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Symmetric Homogeneous Local Interaction

by

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Abstract.

Modelling social interdependence has to deal with the fact that interaction, communication, and competition is mainly limited to other people located in a neighbourhood. The concept of social space uses geometric structure for describing neighbourhoods. The evolution of social processes like segregation or the decay and rise of conventions can then be described by corresponding cellular automata.

Studies in local interaction by psychologists, sociologists, philosophers and economists (cp. Lewenstein/Novak/Latané 1992, Hegselmann 1992, Kandori/Mailath/Rob 1993, Ellison 1993, Berninghaus/Schwalbe 1993) focus on only two special cases of finite homogeneous spaces: the circle and the torus endowed with the „natural" metric.

The following study was motivated by the discovery of some counterexamples showing other kinds of attractors in the evolution of coordination problem as derived in Ellison 1993 and Kandori/Mailath/Rob 1993. In order to identify the causes of such strange behaviour we redefine the concept of local interaction with the help of geometrical axioms. We classify all possible symmetric homogeneous local interaction structures for small numbers and develop some tools that can be used for describing the dynamics of evolutionary processes in such spaces.

Content.

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1. Definition, notation, examples

Imagine a finite **population** $P=\{1,2,...,v\}$ of individuals locally interacting with each other. Let N(p) be the set of individuals a fixed individual p interacts with. The set N(p) is called the **neighbourhood** or the reference group of p. N=N(.) can be seen as a correspondence assigning a set N(p) to each individual p. In order to get simple formulas we set the assigning convention that p is element of N(p).

The term „local interaction" refers not only to $\#N(p)<v$ for all individuals p, but also to the property that for every pair of individuals the reference groups are not identical. Formally we state the following overlap axiom: if $p\neq q$ and $p\in N(q)$, then $N(p)-N(q)\neq\varnothing$.

An ordered pair (p,q) is called (forward-)*connected* if there exists a number n and an n-vector of individuals $(r_0,...,r_n)$ such that $r_0=p$, $r_n=q$ and $r_{i-1}\in N(r_i)$ for all $i\in\{1,2,...,n\}$. Usually in models for local interaction processes within a population it is assumed that a signal given by an individual can spread out all over the population. We say that the connectivity axiom is fulfilled if every ordered pair (p,q), $p,q\in P$ is connected.

Whereas in empirical studies on social networks the correspondence N rarely shows further regularities in theoretical studies we usually find two strong regularity conditions. The first one can be called the balancedness axiom stating that all reference groups are of the same size. Formally: $\#N(p) = k$ for all $p \in N$.

The second one, called two-sidedness, is only assumed in models representing situations in which individuals of every pair are related to each other in the same way. Formally : $p\in N(q)$ iff $q\in N(p)$. Such symmetry assures that the „forward neighbourhood" N(p) and the „backward neighbourhood" $N^*(p):=\{q;\ p\in N(q)\}$ coincide.

Definition: (P,N) is called an **finite local interaction structure (FLIS)** if P is a finite set and N a correspondence on P fulfilling the following axioms:
(I1) assigning convention
(I2) overlap
(I3) connectivity
(I4) balancedness
The FLIS is called **symmetric** if additionally the following axiom is valid:
(I5) two-sidedness

A FLIS (P,N) induces an **incidence structure** $(P,\mathfrak{B},\in)$ by $\mathfrak{B} = \{N(p),p\in P\}$. Moreover $(P,\mathfrak{B},\in)$ is a tactical configuration (abbr. TC; sometimes called 1-design; its defining property translates to axiom (I4)). In this paper we restrict our analysis to symmetric FLIS. In this case N induces a polarity on the TC.

Definition: A bijection (one-to-one mapping) g from one incidence structure $(P,\mathfrak{B},\in)$ onto another $(P',\mathfrak{B}',\in)$ is called an **isomorphism** iff $gB \in \mathfrak{B}'$ for all $B \in \mathfrak{B}$. An isomorphism from an incidence structure $(P,\mathfrak{B},\in)$ onto itself is called an **automorphism**. The group of automorphisms is denoted by $\Gamma:=\mathrm{Aut}\ (P,\mathfrak{B},\in)$.

Definition: a one-to-one mapping π from P onto $\mathfrak{B}$ and from $\mathfrak{B}$ onto P is called a **polarity** if π^2=id and it is incidence preserving, i.e. $p \in B$ implies $\pi(B) \in \pi(p)$.

Proposition: N induces a polarity.
Proof: Let $\pi(p)=N(p)$ and $\pi(B)=p$ if $B=N(p)$. By axiom (I2) π is well-defined. By definition of π the equality π^2=id is guaranteed. W.l.o.g. let $p \in B=N(q)$. We get $\pi(B)=q$ and $\pi(p)=N(p)$. By axiom (I5) we get $\pi(B) \in \pi(p)$.

Definition: Let π be a polarity. A point $p \in \pi(p)$ is called an **absolute point** (or pole) of π. Correspondingly $\pi(p)$ is called a polar.

