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A Note on „Renegotiation in Repeated Games" [Games Econ. Behav. 1 (1989) 327-360]

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A Note on “Renegotiation in Repeated Games” [Games Econ. Behav. 1 (1989) 327–360]*

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Abstract

In Farrell and Maskin (1989), the authors present sufficient conditions for weakly renegotiation-proof payoffs in their Theorem 1 (p. 332). We show that a step in the proof of this theorem is not correct by giving a counterexample. Nevertheless, the sufficient conditions remain true, and we offer a correction of the proof.

Keywords: (Weak) Renegotiation-Proofness; Infinitely Repeated Games

JEL classification: C72, C73

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1 Introduction

In Farrell and Maskin (1989), the concept of *weak renegotiation-proofness* (subsequently abbreviated as WRP) is introduced and the authors provide a characterization of WRP payoffs for general two-player games. In their Theorem 1 (p. 332), the authors give both sufficient and necessary conditions for a strictly individual rational payoff to be weakly renegotiation-proof. In this note, we use a counterexample to show that their proof of the sufficient conditions fails at a particular step. While Farrell and Maskin (1989) are very careful in many steps of the proof, they implicitly assume more of a structure on the set of payoffs than actually exists. More specifically, they claim to obtain a payoff with *independent randomization*, which is only obtainable with correlated strategies. However, if correlated strategies were allowed, large parts of the proof would be unnecessary.

First, we introduce the basic notation as given in Farrell and Maskin (1989). Then, we go through the arguments of the original proof before we point to the crucial and erroneous claim in that proof. We use a counterexample to illustrate the problem, and then prove an alternative result that replaces the erroneous claim and ultimately fixes the proof.

2 Basics and Original Result

We adopt most of the original notation from Farrell and Maskin (1989), but denote sets by calligraphy instead of regular letters since we need a more elaborate notation for our proof.

Consider a two-player, single-stage game with players $i = 1, 2$. Each player i possesses a finite set of actions, and we denote the simplex consisting of player i's mixed actions by $\mathcal{A}_i$. We denote the set of both players' actions by $\mathcal{A} \equiv \mathcal{A}_1 \times \mathcal{A}_2$. Let $g : \mathcal{A} \longrightarrow \mathbb{R}^2$ be the vector of continuous payoff functions $g_i : \mathcal{A}_i \longrightarrow \mathbb{R}$. The single-stage game g is then defined by the set of payoffs and actions. We will denote the set of mixed-strategy payoffs, i.e., the image of g, by

$$\mathcal{U} = \Big\{ (v_1, v_2) \in \mathbb{R}^2 \,|\, \exists\, a \in \mathcal{A} \text{ with } g\,(a) = (v_1, v_2) \Big\}$$

and the set of feasible payoffs in the repeated game by

$$\mathcal{V} = \mathrm{co}\,(\mathcal{U})\,.$$

For player i, the profit-maximizing deviation from action pair $a = (a_1, a_2)$ is defined by $c_i(a) = \max_{a_i} g_i(a_i, a_j)$, $i \neq j$, the minimax payoff[1] is defined by $\underline{v}_i = \min_{a_j} \max_{a_i} g_i(a_i, a_j)$

[1] While Farrell and Maskin (1989) normalize the minimax payoff to zero for both players, we omit this normalization in the subsequent sections for a better illustration. This is immaterial to our results.

and $v_i^{max} = \max_{a_1,a_2} g_i(a_1, a_2)$ is the maximal attainable payoff. The set of strictly individual rational payoffs in the repeated game is given by

$$\mathcal{V}^* = \{(v_1, v_2) \in \mathcal{V} \,|\, v_1 > \underline{v}_1, v_2 > \underline{v}_2\}.$$

In the repeated game, we consider the infinite repetition of the single-stage game g, which will be denoted by g^*. Let $t = 1, 2, \ldots, \infty$ denote the periods and the sequence $\{a_i(t)\}$ denote a player's action profile with $a_i(t) \in \mathcal{A}_i^t$. Note that we assume constant action spaces $\mathcal{A}_i^t = \mathcal{A}_i$ for all t. A t-history will be denoted by $h^t = (a(1), \ldots, a(t))$, and $\mathcal{H}$ is the set of all such possible t-histories. A strategy σ_i for player i in the repeated game is a function that defines an action $a^i \in \mathcal{A}_i$ for every date t and history $h^t \in \mathcal{H}$.[2] In every period, players receive the stage-game payoffs. Player i's discounted average payoff at time t is then given by $(1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t} g_i(a_1(\tau), a_2(\tau))$, where $\delta < 1$ is the common discount factor for all players. The expected payoffs of strategy σ with discount factor δ will be denoted by $g^*(\sigma, \delta)$, but we often omit δ and simply write $g^*(\sigma)$.

A weakly renegotiation-proof equilibrium is defined as follows.

Definition 1 (Farrell and Maskin, 1989)**.** A subgame perfect equilibrium σ is *weakly renegotiation-proof* if there do not exist continuation equilibria σ^1, σ^2 of σ such that σ^1 strictly Pareto-dominates σ^2. If an equilibrium σ is WRP, then we also say that the payoffs $g^*(\sigma)$ are WRP.

2.1 Sufficient Conditions for Weakly Renegotiation-Proof Payoffs

Let us cite the conditions that Farrell and Maskin (1989) propose as sufficient for WRP payoffs, which is the first part of their Theorem 1 (p. 332).

Theorem 1 (Farrell and Maskin, 1989)**.** *Let $v = (v_1, v_2)$ be in $\mathcal{V}^*$. If there exist action pairs $a^i = (a_1^i, a_2^i)$ (for $i = 1, 2$) in g such that (i) $c_i(a^i) < v_i$, while (ii) $g_j(a^i) \geq v_j$ for $j \neq i$, then the payoffs (v_1, v_2) are WRP for all sufficiently large $\delta < 1$.*

To prove this result, two steps have to be completed. First, one needs to construct a sequence of actions to obtain v as a payoff of the repeated game such that no two continuation payoffs along this path can be strictly Pareto-ranked. If the players could use correlated strategies, this step would be trivial. As they can only use independent randomizations, and the set of mixed-strategy payoffs is a peculiar subset of feasible payoffs, this is not straightforward, as we show in the following section. Given this sequence of actions for the normal phase of the game, one then needs to design punishment paths such that v is a subgame perfect equilibrium and no continuation payoffs of the equilibrium strategy can be strictly Pareto-ranked.

[2] Note that by the definition of a strategy σ, we ultimately assume that players can not only observe the realized actions, but also the mixed strategies in the repeated game. Players can therefore condition their strategies on all past *private randomizations*. This assumption is also made by Farrell and Maskin (1989), but they remark that it is not strictly necessary (see their footnote 2 on p. 329).

3 The Error in the Proof of Farrell and Maskin (1989)

In the following text, we will go along the original proof of Theorem 1 and first discuss the simple cases where the proof of Farrell and Maskin (1989) works. Then, we will give a counterexample for the crucial step in their proof and offer a correction.

Clearly, for a mixed-strategy payoff $v \in \mathcal{U}$, i.e., if there exists an action a such that $g(a) = v$, there is not much to do as v can be obtained by playing action a in every period, and trivially, all continuation payoffs along the path are equal to v. For a payoff $v \in \mathcal{V}^* \setminus \mathcal{U}$, the folk theorem for observable mixed strategies without public randomization given in Fudenberg and Maskin (1991, p. 434) yields that for a sufficiently large δ, we can find a sequence of action pairs $\{\hat{a}(t)\}$ such that $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g\left(\hat{a}(t)\right) = v$. However, this does not ensure that any two continuation payoffs of this sequence are Pareto-undominated, as required by the definition of weak renegotiation-proofness.

Therefore, Farrell and Maskin (1989) construct normal-phase actions using one of the action pairs a^1 or a^2 that are given by the hypotheses of the theorem. Given the vectors $g(a^1)$ and v, one can construct the line l^1 that starts in $g(a^1)$ and runs through v. If all payoffs of the sequence $\{\hat{a}(t)\}$ were on this line, Lemma 1 of Farrell and Maskin (1989, p. 355, subsequently denoted as Lemma FM1) yields that the Pareto condition is satisfied. If, however, not all payoffs lie on l^1, there must be actions a^* and a^{**} with payoffs $g(a^*)$ above and $g(a^{**})$ below l^1.

So far, everything is true and works in all two-player games. However, on page 334, Farrell and Maskin (1989) implicitly claim the following.

Claim 1. *Let $v \in \mathcal{V}^* \setminus \mathcal{U}$ and suppose that a^1, a^2 in $\mathcal{A}$ satisfy the hypotheses of Theorem 1; that is, a^i satisfies*

(i) $g_j(a^i) \geq v_j$, $j \neq i$,

(ii) $c_i(a^i) < v_i$,

then, without loss of generality, there exists an action pair $\tilde{a}$ in $\mathcal{A}$ that satisfies

(a) $g_1(a^1) < v_1 < g_1(\tilde{a})$, $g_2(a^1) > v_2 > g_2(\tilde{a})$,

(b) v is a convex combination of $g(a^1)$ and $g(\tilde{a})$.

If the players were able to play correlated strategies, they could easily randomize between $g(a^*)$ and $g(a^{**})$ to obtain a payoff on l^1. However, as there is no public randomization device, players cannot play correlated strategies and can only randomize independently. As Farrell and Maskin (1989) rightly continue, if players randomize independently between a^* and a^{**} with probabilities $p \in (0,1)$ and $1-p$, the obtained payoffs, denoted by $\Gamma(p) = (\Gamma_1(p), \Gamma_2(p))$, will lie above l^1 for a sufficiently large p and below l^1 for a low p. As $\Gamma(p)$ is continuous in p, they argue correctly that there must exist a p^* such that $\Gamma(p^*)$ lies on l^1. Clearly, if $\Gamma(p^*) = v$, the normal phase can be implemented by requiring randomization between a^* and a^{**}, but as we assumed $v \in \mathcal{V}^* \setminus \mathcal{U}$, this is not relevant here.

To obtain v as a convex combination of $\Gamma(p^*)$ and $g(a^1)$, we must have $\Gamma_1(p^*) > v_1$. However, in the following counterexample, we show that there is no mixed-strategy payoff $\Gamma(p^*)$ on l^1 with $\Gamma_1(p^*) > v_1$. Moreover, contrary to the claim by Farrell and Maskin (1989) in their footnote 6 on page 334, the analogous construction with $g(a^2)$ and l^2 does not work either, which ultimately rejects Claim 1.

3.1 Counterexample to Claim 1

Consider the two-player game where Players 1 and 2 can choose between two pure actions $\{u, d\}$ and $\{l, r\}$, and the stage-game payoffs of the pure strategies are given by the payoff matrix shown in Table 3.1.

	l	r
u	$(0, 0)$	$(2, 2)$
l	$(4, 0)$	$(0, 0)$

Table 3.1: Payoff matrix of the two-player strategic game.

For $p, q \in [0, 1]$, we denote by $a = (p, q)$ the mixed strategy in which Player 1 randomizes between u and d with probabilities $1 - p$ and p, respectively, and Player 2 randomizes between l and r with probabilities $1 - q$ and q.

The set of feasible payoffs $\mathcal{V}$ is the convex hull of the payoff vectors $(0, 0), (2, 2)$ and $(4, 0)$; i.e.,

$$\mathcal{V} = co\left(\{(0, 0), (2, 2), (4, 0)\}\right)$$

and the set of strictly individually rational payoffs is given by

$$\mathcal{V}^* = \left\{v \in \mathcal{V} | v_1 > \tfrac{4}{3}, v_2 > 0\right\}.$$

In Figure 3.1, we illustrate how the set of mixed-strategy payoffs $\mathcal{U}$ is included in the set of feasible payoffs $\mathcal{V}$.

Let us consider the strictly individually rational payoff $v = (\frac{5}{2}, 1)$, which is not obtainable with mixed strategies, i.e., $v \in \mathcal{V}^* \setminus \mathcal{U}$. Then, consider the action pairs $a^1 = (\frac{1}{4}, \frac{7}{8})$ and $a^2 = (1, \frac{5}{16})$. First, we show that a^1 and a^2 satisfy the conditions of Theorem 1. For action a^1, $g(a^1) = (\frac{23}{16}, \frac{21}{16})$, and thus $g_2(a^1) > 1$. For Player 1, the maximal deviation payoff is given by $c_1(a^1) = \frac{7}{4}$, which is also in accordance with the conditions. For the action pair a^2, we obtain $g(a^2) = (\frac{11}{4}, 0)$, and therefore $g_1(a^2) > \frac{5}{2}$. Finally, Player 2's maximal deviation payoff is given by $c_2(a^2) = 0$. Thus, the two actions both satisfy Conditions (i) and (ii) of the theorem.

