type* I. *Let G^D be the difference game, where the payoff to G^D at (the terminal note) X is the value of G^X. Assume that G^D is of type* I. *Then G is of type* I, *$NEP(G) = \nu(G^D)$, and the composition of saddle points in G^X and G^D yields a saddle-point in G.*

PROOF OF THE THEOREM
Let s be a strategy profile. We denote by s^X the couple of strategies obtained by restricting s to G^X. We define similarly s^D. We denote by u_i^Γ the payoff of player i in the game Γ.
We need the following result

Theorem 2 *Let G decomposes at X, and let s be a strategy such that (i) s^X is a strong twisted equilibrium of G^X, and (ii) $s^{G\setminus X}$ is a strong twisted equilibrium of $G \setminus X$ with payoff $u(s^X)$ assigned to the terminal payoff X. Then s is a strong twisted equilibrium of G.*

Proof: Let s be an n-tuple of strategies which verifies (i) and (ii), $i \in I$ and $s'_{-i} \in S_{-i}$. From (i), we have

$$u_i(s_i^X, s_{-i}^{\prime X}) \geq u_i(s^X) \tag{4.6}$$

We denote by G_s^D the demand game where the payoff associated to the (terminal) node X is $u(s^X)$. From (ii), we have in the game G_s^D:

$$u_i(s_i^D, s_{-i}^{\prime D}) \geq u_i(s^D) \tag{4.7}$$

But the payoff of player i induced by (s_i, s'_{-i}) is greater in $G^D_{s_i, s'_{-i}}$ than in G_s^D. Hence, $u_i(s_i, s'_{-i}) \geq u_i(s)$ (Owen 1995, Theorem I.4.3). □

Lemma 1 *Let ν be the unique equilibrium payoff of G^D. Then every equilibrium payoff in G is equal to ν.*

Proof: The proof in the n-person case is similar as the proof in the 2-person case (Aumann 1961). □
Let s be a strong twisted equilibrium in G. We denote by s^X the strategy obtained by restricting s to G^X. We denote by P_s the probability over nodes induced by s.

Lemma 2 *Every strong twisted equilibrium payoff of G is equal to ν.*

Proof: First, we prove that: (A) if $P_s(X) > 0$, then s^X is an strong twisted equilibrium of G^X; and (B) s^D is an equilibrium of G^D.
(A): The proof of (A) is the same as for Nash equilibrium (Aumann 1961).
(B): If $P_s(X) > 0$, s^X is a strong twisted equilibrium of G^X, then $G_s^D = G^D$ and if s^D is not a twisted equilibrium of G^D, we can construct a strategy such that s is not a strong twisted equilibrium of G.
If $P_s(X) = 0$, the payoff in G_s^D is the same as the one in G^D. Let s'_{-i} be such that (2.5) is not satisfied. Let $\bar{s}^X_{-i}$ be a saddle-point of G^X. Then

$$\begin{aligned} u_i^G(s_i,(s'_{-i},\bar{s}^X_{-i})) &\leq u_i^{G^D}(s_i,(s'_{-i},\bar{s}^X_{-i})) \\ &= u_i^{G_s^D}(s_i,(s'_{-i},\bar{s}^X_{-i})) \\ &< u_i^{G_s^D}(s^D) = u_i^G(s) \end{aligned}$$

which is impossible. So (B) is true.
Now, we apply Theorem I.4.3 in Owen (1995): for all $i \in I$, $u_i(s) = u_i^{G^D}(s^D) = \nu(G^D) = \nu$. □
Lemmata 1 and 2 imply that condition b) is satisfied.

Lemma 3 *The composition of a Nash (resp. strong twisted) equilibrium of G^X and of G^D yields a Nash (resp. strong twisted) equilibrium of G.*

PROOF
The proof for the Nash equilibria is the same as in (Aumann 1961). For the strong twisted equilibria, it is a consequence of theorem 2 because $G_s^D = G^D$. □
Let s^X (resp. s^D) be a Nash and a strong twisted equilibrium of G^X (resp. G^D). By Lemma 3, the composition of s^X and s^D is a Nash and a strong twisted equilibrium of G. Hence, condition a) is satisfied and G is of type I. □

5 Examples

5.1 Bertrand's model

n firms produce the same item. The marginal cost is the same for each firm and is equal to c. The firms choose simultaneously their prices $p_1,\dots,p_n \geq c$. The demand of the consumers is represented by a function $D(p)$ where $p = (p_1,\dots,p_n)$ is the profile of prices chosen by the firms. (Kreps 1990). The profit of firm i is

$$\Pi_i(p_1,\dots,p_n) = (p_i - c)D_i(p_1,\dots,p_n)$$

where $D_i(p) = \frac{D(p_i)}{|\arg\min\{p_k\}_{k=1,\dots,n}|}\mathbf{1}_{\{p_i \in \arg\min\{p_k\}_{k=1,\dots,n}\}}$, $|L|$ denoting the cardinality of the finite set L.
The aim is to show that Bertrand's model is a game of type I, but not of type C.