Note: Every individual p of a FLIS (P,N) is an absolute point w.r.t. the above used polarity induced by N.

Definition: A bijection (one-to-one mapping) g from one FLIS (P,N) onto another one, say (P',N') is called an **isometry** iff g translates neighbourhoods into neighbourhoods, formally:
$gN(p) = N'(gp)$, or
$g\pi p = \pi' gp$ with π and π' representing the respective polarities.
The group G=Aut(P,N) of isometries from (P,N) on itself is called **the isometric group** of the respective FLIS.

Note: The isometric group G=Aut (P,N) is the subgroup of the group Γ=Aut $(P,\mathfrak{B},\in)$ of automorphisms g of the induced tactical configuration $(P,\mathfrak{B},\in)$, which commute with the given polarity π, formally $g\pi p = \pi g p$.

Definition: A FLIS is called **homogeneous** iff
(H) the isometric group acts transitively on P

Note that for a homogeneous FLIS we can drop axiom (I4) - it is implied by (H): the number #N(p) is fixed. We use the following symbols:

$k := \#N(p)$
$m := k\text{-}1 = \#(N(p)\text{-}\{p\})$
$v := \#P$
$P(r) := \{S \subseteq P; \#S=r\}$

Lemma 1: If the FLIS is symmetric, then $k>2$.
Proof: It follows trivially from (I3) that $k>1$. Suppose that there is a symmetric FLIS for $k=2$. Let $N(p)=\{p,q\}$. Axiom (I5) implies $N(p)=N(q)$. This is a contradiction to axiom (I2).

Lemma 2: $k<v$
Follows trivially from (I2).

Incidence structures and FLIS can be represented by (binary) incidence matrices $A=(a_{B,i})$ and $A=(a_{N(i),i})$ respectively: Blocks and neighbourhoods are represented by rows, points and individuals are represented by columns.

Lemma 3: For v odd and k even there are no symmetric FLIS.
Proof: in each row of the incidence matrix there are v-k zeroes. It follows that the incidence matrix contains an odd number of zeroes. By axiom one there are no zeroes on the diagonal. It follows that it is not possible to symmetricly assign the zeroes. This is a contradiction to axiom (I5).

Lemma 4: Two incidence structures are isomorphic if there exist two permutation matrices P and Q that can transform the incidence matrix A of the first one into the incidence matrix B of the other one; formally $PA = BQ$. Two FLIS are isometric if there exist additionally $P=Q$.
Proof: for the first part see Dembowsky. Remember that P represents permutations of neighbourhoods and Q permutations of individuals. The polarity assures that for neigbourhoods and individuals the same renumbering is used. It follows that $P=Q$.

Corrollary: Isometric FLIS have the same eigenvalues.
From $PA = BP$ we get $B \sim A$ (i.e. $PAP^{-1}=B$).

Lemma 5: Incidence matrices of symmetric FLIS have real eigenvalues.
Symmetric matrices show real eigenvalues. Symmetry is assumed in axiom (I5).

2. Dihedral structures

Definition: A FLIS (P,N) is said to be **dihedral**, if its group of isometries contains the dihedral group D_v with the usual action (on P).
We use the following representation: $P=\{0,1,...,v-1\}$ and D_v generated by the generator (0 1 ... v-1) of the cyclic group C_v and the reflection (01)...((v+3)/2,(v-1)/2) or (01)...((v/2)+1,(v/2)) respectively.

Lemma 6: If the group G of isometries of a symmetric FLIS contains the cyclic group C_v with the usual transitive action on P, then symmetry of the FLIS implies that the structure is dihedral.

Proof: Let C_v be generated by the element g. Let us fix some point p. The following permutation b is an isometry: $b(g^s p)=g^{-s}p$. We have to show that for all q in P there exists some $q'\in P$ such that $bN(q)=N(q')$. First let $q=p$: Let $N(p)=\{g^{a(r)}p \,;\, r=1,...,k\}$ and $a(1)=0$. By applying g we get $p\in N(g^{-a(r)}p)$ from $g^{a(r)}p\in N(p)$. By symmetry we get $g^{-a(r)}p \in N(p)$ and $q'=p$. Second: By transitivity of C_v we can assume $q=g^s p$.
We get $bN(q)=bN(g^s p)= b\{g^s g^{a(r)}p\}=\{g^{-s} g^{-a(r)}p\} = g^{-s} N(p) = N(g^{-s}p)$ and $q'=g^{-s}p$.

For v=k+1 there is only one FLIS, it is dihedral.

For even v in case k is odd all homogeneous symmetric dihedral FLIS can be generated by chosing N(0) and applying the rotations. N(0) can be constructed by doing half of the neighbours, applying the reflection with the fixed point 0 for constructing N(0). If k is even the antipodal point also belongs to the neighbourhood and the remaining k/2-1 neighbours on the one hemisphere have to be distributed asymmetrically with respect to the reflection orthogonal to the axis {0,v/2}.

