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10.Decomposition

The analysis of complex extensive game models can often be facilitated by the application of a decomposition principle. In perturbed games with subgames it is possible to solve the subgames first and then to replace them by their solution payoff vectors in order to obtain a "truncated game" whose solution together with the solutions of the subgames yields the solution of the whole game. The smallest units obtained by the application of this decomposition principle to the game as a whole and to its subgames are called "elementary games".

The decomposition principle has been applied successfully in other game theoretical contexts (e.g., see Selten 1973). However, it is by no means obvious that this principle can be used for the determination of evolutionarily stable strategies in extensive games. As we have seen in the last section a direct ESS of a perturbed game cannot be locally characterized as an LSS.

The results of this section will be based on the assumption that the natural symmetry is subgame preserving. In view of theorem 1 this is a very mild restriction. Eventually, it will be shown that in perturbed games with subgames the analysis of the whole game can be reduced to the successive analysis of its elementary games. A number of auxiliary results must be obtained before this goal can be achieved.

It will be important to distinguish between symmetric and asymmetric subgames. A symmetric subgame coincides with its image under the natural symmetry; an asymmetric subgame is mapped to a different subgame. It will be shown that all information sets of an asymmetric subgame are image detached. This will lead to the conclusion that the restriction of a direct ESS of a perturbed game to one of its asymmetric subgames must be a strong equilibrium point.

An analogous distinction will be made between symmetric and asymmetric elementary games. It will be shown that a direct ESS induces a direct ESS on every symmetric elementary game and that a direct ESS together with its symmetric image induces a strong equilibrium point on every asymmetric elementary game. Moreover, the induced strategies must be such that origins of subsequent subgames are reached with positive probability. These conditions are not only necessary but sufficient for a direct ESS.

Finally, theorem 10 will summarize necessary conditions for a limit ESS which can be obtained as a consequence of decomposition results for perturbed games.

10.1 Symmetric and asymmetric subgames: Let (Γ,f) be a symmetric extensive 2-person game whose natural symmetry is subgame preserving.

Let y be a decomposition point and let Γ_y be the subgame at y. If Γ_y is the symmetric image of itself, i.e. if we have $\Gamma_y = f(\Gamma_y)$ then the restriction f_y of f to the choice set C_y of Γ_y is called the natural symmetry of Γ_y in (Γ,f).

Assume that we have $\Gamma_y = f(\Gamma_y)$. It can be seen easily that the natural symmetry f_y of Γ_y has all the properties of a symmetry of Γ_y. The pair (Γ_y,f_y) is a symmetric game in the sense of definition 5.5. We call (Γ_y,f_y) the symmetric subgame of (Γ,f) at y.

Now assume that we have $\Gamma_y \neq f(\Gamma_y)$. In this case we call Γ_y an asymmetric subgame of (Γ,f). Note that symmetric subgames and asymmetric subgames are different kinds of mathematical objects. Unlike a symmetric subgame and asymmetric subgame is not endowed with a natural symmetry.

Consider an arbitrary subgame Γ_y of Γ. let b be a behavior strategy of player 1 in Γ and let b_y be that behavior strategy of player 1 in Γ_y which assigns the same local strategy as b to every information set u in Γ_y. We call this strategy b_y the strategy _induced by_ b _on_ Γ_y. For behavior strategies b' of player 2 in Γ the strategy b'_y _induced by_ b' _on_ Γ_y is defined analogously.

Let η be a perturbance of (Γ,f). For every subgame Γ_y of (Γ,f) the restriction η_y of η to the choice set C_y of Γ_y is called the _perturbance induced by_ η _on_ Γ_y. If (Γ_y,f_y) is a symmetric subgame of (Γ,f) then Γ_y,f_y and the perturbance η_y induced by η on Γ form a perturbed game $\hat{\Gamma}_y = (\Gamma_y,f_y,\eta_y)$ of (Γ_y,f_y). This perturbed gyme $\hat{\Gamma}_y$ is called the _symmetric subgame of_ $\hat{\Gamma} = (\Gamma,f,\eta)$ _at_ y.

If Γ_y is an asymmetric subgame and η_y is induced by η on Γ_y then the pair $\hat{\Gamma}_y = (\Gamma_y,\eta_y)$ is called an _asymmetric subgame_ of $\hat{\Gamma} = (\Gamma,f,\eta)$. In this case the perturbance η_y induced by η on Γ_y has all the properties of a perturbance of Γ_y as defined in 7.1 except (40) which refers to the natural symmetry f. Perturbances of this kind are called _asymmetric perturbances_. An extensive game Γ together with an asymmetric perturbance η forms an _asymmetric perturbed game_ (Γ,η) of Γ. In this sense an asymmetric subgame (Γ_y,η_y) of $\hat{\Gamma} = (\Gamma,f,\eta)$ is an asymmetric perturbed game of Γ_y.

Local strategies, behavior strategies, best replies, strong best replies, equilibrium points and strong equilibrium points of asymmetric perturbed games are defined in the same way as for perturbed games of symmetric games. The definitions of "dispersed", "permeable" and "pervasive" are also transferred in the obvious way to asymmetric perturbed games.

10.2 Remark: Let (Γ,f) be a symmetric extensive 2-person game and let Γ_y be a subgame at a decomposition point y of Γ. Let b and b' be two behavior strategies for players 1 and 2, respectively in Γ and let b_y and b'_y be the strategies induced on Γ_y by b and b', respectively. We use the symbol E_y for player 1's expected payoff in Γ_y and the symbol E_{yu}

for local payoffs in Γ_y. Let u be an information set in Γ_y which in Γ is not blocked by b'. Obviously, u is not blocked by b'_y in Γ_y either. With the help of (51) it can be seen immediately that local payoffs in Γ and in Γ_y agree in the following sense:

(80) $$E_{yu}(s_u, b_y, b'_y) = E_u(s_u, b, b')$$

for every local strategy s_u at u.

<u>10.3 Lemma 5 (image detachment of asymmetric subgames)</u>: Let (Γ, f) be a symmetric extensive 2-person game with a subgame preserving natural symmetry f and let Γ_y be an asymmetric subgame of (Γ, f). Then every information set of Γ_y is image detached.