Next, as proposed by Farrell and Maskin (1989), we construct the line l^1 and find that if all payoffs of the normal phase sequence $\{\hat{a}(t)\}$ lie on l^1, the average payoff would not be v since there is no stage-game payoff x on l^1 with $x_1 > v_1$. Graphically speaking, there are no stage-game payoffs to the right of v as l^1 does not intersect with $\mathcal{U}$ right of v (see Figure 3.2). Thus, if we select two action pairs a^* and a^{**} with payoffs $g(a^*)$ above and $g(a^{**})$ below l^1, and if players randomize between these two actions with parameters

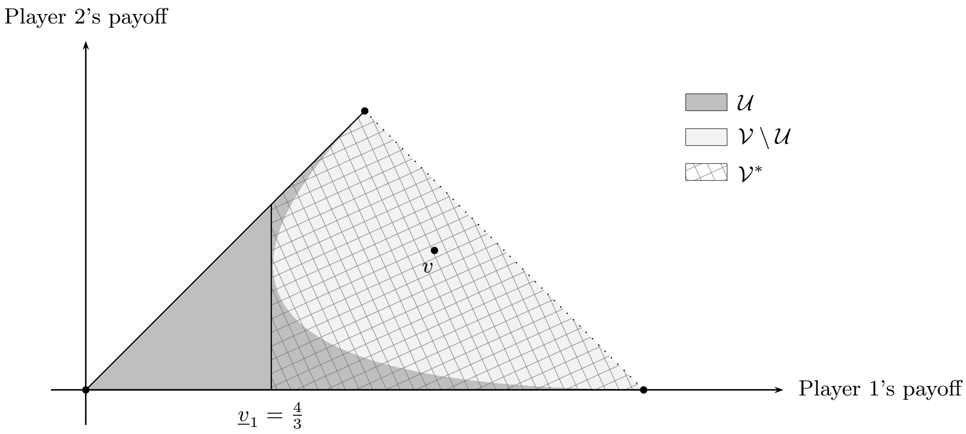

Figure 3.1: Illustration of $\mathcal{U}$ and $\mathcal{V}^*$.

p and q, respectively, the resulting payoff $\Gamma(p, q)$ will certainly lie in $\mathcal{U}$, in the dark-gray area in Figure 3.2. While there exist p^*, q^* such that $\Gamma(p^*, q^*)$ lies on l^1, in our example, this will certainly be to the left of v, that is, $\Gamma_1(p^*, q^*) < v_1$.

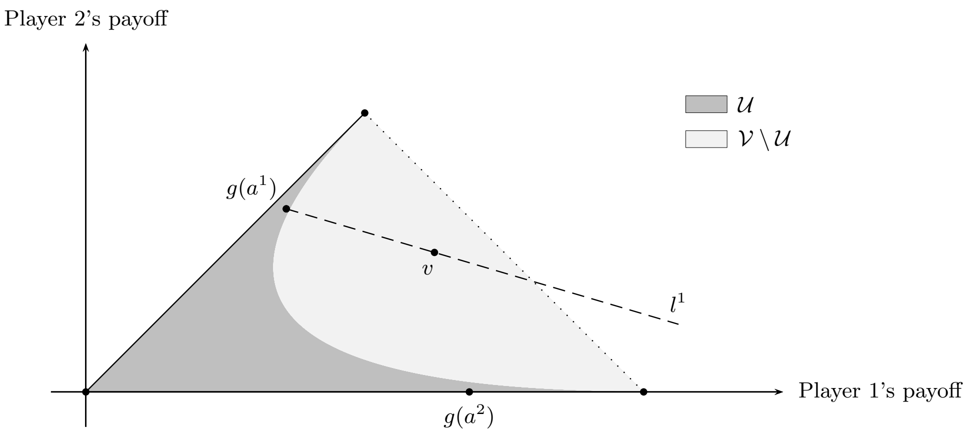

Figure 3.2: Construction with payoffs $g(a^1)$.

Thus, as Farrell and Maskin (1989) claim erroneously in their footnote 6 on page 334, the analogous construction should work for Player 2 and l^2. But, as one can clearly see in Figure 3.3, this does not hold. There are no payoffs x on line l^2 such that $x_1 < v_1$; graphically speaking, l^2 does not intersect with $\mathcal{U}$ left of v. Therefore, it is not clear how to obtain v, and the proof is not correct at this step.

In general, the proof by Farrell and Maskin (1989) fails whenever the two action profiles a^1 and a^2 are such that the constructed vectors $l^1 = g(a^1) + \lambda(v - g(a^1))$ and $l^2 = g(a^2) + \lambda(v - g(a^2))$ do not intersect with the set of mixed-strategy payoffs $\mathcal{U}$

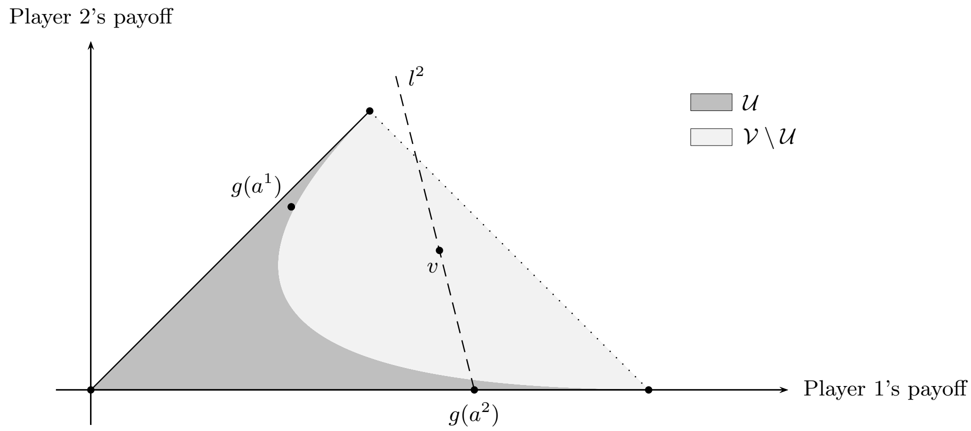

Figure 3.3: Construction with payoffs $g(a^2)$.

for $\lambda > 1$. This is not to say, though, that there are no games where the proposed construction works and Claim 1 holds true.

Note. In this example, v can still be constructed as required. If we choose the action pair $\tilde{a}^1 = (0,1)$ that corresponds to the payoffs $(2,2)$, an extreme point of $\mathcal{V}$, this action pair satisfies the conditions of the theorem. Furthermore, if we construct the line $\tilde{l}^1$ that starts in $g(\tilde{a}^1)$ and runs through v, it intersects with $\mathcal{U}$ right of v, and therefore Claim 1 holds. This, however, does not conflict with our point as the sufficient conditions of Theorem 1 are stated to hold for *any* action pairs a^1, a^2 that satisfy the conditions of the theorem. Moreover, in general $n \times m$ games, one cannot always find such an alternative action pair $\tilde{a}$ that satisfies the conditions of the theorem. Nevertheless, to fix the proof of Theorem 1, we will show that we can always find two action pairs to obtain v as a convex combination, and that this already suffices if we also modify the subsequent steps in the original proof of Farrell and Maskin (1989).

4 Corrected Proof of Theorem 1

For the proof of Theorem 1, we replace Claim 1 with the following proposition.

Proposition 1. *Let $v \in \mathcal{V}^* \setminus \mathcal{U}$. If there exist action pairs a^1, a^2 in $\mathcal{A}$ that satisfy the hypotheses of Theorem 1, that is, a^i satisfies*

(i) $g_j(a^i) \geq v_j$, $j \neq i$,

(ii) $c_i(a^i) < v_i$,

then there exist action pairs a^{1*} and a^{2*} in $\mathcal{A}$ that satisfy

(a) $g_1(a^{1*}) < v_1 < g_1(a^{2*})$, $g_2(a^{1*}) > v_2 > g_2(a^{2*})$,

(b) v is a convex combination of $g(a^{1*})$ and $g(a^{2*})$.

Given the result of Proposition 1, we can continue with the proof of Theorem 1 as follows. By Lemma FM1, we obtain that for a sufficiently large δ there exists a sequence of actions $\{a(t)\}$ with $a(t) \in \{a^{1*}, a^{2*}\}$ that yields discounted average payoffs v. To conclude the proof, we need to show that v can be established as a WRP equilibrium. Therefore, one needs to define punishments to sustain v as a subgame perfect equilibrium and that are such that there is no Pareto-ranking across any continuation equilibria of the strategy.

If a^{i*} satisfies the hypotheses of Theorem 1, it can be used to construct a penance punishment strategy for player i, as suggested by Farrell and Maskin (1989, p. 335), and the rest of the proof then follows their outline.[3] In general, however, this is not the case, and we need to construct a different punishment strategy to sustain v as a WRP equilibrium.

Given the actions a^{1*} and a^{2*} from Proposition 1, we define

$$l^* = \Big\{v \in \mathcal{V} | v = (1-\lambda)g(a^{1*}) + \lambda g(a^{2*}), \lambda \in [0,1]\Big\}$$

as the set of payoffs that lie on the line segment between $g(a^{1*})$ and $g(a^{2*})$. We will first construct Player 1's punishment and assume, without loss of generality, that $g_2(a^1) > v_2$ holds.[4]

As $c_1(a^1) < v_1$, there exists $\delta < 1$ such that

$$(1-\delta)v_1^{max} + \delta c_1(a^1) < v_1$$

and $\epsilon_1 > 0$ such that

$$c_1(a^1) < v_1 - \epsilon_1.$$

Since $g_2(a^1) > v_2$, there also exists $\epsilon_2 > 0$ such that $g_2(a^1) \geq v_2 + \epsilon_2$, and therefore we can find $\hat{\lambda} \in [0,1]$ that satisfies

$$\begin{aligned} v_1 - \frac{\epsilon_1}{2} \leq (1-\hat{\lambda})g_1(a^{1*}) + \hat{\lambda} g_1(a^{2*}) < v_1, \\ v_2 \leq (1-\hat{\lambda})g_2(a^{1*}) + \hat{\lambda} g_2(a^{2*}) \leq v_2 + \epsilon_2. \end{aligned} \tag{1}$$

Let $\tilde{\lambda} = \min_{\hat{\lambda} \in [0,1]}\{\hat{\lambda} \text{ satisfies (1)}\}$ be the minimal value for such $\hat{\lambda}$ and denote the corresponding payoff on l^* by $\tilde{v} = (1-\tilde{\lambda})g(a^{1*}) + \tilde{\lambda} g(a^{2*})$. According to Lemma FM1, there exists a sequence of $g(a^{1*})$ and $g(a^{2*})$, and a discount factor $\delta^p < 1$ such that for all $\delta \in [\delta^p, 1)$, the expected average payoff is $\tilde{v}$ and all continuation payoffs along this sequence can be limited to the line segment between $\tilde{v}$ and v. We denote the sequence of actions by $\{a^p(t)\}$ with $a^p(t) \in \{a^{1*}, a^{2*}\}$ for all t.

[3] It can be shown that for every 2×2 game, one can always identify such an action pair a^{1*}.

[4] If $g_2(a^1) = v_2$ but $g_1(a^2) > v_1$, the subsequent punishment construction can be carried out for Player 2. The case where both $g_2(a^1) = v_2$ and $g_1(a^2) = v_1$ hold is discussed in Appendix A.

Analogously, there exists a sequence of $g(a^{1*})$ and $g(a^{2*})$, and a discount factor $\delta^n < 1$ such that for all $\delta \in [\delta^n, 1)$, the expected average payoff is v and all continuation payoffs along this sequence can be limited to the line segment between v and $\tilde{v}$. This shall be the normal phase of the equilibrium strategy $\sigma(v)$, and we denote the sequence of actions by $\{a^n(t)\}$ with $a^n(t) \in \{a^{1*}, a^{2*}\}$ for all t. Note that, in general, $\delta^n \neq \delta^p$, and we shall therefore take the maximum of the two in the following steps to ensure that the continuation payoffs of $\{a^p(t)\}$ and $\{a^n(t)\}$ are limited to the line segment between $\tilde{v}$ and v. In the following, therefore, let $\delta > \bar{\delta} = \max\{\delta^n, \delta^p\}$.

The punishment of Player 1 shall be carried out as follows: In the first period, play action a^1. Then, from period $t = 1$ on, follow the sequence $\{a^p(t)\}$ with average payoff $\tilde{v}$. If Player 1 cheats during her punishment, restart with action a^1. The payoffs at the beginning of her punishment are then given by $p^1 = (1-\delta)g(a^1) + \delta\tilde{v}$, and all subsequent continuation payoffs lie on the line segment between $\tilde{v}$ and v. This construction is illustrated in Figure 4.1.