For odd v all symmetric dihedral FLIS can be constructed by chosing one point p and half of the neighbours, applying the reflection with the fixed point p for constructing N(p).

3. Homogeneous designs admitting non-dihedral structures

3.1 Basics and overview

The following table lists the number of all TCs (for small k and v) admitting a polarity (up to isomorphy). It includes asymmetric and dihedral structures.

Table 1. The list of Betten/Selzer of TC admitting a polarity

k \ v	7	8	9	10	11	12
3	2	4	3	5	2	11
4	2	6	4	10	4	?
5	?	4	5	9	6	?
6	1	2	3	10	6	?
7		1	2	4	4	?
8			1	7	2	?
9				1	1	7

Theorem: Up to isomorphy there are 14 non-dihedral symmetric homogeneous FLIS for v<11 (they are listed below). The respective isometric groups are either spheric groups (S_4xC_2, $[A_5]C_2$), or torus groups (D_3xC_2, $[D_3xD_3]C_2$, D_5xC_2), with the usual action or - in one exceptional case - a dihedral group with an unusual action (D_8, see 3.2 case 4).

Proof: By construction. Possible candidates for homogeneous FLIS are drawn out of the list of Betten & Selzer. Axiom (I4) is implied by homogenuity. Axioms (I2) and (I3) have to be checked for every polarity that fulfills the two-sidedness axiom (I5). In some cases the incidence structure admits more than one polarity (up to isometry).

The result of the check is given in table two:

Table 2. Number of non-dihedral FLIS

k \ v	6	7	8	9	10
3	0	0	0	0	0
4	1	0	1	0	3
5	0	0	2	1	0
6		0	1	0	2
7			-	1	2

For v=10 the structures are representable by the ikosahedral group (isomorphic to the dodekahedral group), for v=9 by the corresponding torus group, for v=8, by the octahedral group (isomorphic to the hexahedral group), for 7 by the corresponding projective group, and for v=6 by the corresponding small (asymmetric) torus group.

For v=6 let us use the following numbering:

0	1	2
3	4	5

There is only one non-dihedral symmetric structure N6K4 generated by C_3 with the usual action (i.e. (012)(345)), the permutation (05)(14)(23), and the following generating block N(0)

x		

The polarity is represented by the entry x.

For v=7 the only candidates for FLIS are projective structures with k=5 (cp. Lemma 3); the projective plane of 7 points (group PGL(3,2), size 168) admits only asymmetric structures because the corresponding incidence matrices show complex eigenvalues. By lemma 5 there are no symmetric FLIS for v=7.

In the following we list the remaining cases. We refer to the numbering of Betten & Selzer by v_k geo x, for example: 8_4 geo 4 refers to incidence structure no. 4 in the sublist for v=8 and k=4. In case v and k are known from the context v_k is dropped.

3.2 The four octahedral structures (v=8).

Individuals are represented by the 8 vertices of the hexahedron.

0	1
2	3

4	5
6	7

At left we represent the top layer (or hemisphere), at the left the bottom layer of the hexahedron.

Case one: N8K4 (geo 4)
Partners are the 3 adjacent vertices. $G=S_4xC_2$.

For the incidence structure geo 1 we get two FLIS since the two non-isometric polarities x and y. are identical(geo 1).

Case two and three: N8K5 (x) and N8K5D (y)

In N8K5 neighbourhoods consist of the individuals of the own hemisphere and the antipodal one. In N8K5D all individuals of the opposite hemisphere are partners.

Case four: N8K6
Neigbourhoods are generated by the usual action of C_5 , the (antipodal) reflection (07)(16)(25)(34), and the following generator:

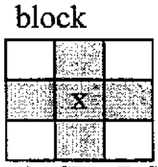
The isometric group is isomorphic to the dihedral group: $[D_4]C_2 \sim D_8$.
The action of the group on the circle is not as usual. Thus the FLIS N8K6 is not dihedral.

3.3 The two torus structures for v=9.

Individuals are represented by the 9 cells of the 3x3-torus.

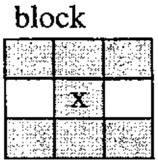
0	1	2
3	4	5
6	7	8

Case one: N9K5 (BS 3479)
The structure is generated by D_3xD_3 with the usual action and the following generating block



$G=[D_3xD_3]C_2$

Case two: N9K7 (geo 1)
The structure is generated by D_3xD_3 with its usual action and the following generating block



$G=D_3xD_3$

3.3 The nine ikosahedral structures for v=10.

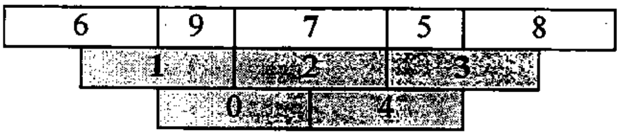
Individuals are represented by the 10 antipodal pairs of the vertices of ikosahedron.