<u>Proof</u>: Let y be the origin of Γ_y and let $\bar{y}$ be the origin of $\Gamma_{\bar{y}} = f(\Gamma_y)$. Obviously, we have $\bar{y} \neq y$. Moreover, as we shall see $\bar{y}$ does not belong to Γ_y. This can be seen as follows. y is the only element of an information set $v = \{y\}$. If $\bar{y}$ comes after y then $\Gamma_{\bar{y}}$ is a subgame of Γ_y which contains fewer information sets than Γ_y. Two different information sets must have different symmetric images, since otherwise $f(f(u)) = u$ could not hold for both of them (see (21) in 5.6). Therefore, $f(\Gamma_y)$ must have the same number of information sets as Γ_y. This shows that $\bar{y}$ cannot come after y. The same argument applied to $\Gamma_{\bar{y}}$ instead of Γ_y shows that y cannot come after $\bar{y}$. Therefore, y and $\bar{y}$ cannot be on the same play. If u is an information set in Γ_y then y is on every play which intersects u. Moreover, f(u) belongs to $\Gamma_{\bar{y}}$ and $\bar{y}$ is on every play which intersects f(u). This shows that there cannot be any play which intersects both u and f(u). All information sets in Γ_y are image detached.

<u>10.4 Lemma 6 (local characterization of strong equilibrium points)</u>: Let $\hat{\Gamma} = (\Gamma, \eta)$ be a asymmetric perturbed game of an extensive 2-person game Γ and let r and r' be behavior strategies of players 1 and 2, respectively in $\hat{\Gamma}$.

The pair (r,r') is a strong equilibrium point of $\hat{\Gamma}$, if and only if both of the following conditions are satisfied:

(i) For every essential information set u in Γ the realization probability $\gamma(u,r,r')$ is positive.

(ii) For every information set u of player 1 or 2 in Γ the local strategy r_u or r'_u assigned to u by r or r', respectively is a strong local best reply to r and r' in $\hat{\Gamma}$.

Proof: Assume that (r,r') is a strong equilibrium point of $\hat{\Gamma}$. Suppose that (i) is not satisfied. Let u be an essential information set of player 1 or 2 with $\gamma(u,r,r')=0$. Let s_u be a local strategy at u in $\hat{\Gamma}$ which is different from the local strategy assigned to u by r or r', respectively. If u is an information set of player 1 then $s = r/s_u$ is an alternative best reply to r' and if u is an information set of player 2, then $s' = r'/s_u$ is an alternative best reply to r. In both cases expected payoffs are not influenced by the local strategy change. Therefore, (i) holds if (r,r') is a strong equilibrium point of $\hat{\Gamma}$.

Suppose that (i) holds and that (ii) is violated for at least one information set u of player 1 or 2. Since the assertion of the lemma is completely symmetric with respect to players 1 and 2, it is sufficient to examine the case that u belongs to player 1. Let u be an information set of player 1, such that the local strategy r_u assigned by r to u is not a strong local best reply to r' in $\hat{\Gamma}$. Let s_u be a local best reply to r' at u in $\hat{\Gamma}$ with $s_u \neq r_u$. The local payoff at u remains unchanged by a local strategy change to $s = r/s_u$. It follows by (b) in 8.4 that player 1's global payoff remains unchanged if he uses s instead of r against r'. This shows that s is a best reply to r' in $\hat{\Gamma}$. In view of $s \neq r$ the equilibrium point (r,r') fails to be strong, contrary to the assumption made above. Therefore, (ii) holds.

Now assume that (i) and (ii) are satisfied. Since the assertion of the lemma is symmetric with respect to players 1 and 2, it is sufficient to examine the case that player 1 has an alternative best reply. Let S be the set of all best replies s with $s \neq r$ to r' in $\hat{\Gamma}$. For any $s \in S$ let s_u be the local strategy assigned to an information set of player 1 by s and let k(s) be the number of information sets of player 1 with $s_u \neq r_u$. Let $s \in S$ be a strategy such that for no $t \in S$ we have $k(t) < k(s)$. We shall proceed in a similar fashion as in the proof of lemma 3.

Let u be an information set of player 1 with $s_u \neq r_u$ such that u does not precede any information set v of player 1 with $s_v \neq r_v$. In view of (51) we must have:

(81) $E_v(t_v,s,r') = E_v(t_v,r,r')$

for every local strategy t_v at v. Since in view of (ii) the local strategy s_v cannot be a local best reply to r and r' in $\hat{\Gamma}$, it cannot be a local best reply to s and r' in $\hat{\Gamma}$, either. This yields:

(82) $E_v(s_v,s,r') < E_v(r_v,s,r')$

Consider the strategy $t = s/r_v$. If $\gamma(v,s,r')$ were positive then t would yield a higher global payoff than s against r' (see (d) in 8.4). Therefore, we must have $\gamma(v,s,r') = 0$. However, in this case a local change at v has no influence on global payoffs. Consequently, t is in S. Moreover, $k(t)=k(s)-1$ contrary to the assumption that s is minimal with respect to k(s). Therefore, r must be a strong best reply to r' in $\hat{\Gamma}$.

We can conclude that (r,r') is a strong equilibrium point of $\hat{\Gamma}$ if (i) and (ii) are satisfied.

<u>10.5 Theorem 6 (strong equilibrium points in asymmetric subgames):</u>
Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) with a subgame preserving natural symmetry f and let $\hat{\Gamma}_y = (\Gamma_y,\eta)$ be an asymmetric subgame of $\hat{\Gamma}$. Let b be a direct ESS of $\hat{\Gamma}$. Let b_y and b'_y be the strategies induced

on Γ_y by b and $b' = f(b)$, respectively. Then (b_y, b'_y) is a strong equilibrium point of $\hat{\Gamma}_y$.

Proof: Let b_u and b'_u be the local strategies assigned by b and b', respectively to information sets of players 1 or 2. In view of lemma 6 it is sufficient to show that the following statements are true.

(i) For every essential information set u in Γ_y the realization probability $\gamma_y(u, b_y, b'_y)$ of u in Γ_y is positive.

(ii) For every information set u of player 1 or 2 in Γ_y the local strategy b_u or b'_u assigned to u by b_y or b'_y is a strong local best reply to b_y and b'_y in $\hat{\Gamma}_y$.

Since b is a direct ESS of $\hat{\Gamma}$ it follows by theorem 2 that b is pervasive. In view of lemma 2 we can conclude that the realization probabilities $\gamma(u, b, b')$ of essential information sets in Γ are positive. Consequently, (i) holds.

Lemma 5 shows that every information set of Γ_y is image detached. It follows by (a) in theorem 4 and by (a') in 9.3 that for every information set u of player 1 or 2 in Γ the local strategy b_u or b'_u is a strong local best reply to b and b' in Γ. Since local payoffs in Γ and Γ_y agree in the sense of (80) in 10.2, we can conclude that (ii) holds.