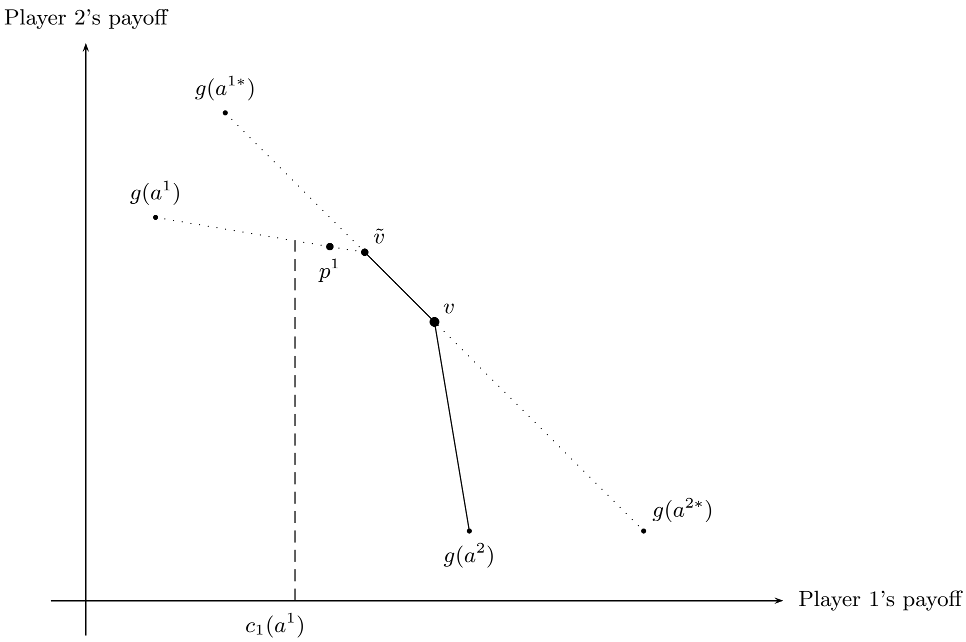

Figure 4.1: Construction of punishment for Player 1.

For the punishment of Player 2, we can adapt the construction for a penance punishment strategy, as suggested by Farrell and Maskin (1989, p. 335): After a single deviation of Player 2, play action a^2 for a suitable number of periods t_2 before returning to the normal phase with expected average payoff v. If Player 2 cheats on her punishment, restart with action a^2 (for details see Farrell and Maskin, 1989).

If Player 1 cheats during the normal phase, she receives $p_1^1 < v_1$, which satisfies

$$(1-\delta)v_1^{max} + \delta p_1^1 < v_1$$

for a sufficiently large δ. As also $p_1^1 > v_1 - \epsilon_1$ for δ sufficiently large, Player 1 has no incentive to cheat in the normal phase or on her own punishment. For Player 2, all her continuation payoffs along Player 1's punishment path are weakly greater than her equilibrium payoff v_2, and thus she has no incentive to deviate from punishing Player 1. As the same holds for Player 2's punishment, this strategy constitutes a subgame perfect equilibrium. Formally, we can define the equilibrium strategy $\sigma(v)$ as follows:

> Play begins in the normal phase, in which players are to follow the sequence $\{a^n(t)\}$. If Player 1 cheats in the normal phase, the continuation equilibrium is "play a^1 for 1 period, then follow the sequence $\{a^p(t)\}$". If Player 2 cheats in the normal phase, the continuation equilibrium is "play a^2 for t_2 periods, then return to the normal phase". If a player cheats during her punishment, the punishment begins again. If player i cheats during the opponent's punishment, then player i's punishment begins immediately.

All continuation payoffs of the normal phase and all continuation payoffs of Player 1's punishment after the first punishment period lie on the line segment between $\tilde{v}$ and v. Furthermore, all continuation payoffs of Player 2's punishment lie on the line segment between between v and $g(a^2)$, and therefore there is no Pareto-ranking between those three equilibrium paths. Finally, as $p_1^1 < \tilde{v_1}$ and $p_2^1 > \tilde{v_2}$, none of the continuation payoffs of $\sigma(v)$ are Pareto-ranked, and therefore v is a WRP equilibrium.

Note. If the punishment for Player 1 as suggested by Farrell and Maskin (1989) was followed, which is to play a^1 for a suitable number of periods and then revert to the sequence $\{a^n(t)\}$, all continuation payoffs of the punishment phase would lie on the line segment between $g(a^1)$ and v, and therefore below the line l^*. Then, even for a sufficiently large δ, it cannot be excluded that there is a strict Pareto-improvement from Player 1's punishment to the normal phase (see our discussion in Appendix B).

5 Proof of Proposition 1

As the proof of Proposition 1 is quite intricate, we will first give an elaborate outline of the proof before we formally proof the result in Subsections 5.1–5.3.

We start with the trivial observation that for every v in $\mathcal{V} \setminus \mathcal{U}$, and therefore in the interior of $\mathcal{V}$, one can always find two payoffs $v^{'}$ and $v^{''}$ in $\mathcal{U}$ such that v is a convex combination. However, it is not straightforward that the line segment between $v^{'}$ and $v^{''}$ has a negative slope to satisfy Condition (a). This will generally depend on the payoff structure of the game, and we therefore have to complete several steps to show that Condition (a) is always satisfied, given the hypotheses of the theorem.

In a first step, we will prove Proposition 1 for 2×2 games. As we are interested only in those games where $\mathcal{U}$ is a strict subset of $\mathcal{V}$, we will first give a general characterization result of the set of mixed-strategy payoffs $\mathcal{U}$ in Subsection 5.1. While the set of feasible payoffs $\mathcal{V}$ is generically a quadrilateral whose extreme points correspond to pure-strategy payoffs, we show in Lemma 1 that any payoff $v \in \mathcal{V} \setminus \mathcal{U}$ will be in a convex set that can be characterized by an edge of $\mathcal{V}$ and a convex curve between the two end-points of this edge. To show that Proposition 1 holds for 2×2 games, we must distinguish between the following two cases.

If the edge is of a positive slope, the construction given by Farrell and Maskin (1989) to show Claim 1 does not fail. That is, for at least one of the two action pairs a^1 or a^2 given by the hypotheses of the theorem, there exists an action pair $\tilde{a} \in \mathcal{A}$ such that v is a convex combination of $g(\tilde{a})$ and $g(a^i)$, and thus Proposition 1 holds immediately.

If the edge is of a negative slope, we can use a parallel line that passes through v. Due to the shape of $\mathcal{V} \setminus \mathcal{U}$, this line must intersect with two edges of $\mathcal{V}$ whose points correspond to payoffs in $\mathcal{U}$. That is, the two points of intersection yield action pairs a^{1*} and a^{2*} that fulfill the conditions of Proposition 1.

In the second step in Subsection 5.3, we extend the results for 2×2 games to general $n \times m$ games. While the set of payoffs is generically a polygon, we can identify for every $v \in \mathcal{V} \setminus \mathcal{U}$ a 2×2 game such that v is in its respective convex hull of payoffs. Then, we can use the result for 2×2 games to finally complete the proof of Proposition 1.

5.1 Characterization of $\mathcal{U}$ in 2×2 games

Consider a general 2×2 game with a payoff-matrix of the following form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2}$$

where $A, B, C, D \in \mathbb{R}^2$. For mixed strategies, we assign probabilities $(1-p)$, p to rows and $(1-q)$, q to columns of (2). To characterize the set of mixed-strategy payoffs $\mathcal{U}$, we will distinguish between four different cases. These cases will be determined by the shape of $\mathcal{V}$, i.e., the convex hull of the pure-strategy payoffs A, B, C and D. In the following text, we will therefore shift our analysis from the set of actions and payoff matrices to the space of payoffs; that is, we will study the graphs produced by the payoff function g.[5]

We will frequently make use of the following definitions.

Definition 2. For $A, B \in \mathbb{R}^2$, we will denote by $\overline{AB}$ the edge or line segment that connects A and B. The infinite line through points A and B will be denoted by $\overleftrightarrow{AB}$, and the vector that starts in A and connects A with B will be denoted by $\overrightarrow{AB}$. The triangle with extreme points A, B and C will be denoted by ΔABC.

Definition 3. We call $a = (a_1, a_2) \in \mathcal{A}$ a *semi-pure strategy* if one player plays a pure strategy, while the other chooses a mixed strategy, in which either a_1 or a_2 is equal to a standard unit vector. The set of payoffs from a semi-pure strategy is called an *inducement correspondence*.[6]

[5]For example, see Robinson and Goforth (2005) for an elaborate discussion of this approach.

[6]It is the set of payoffs that one player can "induce" by playing a pure strategy. See also Robinson and Goforth (2005).

It is straightforward that for two payoffs of the matrix (2) that are either in the same row or column, all payoffs on the edge between these two payoffs are obtainable with a semi-pure strategy. Therefore the payoff matrix (2) yields six edges, $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$ and $\overline{CD}$, and four inducement correspondences, $\overline{AB}$, $\overline{AC}$, $\overline{BD}$ and $\overline{CD}$.

Generically, the convex hull $\mathcal{V}$ of the four payoffs will be a quadrilateral in the payoff space. As we are interested in those cases where $\mathcal{U}$ is a strict subset of $\mathcal{V}$, the cases where $\mathcal{V}$ is not a two-dimensional object are of no interest for the proof of Proposition 1. It is straightforward to see that if all four points of (2) are equal, $\mathcal{V}$ is a singleton, and therefore $\mathcal{U} = \mathcal{V}$. Also, if the four payoffs are such that $\mathcal{V}$ is a line, $\mathcal{U} = \mathcal{V}$ holds.

If $\mathcal{V}$ is a two-dimensional object that is defined by at least three extreme points, the issue is more complicated and $\mathcal{U} = \mathcal{V}$ is generally not true. The set of mixed-strategy payoffs is given by

$$\mathcal{U} = \left\{v \in \mathcal{V} \;\middle|\; v = (1-p)(1-q)A + (1-p)qB + p(1-q)C + pqD; p, q \in [0,1]\right\},$$

where v can be rewritten such that

$$\mathcal{U} = \left\{v \in \mathcal{V} \;\middle|\; v = A + p(1-q)(C-A) + q(1-p)(B-A) + pq(D-A); p, q \in [0,1]\right\}.$$

Without loss of generality, we assume that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent, i.e., A, B and C are not on a line. Then we can find parameters $\beta, \gamma \in \mathbb{R}$ such that we can construct the point D as follows:

$$D = \beta(B-A) + \gamma(C-A).$$

For $\beta, \gamma \in [0,1]$ and $\beta + \gamma \leq 1$, $\mathcal{V}$ is a triangle. That is, the point D is either on the boundary or in the interior of the triangle defined by the extreme points A, B and C. In all other cases, $\mathcal{V}$ will be a quadrilateral defined by the four extreme points A, B, C and D. Two edges between these extreme points will necessarily lie in the interior of $\mathcal{V}$, and each one of these edges subdivides $\mathcal{V}$ into two triangles.

Depending on the parameters β and γ, i.e., on the position of D, there are three different cases. For $\beta < 0$, $\gamma > 0$ and $\beta + \gamma < 1$, $\overline{AC}$ is an interior edge of $\mathcal{V}$. For $\beta > 0$,$\gamma < 0$ and $\beta + \gamma < 1$, $\overline{AB}$ is an interior edge of $\mathcal{V}$. For $\beta, \gamma > 0$ and $\beta + \gamma > 1$, $\overline{AD}$ is an interior edge of $\mathcal{V}$. As we show in Lemma 1, we can neglect the last case in the following definition.

Definition 4. Let $\mathcal{V}$ be a quadrilateral with extreme points A, B, C and $D = \beta(B - A) + \gamma(C - A)$ with $\beta + \gamma < 1$. If $\beta < 0$, $\gamma > 0$, $\overline{AC}$ divides $\mathcal{V}$ into two *subtriangles*, ΔABC and ΔACD. If $\beta > 0$, $\gamma < 0$, $\overline{AB}$ divides $\mathcal{V}$ into two *subtriangles*, ΔABC and ΔABD. We denote these subtriangles by

$$\mathcal{V}^1 = \Delta ABC, \qquad \mathcal{V}^2 = \begin{cases} \Delta ACD, & \beta < 0, \gamma > 0 \\ \Delta ABD, & \beta > 0, \gamma < 0 \end{cases}.$$

Note that by definition, we have $\mathcal{V}^1 \cup \mathcal{V}^2 = \mathcal{V}$. Furthermore, for $\beta < 0 < \gamma < 1 + \beta$, we have $\mathcal{V}^1 \cap \mathcal{V}^2 = \overline{AC}$, and for $\gamma < 0 < \beta < 1 + \gamma$, we have $\mathcal{V}^1 \cap \mathcal{V}^2 = \overline{AB}$. Using this subdivision we obtain the following result.

Lemma 1. *Let $\mathcal{V}$ be the convex hull of payoffs A, B, C and D and let A, B and C not be on a line. Let $\beta, \gamma \in \mathbb{R}$ be such that $D = \beta(B - A) + \gamma(C - A)$.*

1. *In the following cases, $\mathcal{V}$ is a triangle and $\mathcal{V} \setminus \mathcal{U}$ is a convex set at the boundary of $\mathcal{V}$.*
   - *(a) $\beta, \gamma \geq 0$ and $\beta + \gamma < 1$*
   - *(b) $\gamma \leq 0$ and $\beta + \gamma \geq 1$*
   - *(c) $\beta \leq 0$ and $\beta + \gamma \geq 1$*
   - *(d) $\beta < 0$ and $\gamma < 0$*
2. *If $\beta, \gamma > 0$ and $\beta + \gamma \geq 1$, $\mathcal{U} = \mathcal{V}$.*
3. *If $\beta + \gamma < 1$ and $\beta < 0$ or $\gamma < 0$, $\mathcal{V}$ can be characterized such that $\mathcal{V}^1 \setminus \mathcal{U}$ and $\mathcal{V}^2 \setminus \mathcal{U}$ are convex sets at the boundary of $\mathcal{V}^1$ and $\mathcal{V}^2$, respectively.*

If $\mathcal{V}$ is a triangle and $\mathcal{U} \neq \mathcal{V}$, we obtain that $\mathcal{U} \subset \mathcal{V}$ is the set of payoffs between the two edges that are inducement correspondences and a convex curve between their distinct endpoints. For a payoff matrix (2) and the parameters $\beta, \gamma \geq 0$ with $\beta + \gamma < 1$, this is the area between $\overline{AB}$ and $\overline{AC}$ and the curve between B and C that is below $\overline{BC}$. An exemplary graph is given in Figure 5.1, and in the proof of the lemma (in Appendix A), we give an analytical expression for the boundary.