Lemma 4 $(c,\dots,c)$ *is a Nash equilibrium of this game.*

Lemma 5 $(0,\dots,0)$ *is the unique Nash equilibrium payoff.*

PROOF:
Let $p^\star$ be an equilibrium and let us suppose that Player i's payoff is positive for some $i \in I$. This implies that $p_i^\star > c$. But then player i has always incentive to deviate in playing $\min_{j\in I}\{p_j^\star\} - \varepsilon$ for some ε sufficiently small, $\varepsilon > 0$. □

Lemma 6 $(c,\ldots,c)$ *is a strong twisted equilibrium of this game, and each strong twisted equilibrium induces a payoff of* 0 *to every player.*

PROOF:
Let $i \in I$. Then $\Pi_i(c,\ldots,c) = \Pi_i(c,p_{-i}) = 0$ for all p_{-i}. Moreover at every strong twisted equilibrium, at least one player plays c. □
The Bertrand's model is a game of type I: by Lemmas 4 and 6, $NE \cap STE \neq \emptyset$ and by Lemma 5 we have that $NEP = STEP$. But it is not a game of type C: suppose $n = 3$, then $\Pi_1(c,2c,2c) = \Pi_1(3c,2c,2c) = 0$ *and* $\Pi_2(c,2c,2c) = 0 < \Pi_1(3c,2c,2c) = D(2c)/2$.

5.2 Auctions

A divisible item is sold by auction (see for example Wolfstetter (1996)). Player i's valuation of the item is v_i. *We assume that everybody knows the valuation of the other players.* The bid of Player i belongs to the set $S_i = \{1,\ldots,v_i - 1\}$ [4]. Player i bids $s_i \in S_i$. The player who has done the greatest bid wins the auction. If there is more than one winner, the item is divided. The payoff function of Player 1 is equal to $u_i(s) = \frac{1}{\varphi(s)}(v_i - s_i)\mathbf{1}\{s_i = \max_{j\in I} s_j\}$ where $\varphi(s) = |\arg\max\{s_1,s_2,s_3\}|$.
The game $(I,(S_i)_{i\in I},(u_i)_{i\in I})$ fulfills hypothesis 1: (98,97,97) is an equilibrium.

Lemma 7 *This game is a game of type* C.

Proof: Let $i \in I$, $s \in S$ and $s'_{-i} \in S_i$. We denote $W(s) = \{i \in \arg\max_{i\in I} s_i\}$.

1. Suppose that $u_i(s_i,s_{-i}) = u_i(s'i,s_{-i}) = \alpha$.
   (a) $\alpha > 0$. Then $i \in W(s)$, and $s_i = s'_i$ and $u_j(s) = u_j(s'_i,s_{-i})$ for all $j \neq i$.
   (b) $\alpha = 0$. This implies that i does not belong to $W(s)$ and $W(s'_i,s_{-i})$. $W(s) = W(s'_i,s_{-i})$, hence $u_j(s) = u_j(s'_i,s_{-i})$ for all $j \neq i$.
2. Suppose that $u_i(s_i,s_{-i}) > u_i(s'_i,s_{-i})$. Then $i \in W(s)$ and $u_j(s_i,s_{-i}) \leq u_j(s'_i,s_{-i}) = \alpha$ for all j.

□

For other economical examples, see De Wolf (1999).

5.3 "Perturbation" of two-person zero-sum games

Let $\Gamma = (S_1,S_2,u,-u)$ be a two-person zero-sum game and let $\delta_i : S_{-i} \to \mathbf{R}$ for $i = 1,2$.

[4] Note that we restrict here the bids available to Player i.

We associate to Γ the non-zero sum game $G = (S_1, S_2, \mathcal{U}_1, \mathcal{U}_2)$ where $\mathcal{U}_1(s_1, s_2) = u(s_1, s_2) - \delta_1(s_2)$ and $\mathcal{U}_2(s_1, s_2) = -u(s_1, s_2) - \delta_2(s_1)$. G may be considered as a perturbation of the zero-sum game Γ.
It is easy to check that G is a game of type B, and that G and Γ have the same set of Nash equilibria.

Acknowledgements I would like to thank my PhD advisor Sylvain Sorin for the time he spent on this article and for his numerous comments. I thank also James W. Friedman for his advice and for pointing out an error in a first draft of this paper. This work was prepared under the financial support of the TMR network.