10.6 Theorem 7 (induced ESS on symmetric subgames): Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ, f) with a subgame preserving natural symmetry f. Let $\hat{\Gamma}_y = (\Gamma_y, f_y, \eta_y)$ be a symmetric subgame of $\hat{\Gamma}$. Let b be a direct ESS of $\hat{\Gamma}$. Then the strategy b_y induced by b on $\hat{\Gamma}_y$ is a direct ESS of $\hat{\Gamma}_y$.

Proof: Theorem 2 shows that b is pervasive. In view of lemma 2 all realization probabilities $\gamma(u, b, f(b))$ of essential information sets are positive. Consequently, the same is true for all realization probabilities $\gamma_y(u,b_y,f_y(b_y))$ of essential information sets u in Γ_y. In view of lemma 2 we can conclude that b_y is pervasive in (Γ_y,f_y).

Since local payoffs in Γ and Γ_y agree in the sense of (80) in 10.2 it is clear that b_u is a local best reply to b_y and $f_y(b_y)$ in $\hat{\Gamma}_y$, if and only if b_u is a local best reply to b and f(b) in $\hat{\Gamma}$. Since b is a pervasive symmetric equilibrium strategy, it follows by theorem 3 that for every information set u of player 1 the local strategy b_u assigned to u by b is a local best reply to b and f(b) in $\hat{\Gamma}$. We can conclude that for every information set u of player 1 the local strategy b_u assigned to u by b_y is a local best reply to b_y and $f_y(b)$ in $\hat{\Gamma}_y$. Since b_y is pervasive it follows by theorem 3 that b_y is a symmetric equilibrium strategy for (Γ_y,f_y).

It remains to be shown that b_y satisfies the second condition in the definition of a direct ESS for a perturbed game (see 9.6). Assume that r_y with $r_y \neq b_y$ is a best reply to $f_y(b_y)$ in $\hat{\Gamma}_y$ and that in addition to this we have:

$$E_y(b_y,f_y(b_y)) \leq E_y(r_y,f_y(b_y)) \tag{83}$$

Let r be that behavior strategy of player 1 in Γ which agrees with r_y at information sets in Γ_y and with b at information sets outside Γ_y. Local strategy changes at information sets in Γ_y do not influence realization probabilities of endpoints outside of Γ_y. This together with the fact that r_y is a best reply to $f_y(r_y)$ in $\hat{\Gamma}_y$ permits the conclusion that r is a best reply to f(b) in $\hat{\Gamma}$. Moreover, we can conclude that the following is true:

$$E(b,f(r)) \leq E(r,f(r)) \tag{84}$$

In view of the second condition in the definition of a direct ESS for a perturbed game (84) contradicts the assumption that b is a direct ESS of $\hat{\Gamma}$. Consequently, an alternative best reply r_y to $f_y(b_y)$ with (84) cannot be found and b_y is a direct ESS of $\bar{\Gamma}_y$.

10.7 Truncations and elementary games: Let Γ be an extensive 2-person game and let b and b' be behavior strategies of players 1 and 2 for Γ. A multisubgame M of Γ is a non-empty set of subgames of Γ which are pairwise non-intersecting in the sense that two subgames in M do not have any vertex in common. Let M be a multisubgame of Γ.

We shall define a game $\bar{\Gamma} = (\bar{K},\bar{P},\bar{V},\bar{C},\bar{p},\bar{h},\bar{h}')$ which will be called the (b,b')-truncation of Γ with respect to M. The components of $\bar{\Gamma}$ are as follows:

(a) the origin o of $\bar{K}$ is the origin of K. The decision points of $\bar{K}$ are those decision points of K which do not belong to subgames in M. The endpoints of $\bar{K}$ are either endpoints of Γ which do not belong to subgames in M or origins of subgames of M. The origins of subgames in M are also called decomposition endpoints of $\bar{K}$,

(b) P,V,C and $\bar{p}$ are the restrictions of P,U,C and p, respectively to $\bar{K}$.

(c) Let z be an endpoint of $\bar{\Gamma}$ which also belongs to Γ. Then we have:

(85) $\bar{h}(z) = h(z)$

and

(86) $\bar{h}'(z) = h'(z)$

Let y be the origin of a subgame $\Gamma_y \in M$ and let b_y and b'_y be the strategies induced by b and b', respectively on Γ_y. then we have:

(87) $\bar{h}(y) = E_y(b_y,b'_y)$

(88) $\bar{h}'(y) = E'_y(b_y,b'_y)$

where E_y and E'_y denote the expected payoff functions of players 1 and 2 in Γ_y.

Let η b a perturbance for Γ and let $\bar{\eta}$ be the restriction of η to the union $\bar{C}_1 \cup \bar{C}_2$ of the choice sets of players 1 and 2 in $\bar{\Gamma}$. Then $\tilde{\Gamma} = (\bar{\Gamma},\bar{\eta})$ is called the (b,b')-truncation of $\hat{\Gamma} = (\Gamma,\eta)$. Note that b and b' need not belong to $\hat{B}$ and $\hat{B}'$, respectively; (b,b')-truncations of $\hat{\Gamma} = (\Gamma,\eta)$ can be formed, too, if b and b' do belong to Γ but not to $\hat{\Gamma}$.

A subgame Γ_y of Γ is called <u>maximal</u> if Γ_y is not a subgame of another subgame of Γ. It is clear that two maximal subgames cannot have any vertices in common. Therefore, the set of all maximal subgames of Γ is a multisubgame. The (b,b')-truncation of Γ with respect to this multisubgame is called the <u>main (b,b')-truncation</u> of Γ; in the case that Γ has no subgames, this name is applied to Γ itself. The main (b,b')-truncation $\tilde{\Gamma} = (\bar{\Gamma},\bar{\eta})$ of a perturbed game (Γ,η) is formed by the main (b,b')-truncation $\bar{\Gamma}$ of Γ together with the restriction $\bar{\eta}$ of η to the choices in $\bar{\Gamma}$.

Let f be a subgame preserving symmetry of Γ. A multisubgame M of Γ is called <u>symmetric multisubgame of</u> (Γ,f) if for every $\Gamma_y \in M$ the symmetric image $f(\Gamma_y)$ belongs to M, too.

It can be seen easily that the symmetric image $f(\Gamma_y)$ of a maximal subgame Γ_y is maximal; if $f(\Gamma_y)$ were a subgame of a subgame $\Gamma_{\bar{y}}$ then $f(\Gamma_{\bar{y}})$ would contain Γ_y as a subgame and Γ_y could not be maximal. Therefore, the set of all maximal subgames of Γ is symmetric with respect to f.