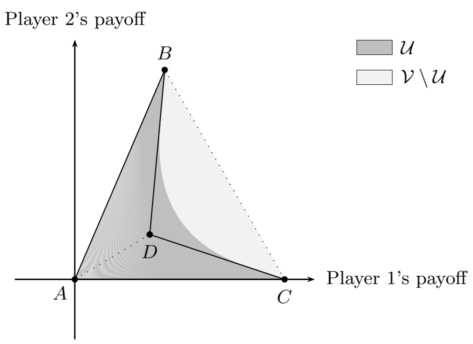

Figure 5.1: $\beta, \gamma \in (0, 1)$ and $\beta + \gamma < 1$.

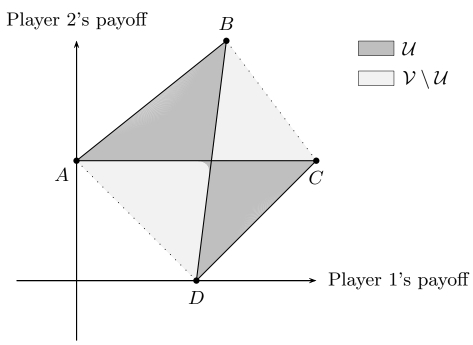

Figure 5.2: $\beta < 0, \gamma > 0$ and $\beta + \gamma < 1$.

If $\mathcal{V}$ is a quadrilateral and $\mathcal{U} \neq \mathcal{V}$, as illustrated in Figure 5.2, the characterization of $\mathcal{U}$ in the two subtriangles $\mathcal{V}^1$ and $\mathcal{V}^2$ is similar to the characterization of $\mathcal{U}$ where $\mathcal{V}$ is a triangle. In the proof of the lemma (in Appendix A), we give an analytical expression for the boundaries of $\mathcal{U}$ in the subtriangles.

Given this characterization for the set of mixed-strategy payoffs, we can now proof Proposition 1 for 2×2 games. As for the characterization, we will first consider those games where $\mathcal{V}$ is a triangle, and then we will consider the general case where $\mathcal{V}$ is a quadrilateral.

5.2 Proposition 1 for 2×2 games

Lemma 2. *Let $\mathcal{V}$ be the convex hull of payoffs A, B, C and D. Let $\beta, \gamma \in \mathbb{R}$ be such that $D = \beta(B - A) + \gamma(C - A)$. Then Proposition 1 holds.*

Proof. Those cases where $\mathcal{U} = \mathcal{V}$ can be excluded here. First, we will proof the lemma for the case where $\mathcal{V}$ is a triangle. From Lemma 1, we have that one of the edges on the boundary of $\mathcal{V}$ is not an inducement correspondence, and that this edge is also a boundary of the set $\mathcal{V} \setminus \mathcal{U}$. In the following, we will distinguish between different cases for the slope of this edge.

Without loss of generality, we assume that A, B and C are not on a line and that $\beta, \gamma \geq 0, \beta + \gamma < 1$. Then, the edge of $\mathcal{V}$ that is not an inducement correspondence is $\overline{BC}$ (see also Figure 5.1). For the characterization, we first normalize A to zero and assume, without loss of generality, that for the payoffs $B = (B_1, B_2)$ and $C = (C_1, C_2)$, we have $B_1 \leq C_1$.

First, consider those cases with $B_1 < C_1$, $B_2 > C_2$, where $\overleftrightarrow{BC}$ has a negative slope, as illustrated in Figure 5.3. As $v \in \mathcal{V}$, the line l^* that is parallel to $\overleftrightarrow{BC}$ and runs through v will intersect with both inducement correspondences $\overline{AB}$ and $\overline{AC}$. Let these points of intersections be $v^{'} = l^* \cap \overline{AB}$ and $v^{''} = l^* \cap \overline{AC}$. Since $v^{'}, v^{''}$ are mixed-strategy payoffs, there are actions a^{1*} and a^{2*} such that $g(a^{1*}) = v^{'}$ and $g(a^{2*}) = v^{''}$. Clearly a^{1*} and a^{2*} satisfy the conditions of Proposition 1. The same holds true when $\overleftrightarrow{BC}$ has an infinite slope, i.e., when $B_1 = C_1$ and $B_2 > C_2$ or $B_2 < C_2$.

Second, assume that $B_1 < C_1$ and $B_2 \leq C_2$, so that $\overleftrightarrow{BC}$ has a non-negative slope. Assume first that $\overleftrightarrow{BC}$ lies above the origin; that is, it crosses the x-axis to the left of the origin or is constant above the x-axis, as, for example, in Figure 5.4. Then, as in the original proof, construct the line l^1 that starts in $g(a^1)$ and runs through v. By the hypothesis of Theorem 1, l^1 has a negative slope and will therefore intersect with $\overline{AC}$ at a point $v^{''}$. Hence, the construction for Claim 1 works, and therefore Proposition 1 follows immediately with $a^{1*} = a^1$ and a^{2*} such that $g(a^{2*}) = v^{''}$. For the case that $\overleftrightarrow{BC}$ lies below the origin, i.e., it crosses the x-axis to the right of the origin or is constant below the x-axis, the analogous construction with l^2 works, and therefore Proposition 1 holds as well.

The proof for the case where $\mathcal{V}$ is a quadrilateral uses the same approach. As in Lemma 1, we consider the two subtriangles $\mathcal{V}^1$ and $\mathcal{V}^2$. Without loss of generality, assume that $v \in \mathcal{V}^1$. Then, one of the edges on the boundary of $\mathcal{V}^1$ is not an inducement correspondence, and this edge is also a boundary of the set $\mathcal{V}^1 \setminus \mathcal{U}$. We can now duplicate the arguments from the case where $\mathcal{V}$ is a triangle to the subtriangle $\mathcal{V}^1$ to show that Proposition 1 holds. □

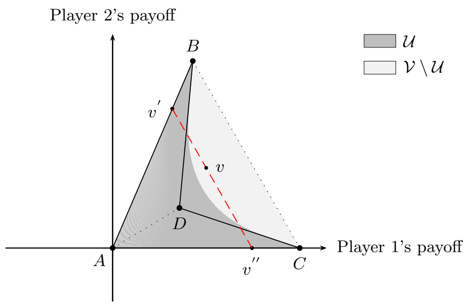

Figure 5.3: $\mathcal{V}$ for $\overleftrightarrow{BC}$ with negative slope.

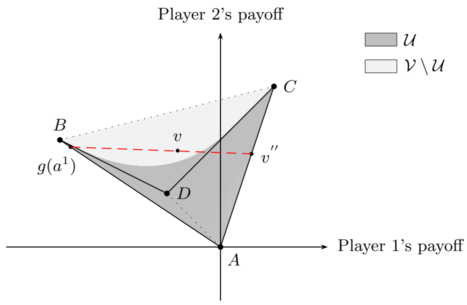

Figure 5.4: $\mathcal{V}$ for $\overleftrightarrow{BC}$ with positive slope.

5.3 Generalization to $n \times m$ games

To completely prove Proposition 1, we have to consider general $n \times m$ games with $n, m \geq 2$. The resulting convex hull of payoffs $\mathcal{V}$ in these games will generally be a polygon, as in Example 1 below. As in the proof for 2×2 games, we first characterize the set $\mathcal{U}$. To do so, we will make use of the following definition.

Definition 5. Let $\Pi = (\pi_{ij}) \in \mathbb{R}^{n \times m}$ be the payoff matrix of the $n \times m$ game g. Then, every two elements π_{ij} and π_{kl} of Π with $i \neq k$, $j \neq l$, will induce a unique 2×2 submatrix

$$\Pi_{ijkl} = \begin{pmatrix} \pi_{ij} & \pi_{il} \\ \pi_{kj} & \pi_{kl} \end{pmatrix}.$$

We define the *induced* 2×2 *game* $g|_{ijkl}$ of g as the 2×2 game restricted to those pure actions that yield payoff matrix Π_{ijkl}. The set of mixed-strategy payoffs obtainable in $g|_{ijkl}$ will be denoted by $\mathcal{U}|_{ijkl}$.

By definition, we have $\mathcal{U}|_{ijkl} \subseteq \mathcal{U}$ for every induced 2×2 game $g|_{ijkl}$ of g, and also

$$\mathcal{U}_{\{2\times 2\}} := \bigcup_{\substack{i,j,k,l \\ i\neq k, j\neq l}} \mathcal{U}|_{ijkl} \subseteq \mathcal{U}. \tag{3}$$

This characterization, together with Lemma 2, suffices to prove Proposition 1.

Proof of Proposition 1. Let $v \in \mathcal{V}^* \setminus \mathcal{U}$. By (3) we have

$$\mathcal{V}^* \setminus \mathcal{U} \subseteq \mathcal{V}^* \setminus \mathcal{U}_{\{2\times 2\}}.$$

Therefore, there exists an induced 2×2 game $g|_{ijkl}$ of g such that $v \in \mathcal{V}^* \setminus \mathcal{U}|_{ijkl}$. By Lemma 2, we have that for every $v \in \mathcal{V}^* \setminus \mathcal{U}|_{ijkl}$ with a^1 and a^2 as given in the hypotheses of Theorem 1, there are always action pairs a^{1*} and a^{2*} that satisfy Conditions (a) and (b) of Proposition 1. $\square$

Example 1. Consider the following 2×4 game with pure-strategy payoffs $A = (0, 2), B = (1, 3), C = (5, 3), D = (5, 1), E = (6, 2), F = (3, 4), G = (1, 1)$ and $H = (3, 0)$ according to the payoff matrix

$$\Pi = \begin{pmatrix} A & B & C & D \\ E & F & G & H \end{pmatrix}. \tag{4}$$

The resulting convex hull of payoffs $\mathcal{V}$ and the set $\mathcal{U}_{\{2\times 2\}}$ are illustrated in Figure 5.5. Consider payoff $v = (5.1, 2.75)$. As illustrated in Figure 5.5, v is not a mixed-strategy payoff. It is close to the edge $\overline{CE}$, which is not an inducement correspondence. This edge induces the 2×2 game $g|_{ACEG}$ with payoff matrix

$$\Pi_{ACEG} = \begin{pmatrix} A & C \\ E & G \end{pmatrix}.$$

As illustrated in Figure 5.5, v is in the convex hull of A, C, E and G. For this 2×2 game, we can apply Lemma 2 to show that Proposition 1 holds. Using the punishment strategy developed in Section 4, v can be sustained as a WRP equilibrium.

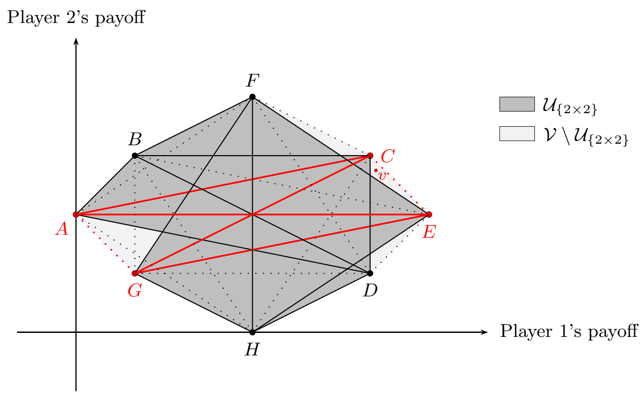

Figure 5.5: $\mathcal{V}$ is a polygon and v induces the 2×2 game $g|_{ACEG}$.

6 Conclusion

We have shown by means of a counterexample that the proposed proof of Theorem 1 in Farrell and Maskin (1989) may fail. Given a strictly individually rational payoff v and two action pairs a^1 and a^2 that satisfy the hypotheses of Theorem 1, these action pairs cannot always be used to construct a sequence that yields an average payoff v. Nevertheless, as we have shown in Proposition 1, given such action pairs a^i, we can always find two alternative actions such that their convex combination yields payoff v and that can be used to define the normal phase of the game. For the punishment strategies, we can use the action pairs a^i, although we need to design a different punishment as in the original proof to ensure that no continuation payoffs of the strategy can be strictly Pareto-ranked. Therefore, we prove that the sufficient conditions of Farrell and Maskin (1989) continue to hold, and that an equilibrium strategy exists that sustains v as a WRP equilibrium.

Appendix

A Proofs

In the proof of Theorem 1 in Section 4, we assumed that for $i \neq j$, at least one of the inequalities $g_j(a^i) \geq v_j$ is strict. In the following segment, we show that this is indeed without loss of generality by discussing the case where $g_2(a^1) = v_2$ and $g_1(a^2) = v_1$ hold, as illustrated in Figure A.1.