0	1	2	3	4

5	6	7	8	9

At the top we represent the pairs induced by a polar cap (= a fixed face).

An alternative representation of the antipodal pairs starts with a central pair, say 2 and its 3 adjacent neighbours 1, 3, 7 (in bold). The former polar cap {0,1,2,3,4} is shaded in the following sketch. The other two faces adjacent to the pole are {1,2,6,7,9} and {2,3,5,7,8}.



For the incidence structure geo 4 we get the following two FLIS.

Case one and two: Torus structures N10K4T (x) and N10K4TD (y)

The isometric group is G = D_5xC_2.

The incidence structure of the following FLIS is geo 7. The isometric group $[A_5]C_2$ is the composition of a reflection and the group A_5 of rotations of the ikosahedron group. The structure can also be generated by composition of the two non-isometric dihedral structures D5K3 and D5K3D (induced by icosahedral structure).

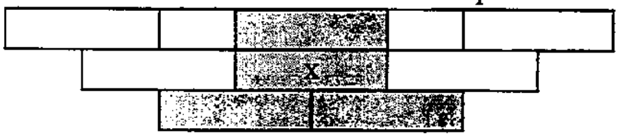
Case three: N10K4

Neigbourhoods are generated by the usual action of C_5 and the following two generators:

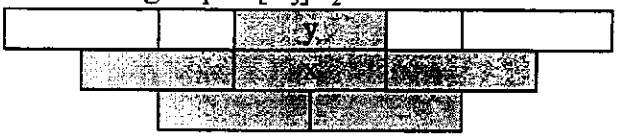
and

Shifting to the other representation we can see that the above generators can be substituted by a single highly symmetric generator.
Moreover it becomes clear that the isometric group is $[A_5]C_2$.

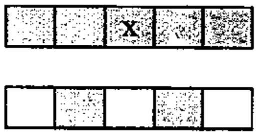
Case three: N10K4 – ikosahedral representation.



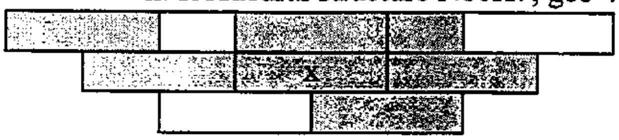
Case four and five: Icosahedral structures N10K6 (x) and N10K6D (y); geo 3. isometric group is $[A_5]C_2$.



Case six: Torus structure N10K7T; geo 1.



Case seven: Icosaedral structure N10K7; geo 4.



4. Binary local interaction processes

Let (P,N) be a homogenious FLIS with neighbourhoods of size k=m+1, and isometric group G. A state of the population P is given by a v-vector $x = (x_p)$. Every point $p \in P$ (or every individual of the population P) can exhibit two states $x_p = 0$ („dead„) and $x_p = 1$ („alive„). Let us identify a subset S of the population with the corresponding state x such that $x_p = 1$ iff $p \in S$.
The counting # induces a measure on the state space. Let $y_p := \#(\{q \in N(p);\, q \neq p \text{ and } x_q=1\})$.

Definition: (P,N,H) is called a (deterministic homogeneous binary) **local interaction process** (in discrete time), if (P,N) is a homogenious FLIS with neighbourhoods of size m+1 and the transition function $H: 2^P \to 2^P$ is induced by N and a function h: $2 \times \{0,1,\dots,m\} \to 2$ in the following way:
$H_p(x)=h(x_p,y_p)$ for all $p \in P$

The group G of isometries of a FLIS (P,N) acts on 2^P, the space of states, and on its subspaces P(r).

Definition: Two states x and x' are said to be **equivalent** if there exists a group element $g \in G$ moving x to x'. Correspondingly $2^P/G$ and P(k)/G are the spaces of equivalence classes. .

Lemma: For every local interaction process (P,N,H) with X(t) = x and x'=g(x) the state X(t+s) after s steps is equal to g^{-1} X'(s), X' being the path with initial state X'(0) = g(x).

Main principle of classification: For deriving the full information on the dynamics, we can restrict our analyses to $2^P/G$.

A main tool for the classification of all possible dynamics is Burnside's Theorem (setting Z to be the set P(r) of all subsets of P of a fixed size r).

Burnside's Theorem: Let G be a finite group that acts transitively on a finite set Z. For each $g \in G$ let Fix g := $\{ z \in Z ;\, g(z)=z \}$. If ζ is the number of orbits of Z under G, then

$$\zeta\, \#G = \Sigma\, \{ \#\, \text{Fix}\, g ;\, g \in G \}$$

For every homogeneous FLIS (P,N) the action (G,P) of its isometry group G and the corresponding space $2^P/G$ shows the following characteristics
- let $S \in P(r)$; the equivalence class GS is a subset of P(r); if r=1 or r=v-1 then #GS=1
- $2^P/G$ inherits the structure of a lattice with complement.

Thus it is enough to consider Z=P(r) for $v/2 \leq r \leq v-2$.