Assume b' = f(b) and let M be a symmetric multisubgame of (Γ,f) where f is subgame preserving. Consider the (b,b')-truncation $\bar{\Gamma}$ of Γ with respect to M. Since f is subgame preserving, the symmetric image f(u) of an information set u in $\bar{\Gamma}$ is an information set of $\bar{\Gamma}$. With the help of (85) to (87) it can be seen that the restriction $\bar{f}$ of f to $\bar{\Gamma}$ is a symmetry of $\bar{\Gamma}$. We call $(\bar{\Gamma},\bar{f})$ the <u>b-truncation of</u> (Γ,f) <u>with respect to</u> M. In view of b' = f(b) it is convenient to use a name which does not explicitly mention b'. The b-truncation of (Γ,f) with respect to the set of all maximal subgames of Γ is called the <u>main b-truncation of</u> (Γ,f).

Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of (Γ, f) and let $\bar{\eta}$ be the restriction of η to the choices in the b-truncation $(\bar{\Gamma}, \bar{f})$ with respect to a symmetric multisubgame M. Then the perturbed game $\tilde{\Gamma} = (\bar{\Gamma}, \bar{f}, \bar{\eta})$ is called the <u>b-truncation of $\hat{\Gamma}$ with respect to</u> M. If M is the set of all maximal subgames of Γ, then $\tilde{\Gamma}$ is the <u>main b-truncation of</u> $\hat{\Gamma}$.

<u>An elementary game of</u> $\hat{\Gamma} = (\Gamma, f, \eta)$ or shortly a <u>b-element of</u> $\hat{\Gamma}$ is a game $\tilde{\Gamma}$ which fits one of the following descriptions (i), (ii) or (iii):

(i) $\tilde{\Gamma} = (\bar{\Gamma}, \bar{f}, \bar{\eta})$ is the main b-truncation of $\hat{\Gamma}$

(ii) $\tilde{\Gamma} = (\bar{\Gamma}_y, \bar{f}_y, \bar{\eta}_y)$ is the main b_y-truncation of a symmetric subgame (Γ_y, f_y, η_y) of $\hat{\Gamma}$ where b_y is the strategy induced by b on Γ_y.

(iii) $\tilde{\Gamma} = (\bar{\Gamma}_y, \bar{\eta}_y)$ is the main (b_y, b'_y)-truncation of an asymmetric subgame $\hat{\Gamma}_y = (\Gamma_y, \eta_y)$ of $\hat{\Gamma}$ where b_y and b'_y are the strategies induced by b and f(b), respectively on Γ_y.

A behavior strategy for a b-element is called <u>induced</u> by a behavior strategy for the whole game if both strategies assign the same local strategies to the information sets in the b-element. The term "induced" will also be used analogously for truncations in general.

A b-element is called <u>symmetric</u>, if it fits one of the descriptions (i) and (ii) and <u>asymmetric</u> in the case of (iii). From what has been said before, it is clear that symmetric b-elements are symmetric games.

Since we do not distinguish between (Γ, f) and the special case of a perturbed game of (Γ, f) where all minimum probabilities are zero, the definition of b-elements also covers the case of an unperturbed symmetric extensive two-person game.

A maximal subgame of a perturbed game $\hat{\Gamma} = (\Gamma, f, \eta)$ either is a symmetric subgame (Γ_y, f_y, η_y) or an asymmetric subgame (Γ_y, η_y) of $\hat{\Gamma}$ where Γ_y is a maximal subgame of Γ.

We say that a subgame is blocked by a behavior strategy if the information set containing the origin of Γ_y is blocked by this strategy. A subgame or more generally an extensive game is called essential, if it contains at least one essential information set (see 8.6 for the definition of "essential").

10.8 Comment: The strategies induced by a direct ESS of a perturved game on subgames are characterized by theorems 6 and 7. It is necessary to derive a similar result for truncations with respect to symmetric multigames. This will be the content of lemma 8. On the basis of lemma 8, theorem 8 will characterize a direct ESS of a perturbed game in terms of the strategies induced on the subgames in a symmetric multisubgame and the truncation with respect to this subgame. Theorem 9 will give a similar characterization in terms of strategies induced on elementary games.

Even if all the results of this section may seem to be fairly obvious at first glance, careful proofs require much more detail than one might think.

It will be necessary to derive a result which is not directly connected to subgames and truncations. If one wants to check whether a symmetric equilibrium strategy is an ESS, one can restrict ones attention to alternative best replies which do not differ from the ESS on information sets blocked by the alternative best reply. This will be the content of lemma 7. One can expect that this lemma is useful not only as a step towards further results, but also as a tool for the analysis of specific game models.

10.9 Lemma 7 (on alternative best replies): Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) and let b be a symmetric equilibrium strategy for $\hat{\Gamma}$. If b is not a direct ESS of $\hat{\Gamma}$, then a best reply r to f(b) in $\hat{\Gamma}$ with $r \neq b$ and

(89) $\qquad E(b,f(r)) \leq E(r,f(r))$

can be found which has the following additional property: If an information set u of player 1 is blocked by r, then b and r assign the same local strategy to u.

<u>Proof</u>: Let R be the set of all best replies r to f(b) in $\tilde{\Gamma}$ with $r \neq b$ and (89). Assume that b is not a direct ESS. It follows by (89) together with condition (b) in definition 7.5 of a direct ESS for a perturbed game that R is not empty. For every $r \in R$ let k(r) be the number of information sets of player 1, blocked by r, where b and r prescribe different local strategies. Let $r \in R$ be a strategy with $k(r) \leq k(t)$ for every $t \in R$.

We have to show $k(r) = 0$. Assume $k(r) > 0$ and let u be an information set of player 1, blocked by r, where the local strategies b_u and r_u assigned to u by b and r, respectively are different from each other.

Consider the strategy $s = r/b_u$. An information set v is blocked by s, if and only if it is blocked by r. It does not matter that different local strategies are prescribed at an information set which is not reached anyhow. Therefore, an information set v is blocked by f(s), if and only if it is blocked by f(r). Consequently, (89) continues to hold, if f(r) is replaced by f(s) on both sides. Moreover, the payoff on the right hand side remains unchanged if in addition to this r is replaced by s, since u is blocked by r and by s. This shows that we have:

(90) $\qquad E(b,f(s)) \leq E(s,f(s))$

Moreover, it is clear that s is a best reply to f(b) in $\hat{\Gamma}$ since in view of the fact that u is blocked by r and s the payoffs E(r,f(b)) and E(s,f(b)) must be equal. Therefore s belongs to R. However $k(s) < k(r)$ contrary to the assumption that r is minimal in R with respect to $k(r)$. Therefore, the assertion of the lemma is true.