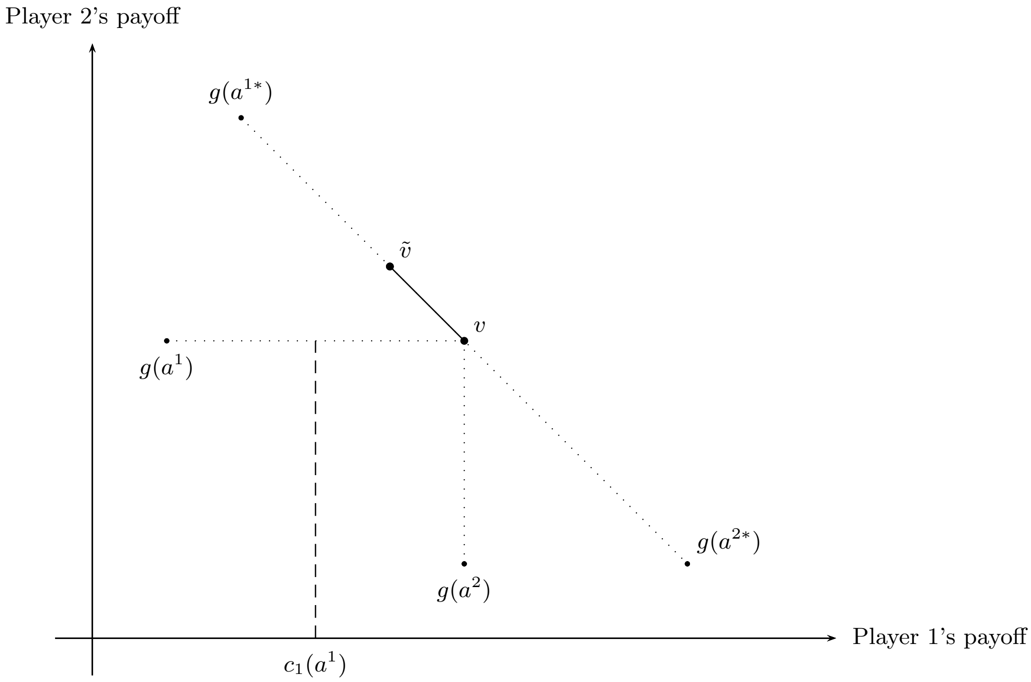

Figure A.1: Boundary case of Theorem 1.

Let $a^1 = (p^1, q^1) \in [0,1]^{n\times m}$, where p^1 and q^1 are vectors of probabilities over pure actions such that $p^1 = (p^1_1, \ldots, p^1_n)$ with $\sum_{i=1}^n p^1_i = 1$, and $q^1 = (q^1_1, \ldots, q^1_m)$ with $\sum_{i=1}^m q^1_i = 1$. By the hypotheses of the theorem, we have that $g_1(p, q^1) < v_1$ for every probability vector $p \in [0,1]^n$. If for Player 1 there is a probability vector $\tilde{p} \in [0,1]^n$ such that $g_2(\tilde{p}, q^1) > v_2$, we can use $\tilde{a}^1 = (\tilde{p}, q^1)$ to construct the punishment of Player 1. This is illustrated in Figure A.2.

If there is no such $\tilde{p}$, as illustrated in Figure A.3, we need to find a different action for Player 1's punishment. We will therefore slightly perturb Player 2's mixed strategy q^1 by $\epsilon > 0$ to obtain an action pair $a^1(\epsilon)$ that we can use to construct Player 1's punishment. We will need the following definition.

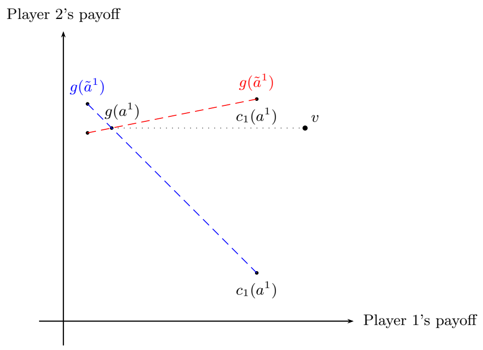

Figure A.2: The alternative payoff $g(\tilde{a}^1)$ lies on $g(\cdot, q^1)$.

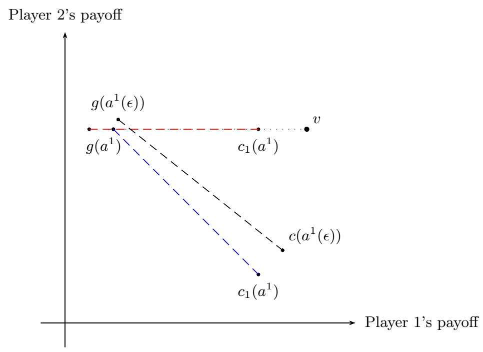

Figure A.3: The alternative action $a^1(\epsilon)$ is a perturbation of a^1.

Definition A.1. Let $q \in [0,1]^m$ be a probability vector. For $\epsilon > 0$, we define $Q(\epsilon)$ to be the set of probability vectors $q(\epsilon)$ that differ from q in every entry by at most ϵ:

$$Q(\epsilon) = \left\{ q(\epsilon) \in [0,1]^m \;\middle|\; |q_j(\epsilon) - q_j| \leq \epsilon \text{ for all } j \in \{1, \ldots, m\}, \sum_{j=1}^{m} q_j(\epsilon) = 1 \right\}.$$

Since $c_1(a^1) < v_1$, there is an entry $i \in \{1, \ldots, n\}$ such that $p_i = 1$ is a best response to q^1 and $\sum_{j=1}^{m} q_j^1 g_1(a_{ij}) < v_1$. Then, there exists an $\epsilon > 0$ such that for all $q^1(\epsilon) \in Q^1(\epsilon)$, we have

$$\sum_{j=1}^{m} q_j^1(\epsilon) g_1(a_{ij}) < v_1.$$

Furthermore, we have that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j^1(\epsilon) g_1(a_{ij}) < v_1$$

for all $p \in [0,1]^n$, and therefore $c_1(\cdot, q(\epsilon)) < v_1$.

If $g_2\Big(\cdot, q^1(\epsilon)\Big) \leq v_2$ for all $q^1(\epsilon) \in Q^1(\epsilon)$, then $g(\cdot, q^1)$ must be either an edge or an extreme point of $\mathcal{V}$, and consequently the construction for Claim 1 holds true. Otherwise, there exists $p \in [0,1]^n$ and $q^1(\epsilon) \in Q^1(\epsilon)$ with $g_2(p, q^1(\epsilon)) > v_2$, and we can use $a^1(\epsilon) = (p, q^1(\epsilon))$ to construct the punishment of Player 1.

Proof of Lemma 1. For the proof of Lemma 1, we first derive the characterization result for those cases where $\mathcal{V}$ is a triangle. For better readability, we state this in a separate lemma.

Lemma A.1. *Let $\mathcal{V}$ be the convex hull of payoffs A, B, C and D, and let A, B and C not be on a line. Let $\beta, \gamma \in [0,1]$ with $\beta + \gamma \leq 1$ be such that $D = \beta(B - A) + \gamma(C - A)$.*

1. If $\beta + \gamma = 1$, $\mathcal{U} = \mathcal{V}$.

2. If $\beta, \gamma \in (0,1)$ and $\beta + \gamma < 1$, $\mathcal{V} \setminus \mathcal{U}$ is a convex set at the boundary of $\mathcal{V}$.

Proof of Lemma A.1. First, and without loss of generality, we assume the payoff A to be normalized to zero, i.e., $A = (0,0)$. For $D = \beta B + \gamma C$ and with abuse of notation, we can rewrite $\mathcal{U}$ as

$$\begin{aligned}\mathcal{U} &= Cp\left((1-q) + q\gamma\right) + Bq\left((1-p) + p\beta\right)\\ &= Cp\left(1 - q(1-\gamma)\right) + Bq\left(1 - p(1-\beta)\right).\end{aligned}$$

Next, we define two functions $x, y : [0,1]^2 \longrightarrow [0,1]$ with $x(p,q) = p\left(1 - q\left(1-\gamma\right)\right)$ and $y(p,q) = q\left(1 - p\left(1-\beta\right)\right)$ such that $\mathcal{U}$ can be rewritten as

$$\mathcal{U} = x(p,q)C + y(p,q)B.$$

In order to determine the set $\mathcal{U}$, we will characterize its boundaries. Clearly, $\mathcal{U}$ is a subset of the convex hull of $\mathcal{V}$, and the sides $\overline{AB}$ and $\overline{AC}$ are obviously boundaries of $\mathcal{U}$. To completely characterize the shape of $\mathcal{U}$, we need to determine the remaining boundary of $\mathcal{U}$ between the two extreme points B and C. Depending on β and γ, $\mathcal{U}$ may not reach the side $\overline{BC}$, but rather lie below this edge. We can characterize this boundary that is as close as possible to $\overline{BC}$ by determining the maximal value of y for each x. Geometrically speaking, for every distance from A along the vector $\overrightarrow{AC}$, we want to find the maximal distance that we can go along the vector $\overrightarrow{AB}$.

In formal terms, we will solve the optimization problem that yields the maximal value of y for every given value of x, subject to p and q being from the unit interval. Given a value x and $\gamma > 0$, p is implicitly defined as a function of x and q by $p(x,q) = \frac{x}{1-q(1-\gamma)}$. Therefore, we can express the optimization problem only in x and q, i.e., $\max_q y(x,q)$, and as we will make use of the Karush–Kuhn–Tucker (KKT) Theorem, we state it in the following standard form:

$$\begin{aligned}&\max_q q\left(1 - \frac{x(1-\beta)}{1-q(1-\gamma)}\right)\\ &\qquad\text{s.t.}\quad q \geq 0\\ &\qquad\qquad 1 - q \geq 0\\ &\qquad\qquad \frac{x}{1-q(1-\gamma)} \geq 0\\ &\qquad\qquad 1 - \frac{x}{1-q(1-\gamma)} \geq 0\end{aligned} \tag{A.1}$$

With the Lagrange multipliers $\alpha_1, \dots, \alpha_4 \geq 0$, the necessary conditions for a solution of (A.1) are given by

$$1 - \frac{x\,(1-\beta)}{\left(1-q\,(1-\gamma)\right)^2} + \alpha_1 - \alpha_2 + \alpha_3 \frac{x(1-\gamma)}{\left(1-q\,(1-\gamma)\right)^2} - \alpha_4 \frac{x(1-\gamma)}{\left(1-q\,(1-\gamma)\right)^2} = 0 \tag{A.2}$$

$$q \geq 0 \tag{A.3}$$

$$\alpha_1 q = 0 \tag{A.4}$$

$$1 - q \geq 0 \tag{A.5}$$

$$\alpha_2(1-q) = 0 \tag{A.6}$$

$$\frac{x}{1-q(1-\gamma)} \geq 0 \tag{A.7}$$

$$\alpha_3 \left(\frac{x}{1-q(1-\gamma)} \right) = 0 \tag{A.8}$$

$$1 - \frac{x}{1-q(1-\gamma)} \geq 0 \tag{A.9}$$

$$\alpha_4 \left(1 - \frac{x}{1-q(1-\gamma)} \right) = 0 \tag{A.10}$$

As $y(x, q)$ is concave in q and all inequality constraints are linear, these necessary conditions are also sufficient. Let us first discuss the general case for $\beta \in (0, 1)$ and $\gamma \in (0, 1)$, that is, D is in the interior of $\mathcal{V}$, as illustrated in Figure 5.1.

(1) Assume $\alpha_1 > 0$ holds.

From (A.4) we obtain $q = 0$ as a possible solution and from (A.6) it follows that $\alpha_2 = 0$. Condition (A.7) is equivalent to $x \geq 0$, and from (A.9) we obtain that $x \leq 1$ must hold. Assume $\alpha_3 > 0$. Then, by (A.8) $x = 0$ must hold, but (A.2) yields a contradiction and thus $\alpha_3 = 0$ must hold. For $\alpha_4 = 0$, (A.2) is equivalent to $1 - x(1-\beta) + \alpha_1 = 0$, which is again a contradiction. Therefore, it remains to check $\alpha_4 > 0$ and hence $x = 1$. Condition (A.2) yields no contradiction, and therefore $q = 0$ is a solution if $x = 1$.

(2) Assume that $\alpha_1 = 0$ and $\alpha_2 > 0$ hold.

From (A.6) we obtain that $q = 1$ is a possible solution and from (A.7) we obtain that $x \geq 0$ must be satisfied. Also, by (A.9) we have that $x \leq \gamma$ has to hold. Assume $\alpha_3 > 0$. From (A.8) we have that $x = 0$, and therefore $\alpha_4 = 0$ must be satisfied. Condition (A.2) becomes $1 - \alpha_2 = 0$, and is therefore satisfied for $\alpha_2 = 1$.