Example 1:

The dihedral group D_6 with the usual action on the circle {0,1,2,3,4,5,6}. The group is acting transitively on P - this is why both P/G and P(v-1)/G are singletons (P=P(1)). Using the mapping which assigns the complement to each set we get that P(r)/G is isomorphic to P(v-r)

Thus we only have to consider the cases $3 = v/2 \le r \le v\text{-}2 = 4$.

Table 3. Fixed elements in P(k)

Type of g	Order of g	Number s, of elements	# Fix g		s # Fix g	
			r=4	r=3	r=4	r=3
Identity	1	1	15	20	15	20
Reflection about a line	2	3	3	4	9	12
Reflection about an imaginary line	2	3	3	0	9	0
Rotation by half	2	1	3	0	3	0
Rotation by thirds	3	2	0	2	0	4
Other rotations	2	2	0	0	0	0

Table 4. Number of orbits for $2^P/G$

	r=0, r=6	r=1, r=5	r=4, r=2	r=3
Σ { # Fix g ; g∈G }			36	36
Number of orbits	1	1	3	3

For example: we get $\zeta = 36/12 = 3$ for P(4) and for P(3).

Example 2:

The dihedral group D_9 with the usual action on the circle {0,2,4,6,8,7,5,3,1}. We have to consider the cases $4.5 = v/2 \le r \le v\text{-}2 = 7$.

Table 3. Fixed elements in P(k)

Type of g	Order of g	Number s, of elements	# Fix g			s # Fix g		
			r=5	r=6	r=7	r=5	r=6	r=7
Identity	1	1	126	84	36	126	84	36
Reflection about a line	2	9	6	4	4	54	36	36
Rotation by thirds	3	2	0	3	0	0	6	0
Other rotations	9	6	0	0	0	0	0	0

Table 4. Number of orbits for $2^P/G$

	r=0, r=9	r=1, r=8	r=2, r=7	r=3, r=6	r=4, r=5
Σ { # Fix g ; g∈G }			72	126	180
Number of orbits	1	1	4	7	10

5. Game induced dynamics

Let A be a 2x2-matrix

a	b
c	d

Using indices $i,j \in \{0,1\}$ we can write A as

A(0,0)	A(0,1)
A(1,0)	A(1,1)

Let (P,N) be a homogeneous FLIS with #N(p)=k=m+1. If every pair of neighbours (i,j), $i \neq j \in N(i)$ is equally likely to meet for a contest given by the 2 by 2 base game given by A, every individual faces the following expected payoff matrix B=(b(i,r))=A(m)

	$y_{N(p)}$		
x_p	0	r	m
0	a	((m-r)a+rb)/m	b
1	c	((m-r)c+rd)/m	d

If for a given state $(x_p(t))_p \in 2^P$ of the population P at time t individual p faces the local aggregate partner state $y_{N(p)}(t) = \Sigma\{(x_q(t)); q \in N(p), p \neq q\}$ the expected payoff of player p at time t in the global game is given $A(m)(x_p(t), y_{N(p)}(t))$. The corresponding v-person game is called a **population game** based on A and (P,N).

By adding a transition function h population games generate processes. One of the most examined transition function h for such population games is the so-called **myopic best reply** (we assume that there are no ties in arg max) defined by
$x_p(t+1) = h(x_p(t), y_{N(p)}(t)) = \arg\max b(\,.\,, y_{N(p)}(t))$.

Note that because of the linear structure of B the are only three best reply structures:
1. h is constant (prisonners' dilemma type)
2. $h(x_p,r)=0$ iff $r<\lambda$ (coordination type, i.e. a>c, d>b)
3. $h(x_p,r)=0$ iff $r>\lambda$ (hawk-dove type, i.e. c>a, b>d)

6. Myopic best reply for coordination games

In the following we consider a base game A of coordination type.
Additionally let
(1) d>a
In this case the equilibrium (1,1) is **payoff-dominant.**
Moreover let
(2) a-c > d-b
I.e. (0,0) is the so-called **risk-dominant** equilibrium.
Note: by (1)+(2) we get d-c > a-c > d-b > a-b, i.e. the possible loss if the partner will

break the convention is larger for the payoff-dominant strategy.
For the mixed strategy extension of the game we get a third equilibrium described by the probability π of deciding for option/strategy 1 (for both players):

(3) $\pi b + (1-\pi)a = \pi d + (1-\pi)c$

or

(4) $\pi = (a-c)/((d-b)+(a-c))$

By considering the local decision problem represented by A(m) we get a limit $\lambda = m\pi$, such that below that value the best response is 0 and above switches to 1:

(5) $\lambda b + (m-\lambda)a = \lambda d + (m-\lambda)c$

(6) $m(a-c) = \lambda((d-b) + (a-c))$

we get by using of (1)

(7) $m < 2\lambda$

Assuming that λ is not an integer we get the corresponding best reply function $h(x_p,r) = 0$ iff $r<\lambda$ (and =1 else).