10.10 Lemma 8 (truncation lemma): Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ, f) with a subgame preserving natural symmetry f. Let M be a symmetric multisubgame of (Γ, f) and let b be a direct ESS for $\hat{\Gamma}$. Then the strategy $\bar{b}$ induced by b on the b-truncation $\tilde{\Gamma} = (\bar{\Gamma}, \bar{f}, \bar{\eta})$ of $\hat{\Gamma}$ with respect to M is a direct ESS of this b-truncation $\tilde{\Gamma}$.

Proof: We use the symbol $\bar{E}_u$ in order to denote local payoffs in $\bar{\Gamma}$ at an information set u of $\bar{\Gamma}$. It can be seen immediately that local payoffs in Γ and $\bar{\Gamma}$ satisfy the following condition:

$$(91) \qquad \bar{E}_u(s_u, \bar{b}, \bar{f}(\bar{b})) = E_u(s_u, b, f(b))$$

for every local strategy s_u at u. Since b is pervasive by theorem 2, it is clear that $\bar{b}$ is pervasive in $\bar{\Gamma}$. Local payoffs are defined everywhere. It follows by theorem 3 that b satisfies local conditions which again by theorem 3 together with (91) permit the conclusion that $\bar{b}$ is a pervasive symmetric equilibrium strategy for $\tilde{\Gamma}$. Assume that nevertheless $\bar{b}$ is not a direct ESS of $\tilde{\Gamma}$. Then we can find an alternative best reply $\bar{r}$ to $\bar{f}(\bar{b})$ in $\tilde{\Gamma}$ with $\bar{r} \neq \bar{b}$ and with the following property:

$$(92) \qquad \bar{E}(\bar{b}, \bar{f}(\bar{r})) \leq \bar{E}(\bar{r}, \bar{f}(\bar{r}))$$

where $\bar{E}$ denotes the expected payoff function for $\bar{\Gamma}$. This follows by the second condition (b) in definition 7.5 of a perturbed game direct ESS. Let r be that behavior strategy of player 1 for $\hat{\Gamma}$ which agrees with $\bar{r}$ on $\bar{\Gamma}$ and with b on all subgames in M. It is clear that we have:

$$(93) \qquad E(r, f(b)) = \bar{E}(\bar{r}, f(\bar{b})) = \bar{E}(\bar{b}, f(\bar{b})) = \bar{E}(b, f(b))$$

This shows that r is an alternative best reply to f(b) in

$\hat{\Gamma}$. Moreover, we have:

(94) $\qquad \bar{E}(\bar{b},\bar{f}(\bar{r})) = E(b,f(r))$

and

(95) $\qquad \bar{E}(\bar{r},f(\bar{r})) = E(r,f(r))$

Inequality (92) together with (94) and (95) shows that r violates the second condition (b) in definition 7.5 of a perturbed game direct ESS. Therefore, no alternative best reply $\bar{r}$ with (92) can be found. $\bar{b}$ is a direct ESS of $\hat{\bar{\Gamma}}$.

<u>10.11 Theorem 8(truncation theorem)</u>: Let $\hat{\Gamma}= (\Gamma,f\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) with a subgame preserving symmetry f. Let M be a symmetric multisubgame of (Γ,f) and let b be a behavior strategy of player 1 for $\hat{\Gamma}$. Then b is a direct ESS of $\hat{\Gamma}$, if and only if the following conditions are satisfied for all $\Gamma_y \in M$ and for the b-truncation $\hat{\bar{\Gamma}}=(\bar{\Gamma},\bar{f},\bar{\eta})$ of $\hat{\Gamma}$ with respect to M:

(i) The strategy $\bar{b}$ induced by b on $\tilde{\Gamma}$ is a direct ESS of $\tilde{\Gamma}$. Moreover if $\Gamma_y \in M$ is essential, then the origin y of Γ_y has a positive realization probability $\bar{\gamma}(y,\bar{b},\bar{f}(\bar{b}))$ under $\bar{b}$ and $f(\bar{b})$) in $\bar{\Gamma}$.

(ii) If $\hat{\Gamma}_y = (\Gamma_y,\eta_y)$ is an essential asymmetric subgame of $\hat{\Gamma}$, then the pair (b_y,b'_y) of strategies induced by b and f(b), respectively on $\hat{\Gamma}_y$ is a strong equilibrium point of $\hat{\Gamma}_y$.

(iii) If $\hat{\Gamma}_y = (\Gamma_y,f_y,\eta_y)$ is an essential symmetric subgame of $\hat{\Gamma}$, then the strategy b_y induced by b on Γ_y is a direct ESS of $\hat{\Gamma}_y$.

<u>Proof</u>: It follows by lemma 8 that $\bar{b}$ is a direct ESS of $\tilde{\Gamma}$ if b is a direct ESS of $\hat{\Gamma}$. Moreover, b could not be pervasive if $\bar{\gamma}(y,\bar{b},f(\bar{b}))$ would not be positive for the origin y of every essential subgame in M. If b is a direct

ESS of $\hat{\Gamma}$, then (ii) and (iii) follow by theorems 6 and 7.

It remains to show that (i), (ii) and (iii) together imply that b is a direct ESS of $\hat{\Gamma}$. It will first be shown that b is pervasive, then that b is a symmetric equilibrium strategy and finally that b is a direct ESS.

In view of theorem 2 it is clear that the strategies $\bar{b}$ and b_y in (i) and (iii) are pervasive. Lemma 6 shows that the realization probabilities $\gamma_y(u,b_y,b_y')$ of essential information sets u in asymmetric subgames $\Gamma_y \in M$ are positive. Since $\bar{\gamma}(y,\bar{b},f(\bar{b}))$ is positive for origins y of essential subgames in M we can conclude that b is pervasive. Note that without the assumption on $\bar{\gamma}(y,\bar{b},f(\bar{b}))$ the pervasiveness of b would not follow.

In order to prove that b is a symmetric equilibrium strategy of $\hat{\Gamma}$ we show that the local conditions required by theorem 3 are satisfied. In this respect, it is important to remember that local payoffs in a subgame agree with local payoffs in the whole game in the sense of (80) in 10.2. Moreover, we can make use of (91) in the proof of lemma 8. Therefore, the local optimality conditions implied by theorem 3 and lemma 6 applied to the induced strategies in (i), (ii) and (iii) are nothing else than the local optimality conditions required for b by theorem 3. This shows that b is a pervasive symmetric equilibrium strategy of $\hat{\Gamma}$.