For $\alpha_3 = 0$, assume first $\alpha_4 > 0$. That is, $x = \gamma$ needs to hold. However, (A.2) then becomes $1 - \frac{1-\beta}{\gamma} - \alpha_2 - \alpha_4 \frac{1-\gamma}{\gamma} = 0$, which is a contradiction as $\beta + \gamma < 1$. Therefore, $\alpha_4 = 0$ needs to hold. Condition (A.2) then reads $1 - x\frac{1-\beta}{\gamma^2} - \alpha_2 = 0$ and yields $x = \frac{(1-\alpha_2)\gamma^2}{1-\beta}$. This is not in conflict with (A.7) and (A.9) for $\alpha_2 \leq 1$, and therefore $q = 1$ is a solution if $x \in [0, \frac{\gamma^2}{1-\beta})$.

(3) Finally, assume that $\alpha_1 = 0$ and $\alpha_2 = 0$ hold.

First, assume $\alpha_3 > 0$. Then, by (A.8) $x = 0$ has to hold, but this yields a contradiction of (A.2). Therefore, $\alpha_3 = 0$ must hold. If we assume $\alpha_4 = 0$, we receive from (A.2) that

$q^* = \frac{1-\sqrt{x(1-\beta)}}{1-\gamma}$ is a possible solution. Condition (A.9) is only satisfied for $x \leq 1-\beta$. We now check whether this is in accordance with conditions (A.3) and (A.5), i.e., that $q^* \in [0,1]$. First, $q^* \geq 0$ if and only if $x \leq \frac{1}{1-\beta}$, which is already implied by $x \leq 1-\beta$, and which is therefore no additional constraint. Second, $q^* \leq 1$ if and only if $x \geq \frac{\gamma^2}{1-\beta}$, and therefore q^* is a solution for $x \in [\frac{\gamma^2}{1-\beta}, 1-\beta]$.

Finally, assume $\alpha_4 > 0$. Then, (A.10) yields $q^{**} = \frac{1-x}{1-\gamma}$ as a solution candidate. Inserting this into (A.2) yields that $x > 1-\beta$ must hold. Condition (A.3) is satisfied if and only if $x \leq 1$, and (A.5) holds if and only if $x \geq \gamma$. The latter is already implied by $x > 1-\beta$, and therefore q^{**} is a solution for $x \in (1-\beta, 1]$.

Summarizing, we have obtained the following optimal q^{max} as a function of x

$$q^{max}(x) = \begin{cases} 1, & x \in [0, \frac{\gamma^2}{1-\beta}) \\ \frac{1-\sqrt{x(1-\beta)}}{1-\gamma}, & x \in [\frac{\gamma^2}{1-\beta}, 1-\beta] \\ \frac{1-x}{1-\gamma}, & x \in (1-\beta, 1] \end{cases},$$

which yields for a given x the optimal y^{max} defined by

$$y^{max}(x) = \begin{cases} 1 - \frac{x(1-\beta)}{\gamma}, & 0 \leq x < \frac{\gamma^2}{1-\beta} \\ \frac{\left(1-\sqrt{x(1-\beta)}\right)^2}{1-\gamma}, & \frac{\gamma^2}{1-\beta} \leq x \leq 1-\beta \\ \frac{(1-x)\beta}{1-\gamma}, & 1-\beta < x \leq 1 \end{cases}.$$

Therefore the remaining boundary of $\mathcal{U}$, denoted by $\mathcal{U}^{max}$, is defined as

$$\mathcal{U}^{max} = \left\{ v \in \mathcal{U} \;\middle|\; xC + y^{max}(x)B, x \in [0,1] \right\}. \tag{A.11}$$

That is, we can describe this boundary of $\mathcal{U}$ between the points B and C as a tripartite curve in the Cartesian plane with two linear parts, where either $q = 1$ or $p = 1$ holds, and therefore the edge $\overline{BD}$ or $\overline{CD}$ is the boundary, respectively, and a non-linear part defined by the curve

$$xC + \frac{\left(1 - \sqrt{x(1-\beta)}\right)^2}{1-\gamma} B$$

for $x \in [\frac{\gamma^2}{1-\beta}, 1-\beta]$.

In the following boundary cases, some of the previous steps are not necessary or need to be considered differently. We will discuss them briefly.

If D is equal to one of the two extreme points B or C, the problem simplifies. For $\beta = 0, \gamma = 1$, that is, $D = C$, the function $x(p,q)$ reduces to p and $y(p,q) = q(1-p)$. Thus, given a fixed value p, $q = 1$ is always the maximizer that corresponds to the edge $\overline{BD} = \overline{BC}$. Therefore, the third boundary of $\mathcal{U}$ corresponds to the third side of $\mathcal{V}$, and therefore $\mathcal{U} = \mathcal{V}$ holds. Analogously, for $\beta = 1, \gamma = 0$, that is, $B = D$, the same approach yields $\mathcal{U} = \mathcal{V}$.

If $\beta + \gamma = 1$ holds, the point D lies on the edge $\overline{BC}$, and again $\mathcal{U} = \mathcal{V}$ holds. For values of $x \in [0, 1-\beta]$, $q = 1$ is the feasible maximizer of y, and therefore the edge $\overline{BD}$ is the boundary of $\mathcal{U}$. For $x \in (1-\beta, 1]$, $q = \frac{1-x}{\beta}$ is the feasible maximizer of y that corresponds to $p = 1$. Therefore, the edge $\overline{CD}$ is the boundary of $\mathcal{U}$ and consequently $\mathcal{U} = \mathcal{V}$.

If $\beta = 0$ and $\gamma \in (0, 1)$, the point D lies on the edge $\overline{AC}$. The tripartite boundary $\mathcal{U}^{max}$ reduces to a bipartite one. For $x \in [0, \gamma^2)$, $q = 1$ is the feasible maximizer of y, and therefore the edge $\overline{BD}$ is the boundary of $\mathcal{U}$. For $x \in [\gamma^2, 1]$, $q^* = \frac{1-\sqrt{x}}{1-\gamma}$ determines the boundary of $\mathcal{U}$.

Finally, if $\gamma = 0$, the implicit function $p(x, q)$ is not well-defined for $q = 1$. Only if $x = 0$ is this the case, and this directly yields that $q = 1$ is a feasible maximizer of y if and only if $x = 0$. If also $\beta = 0$, then for $x \in (0, 1]$, $q^* = 1 - \sqrt{x}$ is the maximizer, and therefore the boundary is completely determined by the corresponding curve. If $\beta \in (0, 1)$ holds, for $x \in (0, 1-\beta]$, $q^* = 1 - \sqrt{x(1-\beta)}$ and for $x \in (1-\beta, 1]$, $q^{**} = 1 - x$ are the respective solutions of (A.1).

It now remains to be shown that $\mathcal{V} \setminus \mathcal{U}$ is a convex set for $\beta + \gamma < 1$. First, note that $\mathcal{V} \setminus \mathcal{U}$ is determined by the edge $\overline{BC}$ and the boundary $\mathcal{U}^{max}$ derived above. The set $\mathcal{V} \setminus \mathcal{U}$ can therefore be interpreted as a simple closed curve.[7]

A closed regular plane simple curve is convex if and only if its signed curvature is either always non-negative or always non-positive (see, for example, Gray et al., 2006, pp. 163–165). If we interpret $\mathcal{U}^{max}$ as a vector function

$$f : [0, 1] \longrightarrow \mathbb{R}^2, f(x) = xC + y^{max}(x)B,$$

we can easily show that $f(x)$ is $\mathcal{C}^2$. Also, $\frac{\partial^2 y^{max}(x)}{\partial^2 x} \leq 0$ holds for all $x \in [0, 1]$, and therefore the signed curvature $\kappa(x) = \frac{f''(x)}{(1+[f'(x)]^2)^{3/2}}$ is non-positive for all $x \in [0, 1]$. As the signed curvature of the edge $\overline{BC}$ is also non-positive, the set $\mathcal{V} \setminus \mathcal{U}$ is convex. $\square$

Now, we turn to the proof of Lemma 1. For those cases where $\mathcal{V}$ is a triangle, we can use the characterization of the previous Lemma A.1. For the remaining cases, we use an analogous approach for the two triangles $\mathcal{V}^1$ and $\mathcal{V}^2$.

Without loss of generality, we assume A to be normalized to zero, that is, $A = (0, 0)$ and $D = \beta B + \gamma C$ with $\beta, \gamma \in \mathbb{R}$. First, we show that all parameter constellations given in 1.(a) – 1.(d) yield that $\mathcal{V}$ is a triangle.

In 1.(a), for $\beta, \gamma \in [0, 1]$ with $\beta + \gamma \leq 1$, the convex hull of payoffs $\mathcal{V}$ is a triangle, and therefore we can apply Lemma A.1. For $\gamma = 0, \beta \in \mathbb{R}$, the point D lies on the straight line $\overleftrightarrow{AB}$. For $\beta = 0, \gamma \in \mathbb{R}$, D lies on $\overleftrightarrow{AC}$ and for $\beta + \gamma = 1$, D lies on $\overleftrightarrow{BC}$.

[7] A curve is a simple closed curve if it is a connected curve that does not cross itself and ends at the same point where it begins.

In 1.(b), $\frac{1}{\beta}, \frac{-\gamma}{\beta} \in [0,1]$ and $\frac{1}{\beta} - \frac{\gamma}{\beta} < 1$. Therefore, $B = \frac{1}{\beta}D - \frac{\gamma}{\beta}C$ is in the interior of the triangle ΔADC. For 1.(c), analogous considerations yield that $C = \frac{1}{\gamma} - \frac{\beta}{\gamma}B$ is in the interior of the triangle ΔABD.

Finally, in 1.(d), if $\beta, \gamma \leq 0$, the origin, that is, $A = (0,0) = D - \frac{\beta}{1-\beta-\gamma}(B-D) - \frac{\gamma}{1-\beta-\gamma}(C-D)$ is in the interior of the triangle ΔDBC.

Next, we consider case 2 with $\beta, \gamma > 0$ and $\beta + \gamma \geq 1$. If $\beta + \gamma = 1$, Lemma A.1 yields that $\mathcal{U} = \mathcal{V}$. If $\beta + \gamma > 1$, D is clearly above the edge $\overline{BC}$, and this edge is an interior edge of the quadrilateral $\mathcal{V}$. Analogously to the proof of Lemma A.1, we solve the optimization problem (A.1) to show that in this case $\mathcal{U} = \mathcal{V}$ holds.

First, for $\gamma < 1$, we obtain

$$q^{max}(x) = \begin{cases} 1, & x \in [0, \gamma) \\ \frac{1-x}{1-\gamma}, & x \in [\gamma, 1] \end{cases}$$

as the solution for (A.1), which yields the following optimal y^{max} for a given x:

$$y^{max}(x) = \begin{cases} 1 - \frac{x(1-\beta)}{\gamma}, & 0 \leq x < \gamma \\ \frac{(1-x)\beta}{1-\gamma}, & \gamma \leq x \leq 1 \end{cases}.$$

If $\beta < 1$, the function $y(x,q)$ is concave in q, and therefore the Karush–Kuhn–Tucker (KKT) conditions (A.2) through (A.10) are both necessary and sufficient. If $\beta \geq 1$, the KKT conditions yield only necessary, but not sufficient, conditions. More specifically, $y(x,q)$ is strictly increasing in q, and therefore $q = 1$ is the feasible solution as long as $x < \gamma$. For all $x \geq \gamma$, the maximal value is determined by the linear function $q(x) = \frac{1-x}{1-\gamma}$, and therefore coincides with $y^{max}(x)$. Thus, the sides $\overline{BD}$ and $\overline{DC}$ are also boundaries of $\mathcal{U}$, and therefore $\mathcal{U} = \mathcal{V}$.

If $\gamma = 1$, $x(p,q)$ reduces to p and $y(p,q) = q(1-p)$. Thus, given a fixed value p, q=1 is always the maximizer that corresponds to the edge $\overline{BD}$. Therefore, the sides $\overline{BD}$ and $\overline{DC}$ are also boundaries of $\mathcal{U}$, and therefore $\mathcal{U} = \mathcal{V}$.

For $\gamma > 1$, we obtain for $x \in [0,1]$

$$q^{max}(x) = 1$$

as the solution for (A.1), which yields the following optimal $y^{max}(x)$ for a given x:

$$y^{max}(x) = 1 - \frac{x(1-\beta)}{\gamma}.$$

If $\beta > 1$, the function $y(x,q)$ is concave in q, and therefore the KKT conditions (A.2) through (A.10) are both necessary and sufficient. If $\beta \leq 1$, $y(x,q)$ is strictly decreasing in q, and therefore we need to compare $y^{max}(x)$ with $y(0,x)$. As $1 - \frac{x(1-\beta)}{\gamma} > 0$ for all $x \in [0,1]$, $y^{max}(x)$ is the solution of the optimization problem (A.1). Thus, the sides $\overline{BD}$ and $\overline{DC}$ are also boundaries of $\mathcal{U}$, and thus $\mathcal{U} = \mathcal{V}$.

Finally, let us now consider the remaining parameter constellations of case 3, which we will group into four different cases: i) $\beta < 0, \gamma \in (0,1]$, ii) $\beta < 0, \gamma > 1$ and $\beta + \gamma < 1$, iii) $\beta \in (0,1], \gamma < 0$ and iv) $\beta > 1, \gamma < 0$ and $\beta + \gamma < 1$. Four exemplary graphs for the resulting quadrilaterals are given in Figures A.4–A.7. We will only discuss the cases i) and ii), as iii) and iv) are obviously analogous.