Example 1:
Let (a,b,c,d)=(4,3,1,5). We get $\pi = \lambda/m=0.6$ and the following myopic best replies:
... for k=m+1=3:

r	0	1	2
$h(x_p,r)$	0	0	1

... for k=m+1=5:

r	0	1	2	3	4
$h(x_p,r)$	0	0	0	1	1

For our purpose more adequate is the following

Example 2:
Let (a,b,c,d)=(12,7,0,14). We get $\pi = \lambda/m=12/19=0.63157$ and no integer thresholds within a wide range. Let $\lambda^+ = \min\{r\in \mathcal{N};\ r\geq \lambda\}$. The myopic best reply, given by $h(x_p,r)=1$ iff $r\geq\lambda$, can be read from the table 5:

Table 5. Reaction threshold for the cooperation game

m	2	3	4	5	6	7	8	...	14	15
λ	1.26	1.89	2.53	3.16	3.79	4.42	5.05	...	8.84	9.47
λ^+	2	2	3	4	4	5	6	...	9	10

Note that for m=2 and m=4 the best reply function for this game coincides with that of example 1.

Let us denote a given initial state by $z(0)=(x_p(0))_{p\in P}$ and the corresponding state at time t by z(t). Let us simply say „z dies out„ if the state of the population reaches the zero-vector in finite time.

7. An example that the dynamic flow for very similar structures can be very different

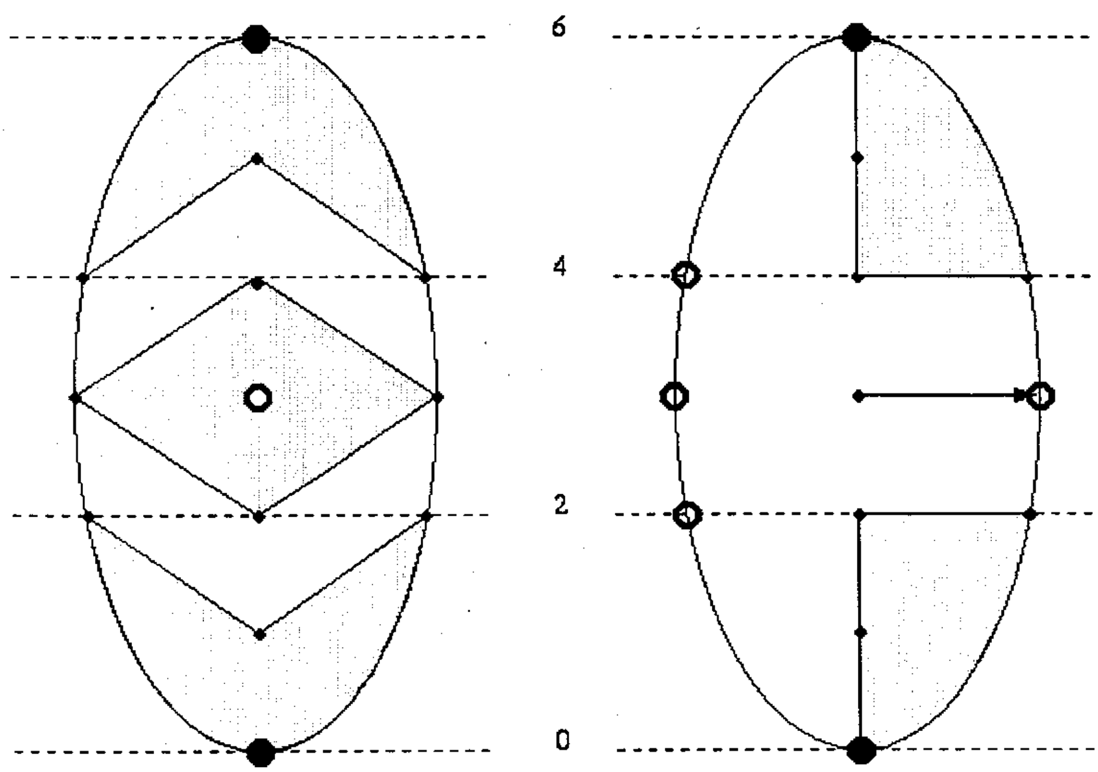
In the following we consider the two dihedral structure and the one non-diedral structure with v=6 and k=4. The both diedral structures are governed by the same group. Thus we can directly compare the dynamical flow in the quotient space $2^N/G$ (as described in section 3). There are 13 = 1+1+3+3+3+1+1 classes of states. Note that for the non-dihedral structure we get a very similar quotient space with 12 = 1+1+3+2+3+1+1 classes of states (see appendix). Thus a comparison of the flows is also possible for this case. For the coordination game considered in example 2 of the previous section we get the following classification of the dynamics:

Table 6. Comparison of dynamics.

structure	attractor types	fixed types	blinker types	transient types
D6K4	3	2	1	10
D6K4D	3	3	3	7
N6K4	2	3	4	5

An **attractor** type is an equivalence class of a fixed set or a within-class blinker that attracts flow from at least one other class. Fixed, blinker, and transient types sum up to $\#2^N/G$. The flow of the diedral structure is sketched in the following figure. Fixed types are represented by filled circles, blinkers by (unfilled) circles. Shaded regions contain those types that move to the respective attractor in one step.