It remains to show that the second condition in definition 7.5 of a perturbed game direct ESS is satisfied for b. We can restrict our attention to best replies to f(b) in $\hat{\Gamma}$ with $r \neq b$ and the additional property from lemma 7. Let r be a strategy of this kind. For every subgame $\Gamma_y \in M$ let r_y be the strategy induced by r on Γ_y. Let $\bar{r}$ be the strategy induced by r on $\bar{\Gamma}$.

If a subgame $\Gamma_y \in M$ is blocked by r, then for this subgame r_y agrees with b_y in view of the additional property from lemma 7.

Consider an asymmetric subgame $\Gamma_y \in M$ not blocked by r. Then it follows by (iii) that b_y is the only best reply in $\hat{\Gamma}_y$ to the strategy b'_y induced by f(b) on $\hat{\Gamma}_y$. It follows by lemma 6 applied to b and by lemma 3 applied to r that in this case r_y agrees with b_y.

Suppose that for a subgame $\Gamma_y \in M$ we have $r_y \neq b_y$. Then Γ_y is not blocked by r and Γ_y belongs to a symmetric subgame (Γ_y, f_y) of (Γ, f). Moreover, by the additional property of lemma 7 the local strategies assigned to information sets blocked by r_y in Γ_y are the same for b_y and r_y. It follows by lemma 3 that the sufficient local conditions required by lemma 4 for a best reply to $f_y(b_y)$ in $\hat{\Gamma}_y = (\Gamma_y, f_y, \eta_y)$ are satisfied by r_y. Therefore r_y is a best reply to $f_y(b_y)$ in $\hat{\Gamma}_y$.

Consider the (r,f(b))-truncation $\bar{\Gamma}_*$ of $\hat{\Gamma}$ with respect to M. Since r induces best replies to the strategies induced by f(b) in the subgames $\hat{\Gamma}_y$ of $\hat{\Gamma}$ mentioned in (ii) and (iii), the game $\bar{\Gamma}_*$ differs from $\bar{\Gamma}$ only with respect to player 2's payoff. Therefore $\bar{r}$ must be a best reply to $\bar{b}$ in $\bar{\Gamma}$.

For every subgame $\Gamma_y \in M$ let r'_y be the strategy induced by f(r) on Γ_y. It is clear that for asymmetric subgames (r_y, r'_y) agrees with (b_y, b'_y). Wherever (r_y, r'_y) does not agree with (b_y, b'_y) the game $\hat{\Gamma}_y = (\Gamma_y, f_y, \eta_y)$ is a symmetric subgame of $\hat{\Gamma}$ and r_y is an alternative best reply to $f_y(b_y)$ which by the second condition in the direct ESS definition 7.5 satisfies the following inequality:

$$(96) \qquad E_y(b_y, f_y(r_y)) - E_y(r_y, f_y(r_y)) > 0$$

Assume that for at least one subgame $\Gamma_y \in M$ we have $r_y \neq b_y$. As we shall see under this assumption (96) has the following consequence:

(97) $\quad E(b,f(r)) - E(r,f(r)) > \bar{E}(\bar{b},\bar{f}(\bar{r})) - \bar{E}(\bar{r},\bar{f}(\bar{r}))$

where $\bar{E}$ denotes payoffs in $\bar{\Gamma}$. On the right hand side of (97) we find the payoff difference between $\bar{b}$ and $\bar{r}$ against $f(\bar{r})$, if player 1 behaves according to b in all subgames $\Gamma_y \in M$. Inequality (96) shows that this difference is increased if player 1 changes his strategy from b_y to r_y in a subgame $\Gamma_y \in M$ with $b_y \neq r_y$. This yields (97).

$\bar{b}$ is a direct ESS of $\bar{\Gamma}$. If $\bar{r} = \bar{b}$, then the right hand side of (97) is zero. If $\bar{r} \neq \bar{b}$, then it follows by the second condition in 7.5 that the right hand side of (97) is positive. Therefore, the left hand side of (97) is positive. This shows that r does not violate the second condition in 7.5 as an alternative best reply to f(b) in $\hat{\Gamma}$.

Now assume that we have $b_y = r_y$ for all subgames $\Gamma_y \in M$. In this case we have:

(98) $\quad E(b,f(r)) - E(r,f(r)) = \bar{E}(\bar{b},\bar{f}(\bar{r})) - \bar{E}(\bar{r},\bar{f}(\bar{r}))$

Moreover, $\bar{r}$ and $\bar{b}$ are different from each other since otherwise r could not be different from b. Since $\bar{b}$ is a direct ESS for $\tilde{\gamma}$ it follows by the second condition in 7.5 that the right hand side of (98) is positive. Consequently, in this case, too, r does not violate the second condition in 7.5 applied to b. We have shown that b is a direct ESS of $\hat{\Gamma}$.

<u>10.12 Theorem 9 (decomposition theorem)</u>: Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of a symmetric extensive game (Γ, f) with a subgame preserving symmetry f. Let b be a behavior strategy of player 1 for Γ. For every b-element $\tilde{\gamma}$ of $\hat{\Gamma}$ let $\bar{b}$ and $\bar{b}'$ be the strategies induced by b and f(b), respectively on $\tilde{\gamma}$. Then b is a direct ESS of $\hat{\Gamma}$, if and only if the following conditions (a), (b) and (c) are satisfied for all b-elements $\tilde{\gamma}$ of $\hat{\Gamma}$:

(a) If $\tilde{\gamma} = (\bar{\Gamma}, \bar{\eta})$ is an asymmetric b-element of $\hat{\Gamma}$, then $(\bar{b}, \bar{b}')$ is a strong equilibrium point of $\bar{\Gamma}$.

(b) If $\tilde{\Gamma} = (\bar{\Gamma},\bar{f},\bar{n})$ is a symmetric b-element of $\hat{\Gamma}$, then $\bar{b}$ is a direct ESS of $\tilde{\Gamma}$.

(c) If a decomposition endpoint y of $\tilde{\Gamma}$ is the origin of an essential subgame Γ_y of Γ, then the realization probability $\bar{\gamma}(y,\bar{b},\bar{b}')$ of y in $\tilde{\Gamma}$ is positive.

Proof: The decomposition rank of a game Γ is defined recursively by the following two conditions (i) and (ii):

(i) If Γ has no subgame then the decomposition rank of Γ is 1.

(ii) If Γ has at least one maximal subgame of decomposition rank k and all maximal subgames have decomposition ranks of at most k, then the decomposition rank of Γ is k+1.