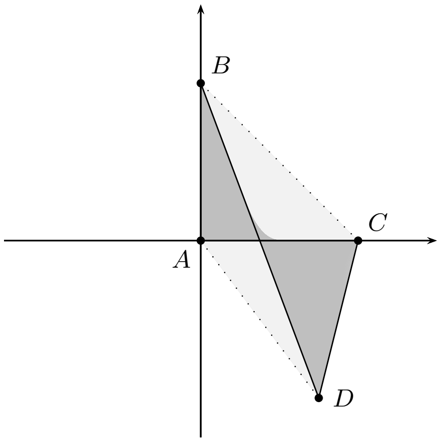

Figure A.4: $\beta < 0, \gamma \in (0, 1)$

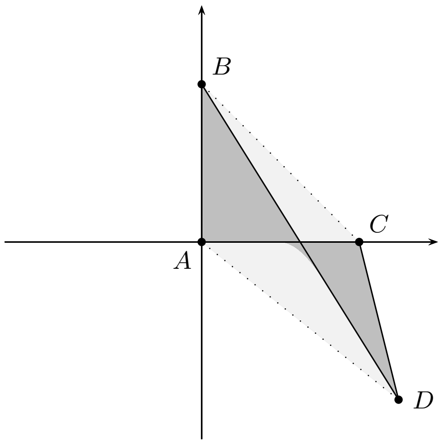

Figure A.5: $\beta < 0, \gamma > 1$

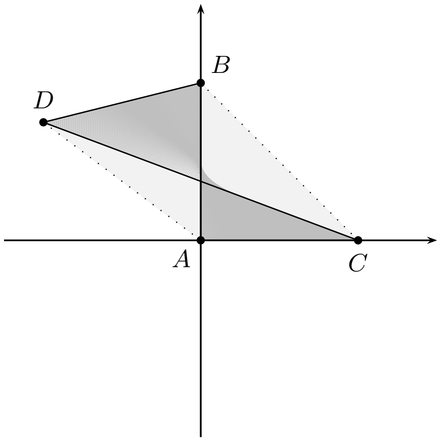

Figure A.6: $\beta \in (0, 1), \gamma < 0$

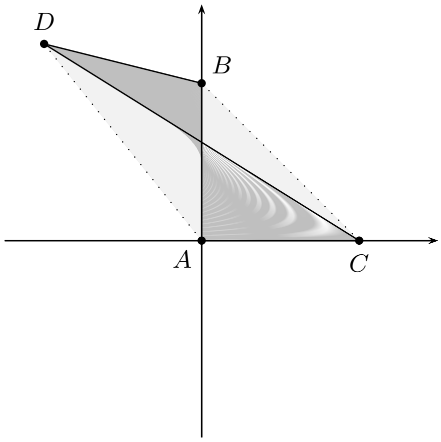

Figure A.7: $\beta > 1, \gamma < 0$

i) First, let $\beta < 0$ and $\gamma \in (0,1]$. In principle, we will follow the same approach as in the proof of Lemma A.1, but we must add several considerations. First, we note that $y(\frac{1}{1-\beta}, q) = 0$ for all $q \in [0,1]$, and since $\beta < 0$, we have to separately study $\mathcal{U}$ for values of $p < \frac{1}{1-\beta}$, $p > \frac{1}{1-\beta}$ and $p = \frac{1}{1-\beta}$. Thus, we effectively split up the convex hull of $\mathcal{V}$ into the two subtriangles $\mathcal{V}^1 = \Delta ABC$ and $\mathcal{V}^2 = \Delta ADC$, and therefore study $\mathcal{U}$ above and below $\overline{AC}$. Clearly, the edge $\overline{AC}$ is an inducement correspondence and is therefore included in $\mathcal{U}$.

We will first determine $\mathcal{U}$ for $p \in [0, \frac{1}{1-\beta})$ and $\gamma \in (0,1)$. We can follow the same steps as in the proof of Lemma A.1, and therefore solve (A.1), but for $p \in [0, \frac{1}{1-\beta})$ instead of $p \in [0,1]$. Then, given the Lagrange multipliers $\alpha_1, \ldots, \alpha_4 \geq 0$, the necessary conditions for a solution of the new optimization problem can be stated as follows:

$$1 - \frac{x\,(1-\beta)}{\left(1-q\,(1-\gamma)\right)^2} + \alpha_1 - \alpha_2 + \alpha_3 \frac{x(1-\gamma)}{\left(1-q(1-\gamma)\right)^2} - \alpha_4 \frac{x(1-\gamma)}{\left(1-q(1-\gamma)\right)^2} = 0 \quad \text{(A.12)}$$

$$q \geq 0 \quad \text{(A.13)}$$

$$\alpha_1 q = 0 \quad \text{(A.14)}$$

$$1 - q \geq 0 \quad \text{(A.15)}$$

$$\alpha_2(1-q) = 0 \quad \text{(A.16)}$$

$$\frac{x}{1-q(1-\gamma)} \geq 0 \quad \text{(A.17)}$$

$$\alpha_3 \left(\frac{x}{1-q(1-\gamma)}\right) = 0 \quad \text{(A.18)}$$

$$\frac{1}{1-\beta} - \frac{x}{1-q(1-\gamma)} \geq 0 \quad \text{(A.19)}$$

$$\alpha_4 \left(\frac{1}{1-\beta} - \frac{x}{1-q(1-\gamma)}\right) = 0 \quad \text{(A.20)}$$

As $y(x,q)$ is a concave function of q, these conditions are also sufficient and we obtain the optimal $y^{max}(x)$ for $x \in [0, \frac{1}{1-\beta})$ as follows

$$y^{max}(x) = \begin{cases} 1 - \frac{x(1-\beta)}{\gamma}, & 0 \leq x < \frac{\gamma^2}{1-\beta} \\ \frac{\left(1-\sqrt{x(1-\beta)}\right)^2}{1-\gamma}, & \frac{\gamma^2}{1-\beta} \leq x < \frac{1}{1-\beta} \end{cases}.$$

Therefore, for $p < \frac{1}{1-\beta}$ and $\gamma \in (0,1)$,

$$\mathcal{U}^{max} = \left\{ v \in \mathcal{U} \;\middle|\; xC + y^{max}(x)B, x \in [0, \tfrac{1}{1-\beta}) \right\}$$

is the boundary of $\mathcal{U}$ between the points B and C. That is, we can describe the boundary of $\mathcal{U}$ between the points B and C as a bipartite curve in the Cartesian plane with a linear part, where $q = 1$ holds, and therefore $\overline{BD}$ is the boundary, and with a non-linear part defined by the curve

$$xC + \frac{\left(1-\sqrt{x(1-\beta)}\right)^2}{1-\gamma} B$$

for $x \in [0, \frac{1}{1-\beta})$. For $\gamma = 1$, as discussed in the special cases for triangles, $x(p,q) = p$, and therefore $q = 1$ is the maximizer of $y(p,q)$ for all $p < \frac{1}{1-\beta}$. Thus, $\mathcal{U}$ is the triangle between $\overline{AB}$, $\overline{AC}$ and $\overline{BD}$. We have now completely characterized $\mathcal{U}$ in $\mathcal{V}^1$, i.e., above $\overline{AC}$ for $\gamma \in (0, 1]$.

For $p > \frac{1}{1-\beta}$, still with abuse of notation, we rewrite $\mathcal{U}$ as follows:

$$\mathcal{U} = C - (1-p)\Big(1 - \frac{q}{\beta}(1-\gamma)\Big)C + \frac{q}{\beta}\Big(1 - p(1-\beta)\Big)(D - C)$$

Next, and analogously to the proof of Lemma A.1, we define two functions $\tilde{x}(p, q) = (1-p)\left(1 - \frac{q}{\beta}(1-\gamma)\right)$ and $\tilde{y}(p, q) = \frac{q}{\beta}\left(1 - p(1-\beta)\right)$ such that

$$\mathcal{U} = C - \tilde{x}(p, q)C + \tilde{y}(p, q)(D - C).$$

As in the proof of Lemma A.1, we now determine for each $\tilde{x}$ the maximal $\tilde{y}$ such that $C - \tilde{x}(p, q)C + \tilde{y}(p, q)(D - C)$ is as close as possible to the edge $\overline{AD}$. Geometrically speaking, for every distance from C along the vector $\overrightarrow{CA}$, we want to find the maximal distance that we can go in the direction of vector $\overrightarrow{CD}$.

In formal terms, we will solve the optimization problem that yields the maximal value of $\tilde{y}$ for every given value of $\tilde{x}$, subject to $p > (\frac{1}{1-\beta}, 1]$ and $q \in [0, 1]$. Given a value $\tilde{x}$, p is implicitly defined as a function of $\tilde{x}$ and q by $p(\tilde{x}, q) = 1 - \frac{\beta\tilde{x}}{\beta - q(1-\gamma)}$.[8]

Therefore, $\tilde{y} = q\Big(1 + \frac{\tilde{x}(1-\beta)}{\beta - q(1-\gamma)}\Big)$, and we can express the optimization problem only in $\tilde{x}$ and q:

$$\begin{aligned} \max_{q}\ & \tilde{y}(\tilde{x}, q) \\ \text{s.t.}\ \ & q \in [0, 1] \\ & p(\tilde{x}, q) \in (\tfrac{1}{1-\beta}, 1] \end{aligned} \tag{A.21}$$

However, for $p > \frac{1}{1-\beta}$, $\tilde{y}$ is a strictly convex function in q, and we therefore only need to consider the two boundary points $q = 0$ and $q = 1$. We have that $\tilde{y}(\tilde{x}, 0) = 0$ and $\tilde{y}(\tilde{x}, 1) = \frac{1}{\beta}\Big(\beta + \frac{\tilde{x}(1-\beta)}{1 - \frac{1-\gamma}{\beta}}\Big) > 0$ for all $\tilde{x} \in [0, 1 - \frac{\gamma}{1-\beta})$. This is equivalent to $p \in (\frac{1}{1-\beta}, 1]$, and thus $q = 1$ is the maximizer. This corresponds to the inducement correspondence $\overline{BD}$, and hence in $\mathcal{V}^2$, $\mathcal{U}$ is the triangle between $\overline{AC}$, $\overline{CD}$ and $\overline{BD}$.

Finally, for $p = \frac{1}{1-\beta}$, we have that $y(p, q) = 0$ and $x \in [\frac{\gamma}{1-\beta}, \frac{1}{1-\beta}]$. That is, the obtainable mixed-strategy payoffs for this value of p is a subset of the inducement correspondence for $q = 0$, i.e., the edge $\overline{AC}$.

In conclusion, we have characterized $\mathcal{U}$ in $\mathcal{V}$ by splitting up $\mathcal{V}$ into two triangles $\mathcal{V}^1$ and $\mathcal{V}^2$ such that $\mathcal{U} \cap \mathcal{V}^1$ is analogous to Lemma A.1 and $\mathcal{U} \cap \mathcal{V}^2$ is a triangle. Therefore, $\mathcal{V}^1 \setminus \mathcal{U}$ and $\mathcal{V}^2 \setminus \mathcal{U}$ are convex sets at the boundary of $\mathcal{V}$.

ii) Now, let $\beta < 0, \gamma > 1$ and $\beta + \gamma < 1$. For this parameter constellation, we can follow the same approach as in i) and split up $\mathcal{U}$ into values of $p < \frac{1}{1-\beta}$, $p > \frac{1}{1-\beta}$ and $p = \frac{1}{1-\beta}$.

[8] As $\beta < 0$ and $\beta + \gamma < 1$, the implicit function $p(\tilde{x}, q)$ is well-defined for all $\tilde{x}$ and q.

For $p < \frac{1}{1-\beta}$ and $\gamma > 1$, $y(x,q)$ is a quadratic, convex function in q, and therefore the KKT conditions (A.2)–(A.10) do not yield sufficient conditions. Thus, it suffices to check the two boundary points $q = 1$ and $q = 0$. We have that $y(x,0) = 0$ and $y(x,1) = 1 - x\frac{1-\beta}{1-\gamma} > 0$ for all $x \in [0, \frac{y}{1-\beta})$. This is equivalent to $p \in [0, \frac{1}{1-\beta})$, and thus, for this range of p, in $\mathcal{V}^1$, $\mathcal{U}$ is the triangle between $\overline{AB}$, $\overline{AC}$ and $\overline{BD}$.