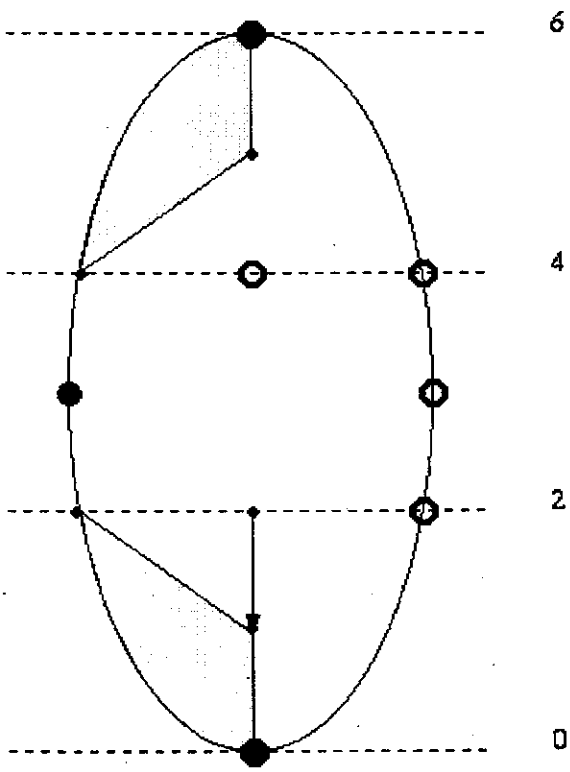
Fig.1: Flow charts for the diedral structures D6K4 (at left) and D6K4D (file: d6k4b.gif)



Note that the central blinker in D6K4 exhibits the largest region of attraction (it attracts the four neighbouring classes). It is evident (by symmetry) that for the usual stochastically pertubed processes of evolution (cp. Diekmann 1995) the limiting asymptotic distribution is not concentrated on the zero-vector as for the special case considered by Ellison 1993. Moreover the bias of the pertubation processes towards sets of mean size (n/2 or $\pm 1 + n/2$) clearly favours the central blinking attractor in case of D6K4.

The following figure shows the non-diedral structure N6K4 that slightly favours paths towards the zero-vector.

Fig.2: Flow charts for the non-diedral structure N6K4 (file: n6k4.gif)



7. Examples of more different structures

In this section we consider the dynamics of the two non-isometric non-dihedral FLIS N8K5 and N8K5D. The isometric group is the octahedral group $G=A_4xD_2$ (#G=48). Remember: A_4 is a simple group isomorphic to AGL(2,2). For constructing the lattice 2N/G it is enough to consider the subsets of size r such that $4 = v/2 \leq r \leq v-2 = 6$.

Table 7. Fixed sets in P(k)

Type of g w.r.t. P	Order of g	Number s of elements g	# Fix g			s # Fix g		
			r=4	r=5	r=6	r=4	r=5	r=6
Identity	1	1	70	56	28	70	56	28
2 4-cycles	4	12	2	0	0	24	0	0
2 3-cycles	3	8	4	2	1	32	16	8
2 2-cycles	2	6	14	12	8	84	72	48
4 2-cycles	2	13	6	0	4	78	0	52
1 2-cycle, 1 6-cycle	6	8	0	0	1	0	0	8

Table 8. Number of orbits for $2^P/G$

	r=0, r=8	r=1, r=7	r=2, r=6	r=3, r=5	r=4
Σ { # Fix g ; g$\in$G }			144	144	288
Number of orbits	1	1	3	3	6