The theorem will be proved by induction on the decomposition rank of Γ. It is trivially true if Γ has decomposition rank 1. Suppose that the assertion holds if Γ has a decomposition rank of at most k. Then the theorem holds for the maximal subgames of Γ. For every maximal subgame Γ_y of Γ let b_y be the strategy induced by b on Γ_y. The b_y-elements of a maximal subgame $\hat{\Gamma}_y$ of $\hat{\Gamma}$ are b-elements of $\hat{\Gamma}$. The main b-truncation of $\hat{\Gamma}$ is the only b-element of $\hat{\Gamma}$ which is not a b_y-element of a maximal subgame $\hat{\Gamma}_y$ of $\hat{\Gamma}$. The decomposition rank connected to the main b-truncation is 1.

Assume that Γ has decomposition rank k+1. If b is a direct ESS of $\hat{\Gamma}$, then it follows by theorem 8 and the induction hypothesis applied to the maximal subgames of $\hat{\Gamma}$ that (a), (b) and (c) hold. If (a),(b) and (c) hold for $\hat{\Gamma}$, then we can conclude that (ii) and (iii) in theorem 8 hold for the maximal subgame of $\hat{\Gamma}$. Theorem 8 together with (b) and (c) applied to the main b-truncation of $\hat{\Gamma}$ permits the conclusion that b is a direct ESS of $\hat{\Gamma}$. Consequently, the assertion of the theorem is true.

<u>10.13 Comment</u>: Theorem 9 characterizes a direct ESS of a perturbed game in terms of the strategies induced on the elementary games generated by the direct ESS. It would be desirable to obtain a similar characterization of a limit ESS. Unfortunately, a result analogous to theorem 9 cannot be derived since the strong inequalities in the definition of a strong equilibrium point and in the second ESS condition need not be preserved by a transition to the limit. A limit ESS cannot be expected to satisfy these strong inequalities but only weak inequalities of the same kind. These weak inequalities are not sufficient for a limit ESS but since they are necessary, they may still serve to exclude many symmetric equilibrium strategies as possible candidates for a limit ESS. In order to be able to express these necessary conditions in a convenient way we shall introduce the notion of a "semistable strategy".

<u>10.14 Semistable strategies</u>: A behavior strategy b for player 1 in a symmetric extensive 2-person game (Γ,f) is called a <u>semistable strategy</u> of (Γ,f) if it is a symmetric equilibrium strategy which satisfies the following additional condition: For every best reply r to f(b) in (Γ,f) we have:

(99) $$E(b,f(r)) \geq E(r,f(r))$$

Analogously a <u>semistable</u> strategy q of a bimatrix game $G = (\Pi,E)$ is a symmetric equilibrium strategy for G which satisfies the following condition:

(100) $$E(q,r) \geq E(r,r)$$

for every best reply r to q.

<u>10.15 Theorem 10 (necessary conditions for a limit ESS)</u>: Let (Γ,f) be a symmetric extensive 2-person game with a subgame preserving natural symmetry f. Let b be a limit ESS of (Γ,f)

and for every subgame Γ_y of Γ let b_y and b'_y be the strategies induced by b and f(b), respectively on Γ_y. Then the following conditions (a) to (e) are satisfied:

(a) If Γ_y is an asymmetric subgame of (Γ,f) then (b_y,b'_y) is an equilibrium point in pure strategies of Γ_y.

(b) If $\bar{\Gamma}$ is a (b_y,b'_y)-truncation of an asymmetric subgame Γ_y of (Γ,f) then the strategies $\bar{b}$ and $\bar{b}'$ induced on $\bar{\Gamma}$ by b and f(b), respectively form an equilibrium point $(\bar{b},\bar{b}')$ in pure strategies of $\bar{\Gamma}$.

(c) If (Γ_y,f_y) is a symmetric subgame of (Γ,f) then b_y is a semistable strategy of (Γ_y,f_y).

(d) If $(\bar{\Gamma},\bar{f})$ is a b-truncation of (Γ,f) with respect to a symmetric multisubgame, then the strategy $\bar{b}$ induced on $\bar{\Gamma}$ by b is a semistable strategy of $(\bar{\Gamma},\bar{f})$.

(e) If $(\bar{\Gamma},\bar{f})$ is a b_y-truncation of a symmetric subgame (Γ_y,f_y) of (Γ,f) with respect to a symmetric multisubgame of (Γ_y,f_y), then the strategy $\bar{b}$ induced on $\bar{\Gamma}$ by b is a semistable strategy of $(\bar{\Gamma},\bar{f})$.

Proof: We can find a test sequence $\hat{\Gamma}^1,\hat{\Gamma}^2,\ldots$ with $\hat{\Gamma}=(\Gamma,f,\eta)$ such that b is a limit ESS of this test sequence, i.e. the limit of a sequence $b^1,b^2,\ldots$ of direct ESS's b^k for the corresponding perturbed games $\hat{\Gamma}^k$. Theorem 6 can be applied to each of the b^k. Let b_y^k and $b_y'^k$ be the strategies induced on Γ_y by b^k and $f(b^k)$, respectively and let η_y^k be the perturbance induced on Γ_y by η^k. Obviously $\hat{\Gamma}_y^k=(\Gamma_y,\eta_y^k)$ is an asymmetric subgame of $\hat{\Gamma}^k$. It follows by theorem 6 that $(b_y^k,b_y'^k)$ is a strong equilibrium point of $\hat{\Gamma}_y^k$. Therefore, the local strategies prescribed by b_y^k and $b_y'^k$ must be extreme in $\hat{\Gamma}_y^k$. Consequently, the limit (b_y,b'_y) of the sequence of the pairs $(b_y^k,b_y'^k)$ must be a pair of pure strategies. Moreover, in view of the continuity of the payoff E_y of Γ_y it can be shown that (b_y,b'_y) is an equilibrium point of Γ_y (compare remark 7.7). Consequently (a) holds.

We now turn our attention to (b). For k = 1,2,... let $\tilde{\Gamma}^k = (\bar{\Gamma}^k, \bar{n}^k)$ be the $(b_y^k, b_y'^k)$-truncation of Γ_y^k with respect to a fixed multisubgame M of Γ_y; moreover, let $\bar{b}^k$ and $\bar{b}'^k$ be the strategies induced by b^k and $f(b^k)$, respectively on $\bar{\Gamma}^k$. Local payoffs in Γ_y and $\bar{\Gamma}^k$ agree in the following sense:

$$(101) \qquad E_{yu}(r_u, b_y^k, b_y'^k) = \bar{E}_u^k(r_u, \bar{b}^k, \bar{b}'^k)$$

where E_{yu} and $\bar{E}_u^k$ are the local payoff functions at u in Γ_y and $\bar{\Gamma}^k$, respectively. It follows by lemma 6 that $(\bar{b}^k, \bar{b}'^k)$ is a strong equilibrium point of $\tilde{\Gamma}^k$. An argument analogous to that used at the end of the proof for statement (a) shows that the limit $(\bar{b}, \bar{b}')$ of the sequence of the $(\bar{b}^k, \bar{b}'^k)$ for $k \to \infty$ is an equilibrium point in pure strategies of the (b_y, b_y')-truncation $\bar{\Gamma}$ of Γ_y with respect to M. Statement (b) holds.