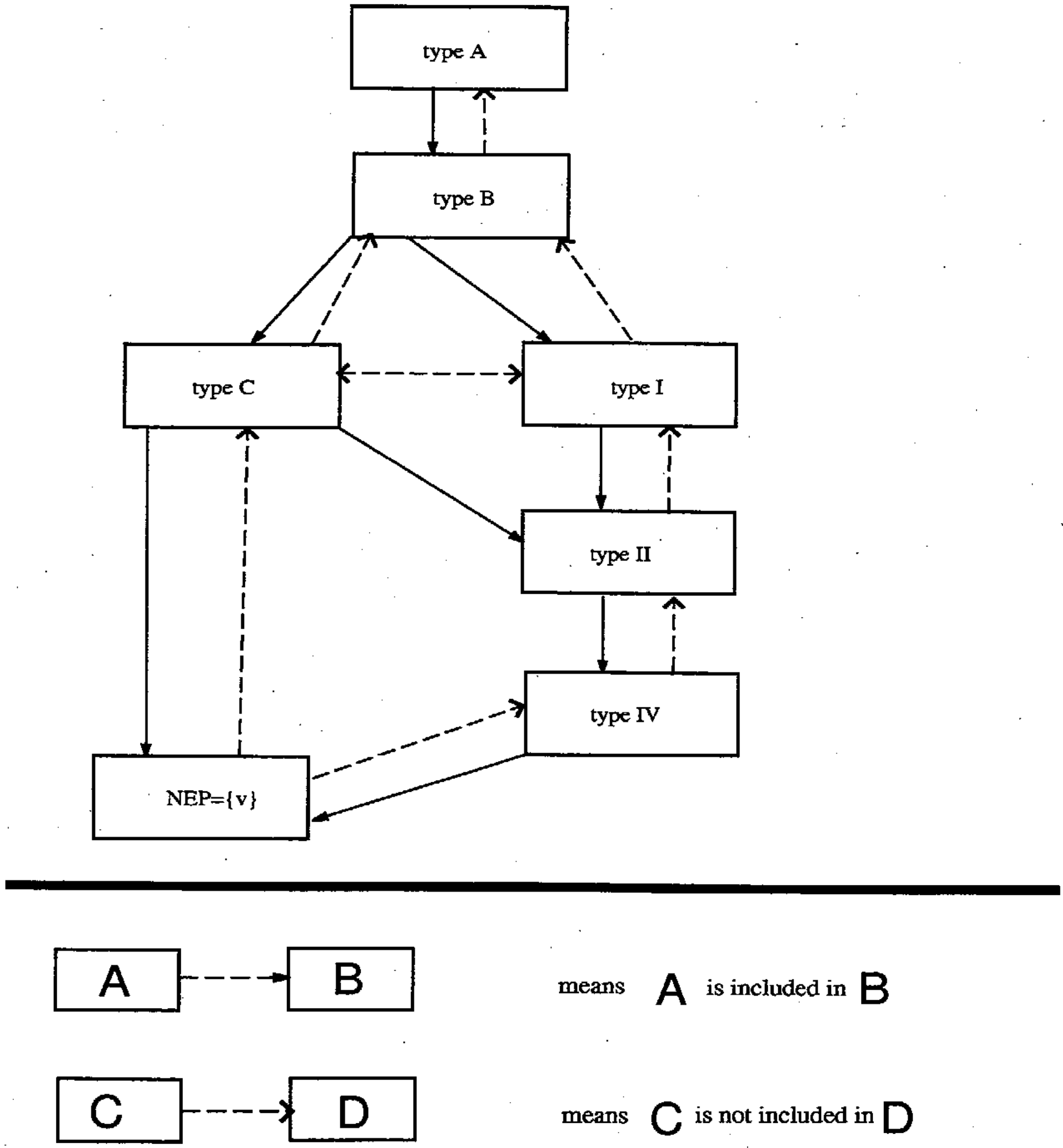

Fig. 2 Connection between the different classes

References

Aumann, R. J. (1961). Almost strictly competitive games, *Journal of the SIAM* **9**: 544–550.

Beaud, J. P. (1999). Antagonistic games, Cahiers du Laboratoire d'Econométrie de l'Ecole Polytechnique 495.

Chin, H., Parthasarathy, T. and Raghavan, T. (1974). Structure of equilibria in N-person non-cooperative games, *International Journal of Game Theory* **3**: 1–19.

De Wolf, O. (1999). Optimal strategies in n-person unilaterally competitive games, C.O.R.E DP 9949.

Friedman, J. (1983). On characterizing equilibrium points in two-person strictly competitive games, *International Journal of Game Theory* **12**: 245–247.

Glicksberg, I. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proceedings of the American Mathematical Society*, number 3, pp. 170–174.

Kats, A. and Thisse, J. (1992). Unilaterally competitive games, *International Journal of Game Theory* **21**: 291–299.

Kreps, D. (1990). *A course in microeconomic theory*, Princeton University Press.

Moulin, H. (1976). Cooperation in mixed equilibrium, *Mathematics of Operations Research* **1**: 273–286.

Nash, J. (1951). Non-cooperative games, *Annals of Mathematics* **54**: 286–295.

Owen, G. (1995). *Game Theory*, Academic Press.

Rockafellar, R. T. (1970). *Convex Analysis*, Princeton University Press, Princeton.

Wolfstetter, E. (1996). Auctions: an introduction, *Journal of Economic Surveys* **10**: 367–420.