For $p > \frac{1}{1-\beta}$, $\tilde{y}(\tilde{x},q)$ is a concave function in q. Thus, the KKT Theorem yields sufficient and necessary conditions for a solution of the optimization problem (A.21). Given the Lagrange multipliers $\alpha_1,\ldots,\alpha_4 \geq 0$, these are:

$$1 + \frac{\tilde{x}\,(1-\beta)\,\beta}{(\beta - q\,(1-\gamma))^2} + \alpha_1 - \alpha_2 - \alpha_3\frac{\tilde{x}\beta(1-\gamma)}{(\beta - q(1-\gamma))^2} + \alpha_4\frac{\tilde{x}\beta(1-\gamma)}{(1 - q(\beta-\gamma))^2} = 0 \quad \text{(A.22)}$$

$$q \geq 0 \quad \text{(A.23)}$$

$$\alpha_1 q = 0 \quad \text{(A.24)}$$

$$1 - q \geq 0 \quad \text{(A.25)}$$

$$\alpha_2(1-q) = 0 \quad \text{(A.26)}$$

$$\frac{1}{1-\beta} + \frac{\tilde{x}}{\beta - q(1-\gamma)} \geq 0 \quad \text{(A.27)}$$

$$\alpha_3\left(\frac{1}{1-\beta} + \frac{\tilde{x}}{\beta - q(1-\gamma)}\right) = 0 \quad \text{(A.28)}$$

$$\frac{\beta\tilde{x}}{\beta - q(1-\gamma)} \geq 0 \quad \text{(A.29)}$$

$$\alpha_4\left(\frac{\beta\tilde{x}}{\beta - q(1-\gamma)}\right) = 0 \quad \text{(A.30)}$$

(1) Assume $\alpha_1 > 0$.

From (A.24) we obtain $q = 0$ as a possible solution, and from (A.26) it follows that $\alpha_2 = 0$. Condition (A.27) is equivalent to $\tilde{x} \leq \frac{\beta}{\beta-1}$, and from (A.29) we obtain that $\tilde{x} \geq 0$ must hold. Assume $\alpha_3 > 0$, then $\tilde{x} = \frac{\beta}{\beta-1}$ must hold and (A.22) is satisfied for suitable $\alpha_1, \alpha_3, \alpha_4 \geq 0$. For $\alpha_3 = 0$ and $\alpha_4 > 0$, $\tilde{x} = 0$ must hold, but then (A.22) yields a contradiction as $1 + \alpha_1 > 0$. For $\alpha_3 = 0$ and $\alpha_4 = 0$, $0 \leq \tilde{x} \leq \frac{\beta}{\beta-1}$ must hold, but then $1 + \alpha_1 + \tilde{x}(1-\beta) > 0$, which also conflicts with (A.22). Therefore, $q = 0$ is a solution if $\tilde{x} = \frac{\beta}{\beta-1}$.

(2) Assume that $\alpha_1 = 0$ and $\alpha_2 > 0$ hold.

From (A.26) we obtain that $q = 1$ is a possible solution. As $\beta + \gamma < 1$, (A.29) yields that $\tilde{x} \geq 0$ must be satisfied, and by (A.27) we have that $x \leq \frac{\beta-1+\gamma}{\beta-1}$ has to hold. Assume $\alpha_4 > 0$. From (A.30) we have that $\tilde{x} = 0$ must hold, and therefore $\alpha_3 = 0$. Condition (A.22) is satisfied for $\alpha_2 = 1$.

For $\alpha_4 = 0$, assume first that $\alpha_3 > 0$. That is, $\tilde{x} = \frac{\beta-1+\gamma}{\beta-1}$ needs to hold. But then, (A.22) becomes $1 - \frac{\beta}{\beta-1+\gamma} - \alpha_2 + \alpha_3\frac{\beta(1-\gamma)}{(1-\beta)(\beta-1+\gamma)} = 0$, which is a contradiction as $\beta + \gamma < 1$ and $\gamma > 1$. Therefore, $\alpha_3 = 0$ needs to hold and Condition (A.22) reads

$1-\frac{\tilde{x}(1-\beta)\beta}{(\beta-1+\gamma)^2}-\alpha_2=0$. Thus, $\tilde{x}=\frac{(1-\alpha_2)(1-\beta-\gamma)^2}{\beta(\beta-1)}$, and for $\alpha_2\leq 1$, (A.27) and (A.29) are satisfied. Therefore, $q=1$ is a solution if $\tilde{x}\in[0,\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)})$.
(3) Finally, assume that $\alpha_1=0$ and $\alpha_2=0$ hold.

First, assume that $\alpha_4>0$. Then, (A.30) yields that $\tilde{x}=0$ has to hold, which conflicts with (A.22). Therefore, $\alpha_4=0$ must hold.

If we assume that $\alpha_3>0$, we receive from (A.28) that $q^{**}=\frac{\beta-\tilde{x}(\beta-1)}{1-\gamma}$ is a possible solution. For (A.23) and (A.25) to be fulfilled, $\tilde{x}$ must satisfy $\tilde{x}\leq\frac{\beta}{\beta-1}$ and $\tilde{x}\geq\frac{\beta-1+\gamma}{\beta-1}$. However, for this range of $\tilde{x}$, (A.22) is not fulfilled, and therefore q^{**} is not a feasible solution.

It remains to check whether $\alpha_3=0$. In this case, (A.22) yields two possible candidates: $q^{*a}=\frac{\beta+\sqrt{\tilde{x}(\beta-1)\beta}}{1-\gamma}$ and $q^{*b}=\frac{\beta-\sqrt{\tilde{x}(\beta-1)\beta}}{1-\gamma}$. The candidate q^{*a} does not satisfy (A.25), but q^{*b} is in the unit interval for $\tilde{x}\in[\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)},\frac{\beta}{\beta-1}]$.

Summarizing, we have obtained the following optimal $q^{max}(x)$ for $x\in[0,\frac{\beta}{\beta-1}]$:

$$q^{max}(x)=\begin{cases}1, & \tilde{x}\in[0,\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)})\\ \frac{\beta-\sqrt{\tilde{x}(\beta-1)\beta}}{1-\gamma}, & \tilde{x}\in[\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)},\frac{\beta}{\beta-1}]\end{cases},$$

which yields

$$\tilde{y}^{max}(x)=\begin{cases}1+\frac{\tilde{x}(1-\beta)}{\beta-1+\gamma}, & \tilde{x}\in[0,\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)})\\ \frac{\beta-\sqrt{\tilde{x}(\beta-1)\beta}}{1-\gamma}(1+\frac{\tilde{x}(1-\beta)}{\sqrt{\tilde{x}(\beta-1)\beta}}), & \tilde{x}\in[\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)},\frac{\beta}{\beta-1}]\end{cases}.$$

Therefore, for $p>\frac{1}{1-\beta}$ we can describe the boundary of $\mathcal{U}$ below $\overline{AC}$ between the points B and C as a function with a linear part, where $q=1$, and a non-linear part, described by the curve

$$C+\tilde{x}C+\frac{\beta-\sqrt{\tilde{x}(\beta-1)\beta}}{1-\gamma}\left(1+\frac{\tilde{x}(1-\beta)}{\sqrt{\tilde{x}(\beta-1)\beta}}\right)(D-C)$$

for $\tilde{x}\in[\frac{(1-\beta-\gamma)^2}{\beta(\beta-1)},\frac{\beta}{\beta-1}]$.

Finally, for $p=\frac{1}{1-\beta}$, we have that $y(p,q)=0$ and $x\in[\frac{\gamma}{1-\beta},\frac{1}{1-\beta}]$. That is, the obtainable mixed-strategy payoffs for this value of p is a subset of the inducement correspondence for $q=0$, i.e., the edge $\overline{AC}$.

In conclusion, we have characterized $\mathcal{U}$ in $\mathcal{V}$ by splitting up $\mathcal{V}$ into the two subtriangles $\mathcal{V}^1$ and $\mathcal{V}^2$ such that $\mathcal{U}\cap\mathcal{V}^1$ is a triangle and $\mathcal{U}\cap\mathcal{V}^2$ is analogous to Lemma A.1. Therefore, $\mathcal{V}^1\setminus\mathcal{U}$ and $\mathcal{V}^2\setminus\mathcal{U}$ are convex sets at the boundary of $\mathcal{V}$. □

B On The Punishment for Player 1

Consider the following 2×2 game where Players 1 and 2 can choose between two pure actions $\{u,d\}$ and $\{l,r\}$ and mix between them with probabilities $(1-p),p$ and $(1-q),q$, respectively. Recall that we denote a mixed-strategy action by $a = (p, q)$. The stage-game payoffs of the pure strategies are given by the payoff matrix shown in Table B.1.

	l	r
u	$(0, 0)$	$(4, 0)$
l	$(0, 4)$	$(0, 0)$

Table B.1: Payoff matrix of the two-player strategic game.

Assume that $v = (1.5, 1.5)$. Then, $a^1 = (\frac{1}{2}, 0)$, $a^2 = (0, \frac{1}{2})$ satisfy the hypotheses of Theorem 1: $g(a^1) = (2, 0)$ and $g(a^2) = (0, 2)$. However, as illustrated in Figure B.1, these actions cannot be used to construct the normal phase. Nevertheless, we can easily show that $a^{1*} = (1, \frac{1}{4})$ and $a^{2*} = (0, \frac{3}{4})$ satisfy the conditions of Proposition 1, and that we can use these actions to construct the normal phase with expected average payoff v.

If we were to construct the penance punishment for Player 1 according to Farrell and Maskin (1989, p. 335), the respective continuation payoffs of the punishment phase are on the line segment between $g(a^1)$ and v, and the continuation payoff at time t is given by

$$p^1(t) = (1 - \delta^{t_1 - t})g(a^1) + \delta^{t_1 - t}v \tag{B.1}$$

for $\delta < 1$ and a sufficiently large t_1.

Let $\epsilon > 0$ and $v(\epsilon)$ be the point on the line segment between $g(a^{1*})$ and $g(a^{2*})$ with $v_1(\epsilon) = v_1 - \epsilon$. Then, by Lemma FM1, there exists a sequence of actions a^{1*} and a^{2*}, and a $\hat{\delta} < 1$ such that for all $\delta > \hat{\delta}$, the average expected payoff of the sequence is v, and all continuation payoffs are limited to the line segment between $v(\epsilon)$ and v.

However, all continuation payoffs along Player 1's punishment path will be on the line segment between $g(a^1)$ and v, which lies strictly below the line segment $g(a^{1*})$ and v. That is, when $\epsilon > 0$ is too large, Player 2's payoff on Player 1's punishment path may be smaller than in the normal phase. Since Player 1 is also better off in the normal phase, there may be a period t such that the punishment payoff $p^1(t)$ is strictly Pareto-dominated by a continuation payoff of the normal phase. That is, there is a renegotiation incentive from Player 1's punishment path to the normal phase which therefore contradicts the WRP condition. This is illustrated in Figure B.1 for $\epsilon = 0.5$.

If we decrease ϵ, according to Lemma FM1, we consequently need to increase δ. That is, for $\epsilon \longrightarrow 0$, we have that $v(\epsilon) \longrightarrow v$ but also $\delta \longrightarrow 1$. This in turn implies that, by the construction of p^1 in (B.1), we also have that $p^1(t) \longrightarrow v$. Hence, it is not clear whether in the limit there is still a Pareto-ranking between the punishment and the normal phase. In fact, we show in the following analysis that in our example, there may always be an incentive to renegotiate from the punishment to the normal phase for all $\epsilon > 0$.

According to Lemma FM1, all continuation payoffs of the normal phase satisfy

$$v_1 \in [1.5 - \epsilon(\delta), 1.5], v_2 \in [1.5, 1.5 + \gamma(\delta)]$$

for $\epsilon(\delta), \gamma(\delta) > 0$. Note that due to the selection of a^{1*} and a^{2*}, we have that $\epsilon(\delta) = \gamma(\delta)$, and from the proof of Lemma FM1 (Farrell and Maskin, 1989, p. 356) we can determine

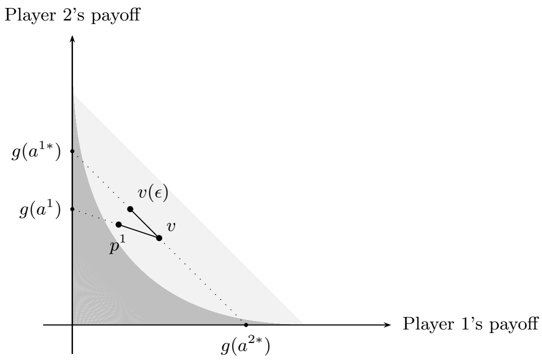

Figure B.1: Punishment- and normal-phase payoffs in the game.

the value of $\epsilon(\delta)$, which is given by

$$\epsilon = \left(\frac{1}{\delta} - 1\right)\left(g_1(a^{2*}) - g_1(a^{1*})\right) = 3\left(\frac{1}{\delta} - 1\right).$$

Let $\delta > 0.9$. Then, from the proof of Farrell and Maskin (1989, p. 335), we obtain that $t_1 = 3$ is sufficient for punishment. By (B.1), Player 1's continuation payoff on her respective punishment path at time t is given by $p_1^1(t) = 1.5\delta^{3-t}$, while Player 2 receives $p_1^2(t) = 2(1-\delta^{3-t}) + 1.5\delta^{3-t}$. Then, for any $\delta > 0.9$, we have that

$$p_1^1(0) = 1.5\delta^3 < 1.5 - \epsilon(\delta)$$
$$p_2^1(0) = 2(1-\delta^3) + 1.5\delta^3 < 1.5 + \epsilon(\delta)$$

holds. Thus, there is an incentive to renegotiate from Player 1's punishment before its start to the continuation payoff $v(\epsilon)$ of the normal phase, as illustrated in Figure B.1.

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