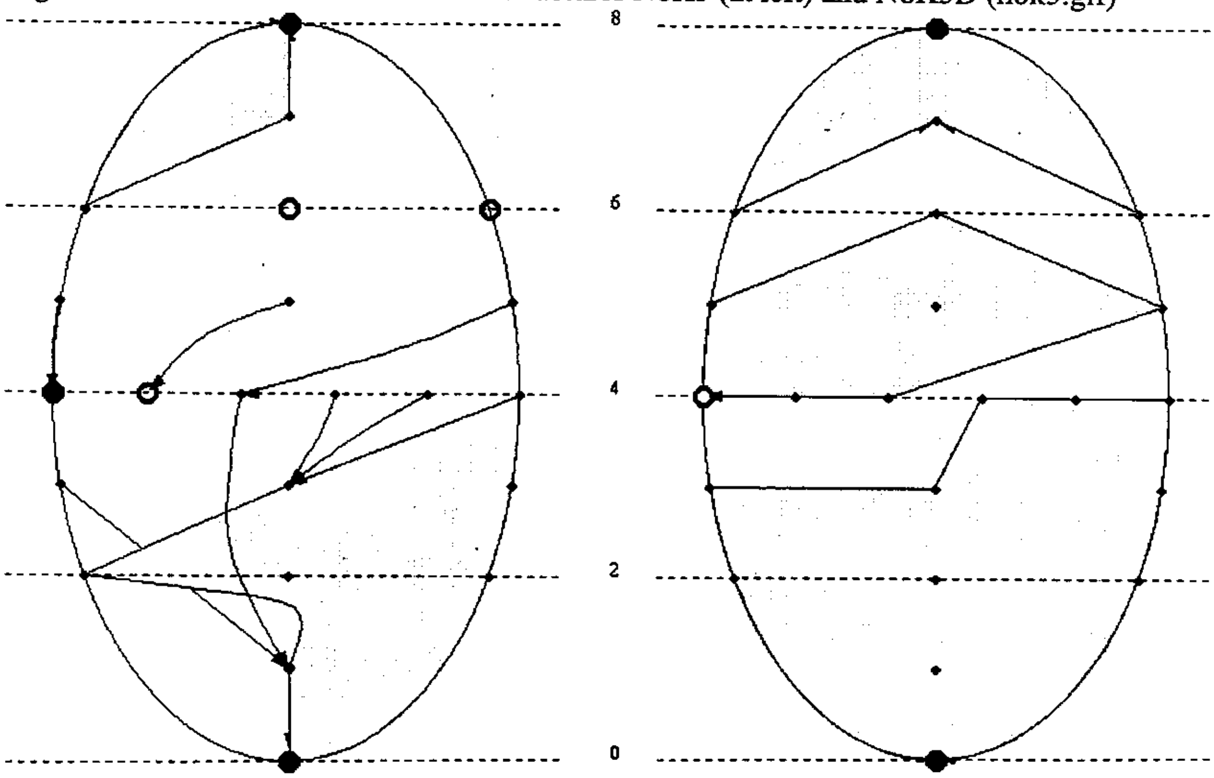
For the same coordination game as considered in the previous section we get the following flow structure:

Table 9. Comparison of dynamics.

structure	attractor types	fixed types	blinker types	transient types
N8K5	4	3	3	17
N8K5D	3	2	1	19

Note that N8K5D is governed by fast attraction: every transient point reaches the attractor in one step, whereas in N8K5 there are 3 intermediate classes. A corresponding sketch is given in the following figure (Fig.3):

Fig.3: Flow charts for the non-diedral structures N8K5 (at left) and N8K5D (n8k5.gif)



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Appendix

Burnside table for dihedral structures:

For even v we get the following table:

Type of g	Order of g	Numbers, of elements	# Fix g	s # Fix g
Identity	1	1	v	v
Reflection about a line	2	v/2	2	v
Reflection about an imaginary line	2	v/2	0	0
Rotations		v-1	0	0

For odd v we get:

Type of g	Order of g	Numbers, of elements	# Fix g	s # Fix g
Identity	1	1	v	v
Reflection about a line	2	v	1	v
Rotations		v-1	0	0

Examples of section 3:

In **example 1** (section 3) we get the following generators:

3 orbits in P(4). Generating subsets:
{0,1,2,3}, {0,1,2,4}, {0,1,3,4}

3 orbits in P(3). Generating subsets:
{0,1,2}, {0,1,3}, {0,2,4}

In **example 2** (section 3) we get the following generators:

10 orbits in P(5). Generating subsets:
{0,1,2,3,4}, {0,1,2,4,5}, {0,1,2,4,7}, {0,3,4,5,6}, {0,1,2,5,6},
{0,3,4,7,8}, {0,1,2,7,8}, {0,1,2,5,8}, {0,1,4,6,7}, {0,1,2,6,8}

7 orbits in P(6). Generating subsets:
{0,1,2,3,4,6}, {0,1,3,4,6,8}, {0,1,2,4,5,8}, {0,1,2,5,6,8},
{0,2,3,5,6,8}, {0,1,2,3,4,8}, {0,1,2,3,6,8}

4 orbits in P(7). Generating subsets:
{0,1,2,3,4,5,6}, {0,1,2,3,4,7,8}, {0,1,2,4,5,7,8}, {0,3,4,5,6,7,8}

Burnside table for the non-dihedral structure N6K4:

Type of g	Order of g	Number s , of elements	# Fix g		s # Fix g	
			r=2,4	r=3	r=2,4	r=3
Identity	1	1	15	20	15	20
Reflection (axis)	2	3	3	0	9	0
Reflection (point)	2	4	3	0	12	0
Cyclic	3	2	0	2	0	4
Cyclic	6	2	0	0	0	0

Number of orbits for $2^P/G$

	r=0, r=6	r=1, r=5	r=4, r=2	r=3
Σ { # Fix g ; g∈G }			36	24
Number of orbits	1	1	3	2