We now prove (c). For k = 1,2,... let $\hat{\Gamma}_y^k = (\Gamma_y, f_y, n_y)$ be the subgame of $\hat{\Gamma}^k$ at y and let b_y^k be the strategy induced by b^k on Γ_y. It follows by theorem 7 that b_y^k is a direct ESS of $\hat{\Gamma}_y^k$. Obviously, the strategy b_y induced on Γ_y by b is the limit of the b_y^k for $k \to \infty$. This shows that b_y is a limit ESS of the test sequence $\hat{\Gamma}_y^1, \hat{\Gamma}_y^2, \ldots$. Therefore, b_y is a symmetric equilibrium strategy of Γ_y (compare remark 7.12). Moreover, in view of the continuity of E_y the second ESS condition in 7.5 applied to b^k secures the semistability of b_y. Statement (c) holds.

The next statement to be proved is (d). For k = 1,2, ... let $\tilde{\Gamma}^k = (\bar{\Gamma}^k, \bar{f}^k, \bar{n}^k)$ be the b^k-truncation of $\hat{\Gamma}^k$ with respect to a fixed symmetric subgame M; moreover, let $\bar{b}^k$ be the strategy induced by b^k on $\bar{\Gamma}^k$. Let $(\bar{\Gamma}, \bar{f})$ be the b-truncation of (Γ, f) with respect to M. It follows by (i) in theorem 8 that $\bar{b}^k$ is a direct ESS of $\tilde{\Gamma}^k$. In view of the continuity of E_y the strategy $\bar{b}$ induced on $\bar{\Gamma}$ by b is the limit of the sequence of $\bar{b}^k$ for $k \to \infty$. the inequalities satisfied by a symmetric equilibrium point remain valid in the transition to the limit. The second ESS condition in 7.5 applied to $\bar{b}^k$ secures the semistability of $\bar{b}$ in $(\bar{\Gamma},\bar{f})$. Statement (d) holds.

It remains to show (e). The proof of (c) has shown that b_y is a limit ESS of the test sequence $\hat{\Gamma}_y^1,\hat{\Gamma}_y^2,\ldots$. Therefore (Γ_y,f_y) instead of (Γ,f) and b_y instead of b satisfy the assumptions of the theorem. We can apply statement (d) to (Γ_y,f_y) and b_y. Thereby we receive statement (e).

<u>10.16 Remark</u>: The proof of (c) has shown that a limit ESS of a symmetric extensive 2-person game (Γ,f) induces a limit ESS of (Γ_y,f_y) on every symmetric subgame of (Γ,f). This property of the notion of a limit ESS will be referred to as <u>subgame consistency</u> (compare Selten 1973).

It is not true in general that a limit ESS b of a symmetric extensive 2-person game (Γ,f) induces a limit ESS of $(\bar{\Gamma},\bar{f})$ on every b-truncation $(\bar{\Gamma},\bar{f})$ of (Γ,f). <u>Truncation consistency</u> in this sense is not a property of the notion of a limit ESS. A counterexample is provided by the male desertion game of figure 9 in 6.5 for the case a+s=1. Consider a perturbed game with minimum probabilities $\varepsilon > 0$ for L at v and v' and with zero minimum probabilities everywhere else. A perturbed game of this kind has exactly one direct ESS which prescribes R at u and v. This shows that the behavior strategy b of player 1 which prescribes R at u and v is a limit ESS of the game of figure 9 with a+s=1. However, the b-truncation with respect to the symmetric multisubgame containing the subgame at x_2 and its symmetric image does not have any limit ESS.

11. Simultaneity games

Theorem 9 shows that one can find the direct ESS's of a perturbed game by a successive analysis of elementary games. One begins with the smallest subgames near the end. It may, of course, happen that these subgames have several direct ESS's. Truncations with respect to the smallest subgames must be formed for each possible case. Then the process is continued by the analysis of the smallest subgames of the truncations, etc. At least, in principle all direct ESS's of the whole game can be found in this way.

The procedure outlined above can be expected to simplify the task of analyzing a game model, in cases where a big complex game can be decomposed into many simple elementary games. In this section we shall turn our attention to a class of games which exhibit a very high degree of decomposibility. Such games arise from models with a period structure where the two players have the opportunity to make simultaneous decisions in each of a finite number of periods. In each period the players know everything which happened in previous periods, but they do not know the random choices and the choices of the opponent in the same period. In the literature such games have been referred to as simultaneity games (Selten 1973).

In this paper a simultaneity game will be formally introduced as a symmetric extensive 2-person game whose elementary games contain at most one information set for each of both players. It is clear that in this way one obtains a class of games which includes the period structure models described above.

It will be shown that for perturbed simultaneity games there is no difference between an LSS in the sense of 9.9 and a direct ESS.

The notion of a regular ESS for a symmetric bimatrix game will be introduced in order to derive sufficient local conditions for a limit ESS of a simultaneity game. A behavior strategy b of player 1 for a simultaneity game is a limit ESS, if strong equilibrium points are induced by b and its symmetric image on asymmetric b-elements and regular ESS's are induced on symmetric b-elements. This will be the final result.

<u>11.1 Simultaneity games</u>: A <u>simultaneity game</u> is a symmetric extensive 2-person game (Γ,f) with a subgame preserving natural symmetry f and with the additional property that every b-element of (Γ,f) has at most one information set for each of both players 1 and 2. This definition does not really depend on b. With the exception of the payoff functions the components of the b-elements are not influenced by b.

Consider a symmetric b-element $(\bar{\Gamma},\bar{f})$ of a simultaneity game (Γ,f). Assume that $(\bar{\Gamma},\bar{f})$ is essential or, in other words, that $\bar{\Gamma}$ has at least one essential information set. Then $\bar{\Gamma}$ has exactly one information set for each of both players 1 and 2. Let u be player 1's information set in $\bar{\Gamma}$. Then f(u) is player 2's information set in $\bar{\Gamma}$. Consider the local game $G_{ub} = (C_u,E_{ub})$ of (Γ,f) at u under b (see 9.8). Obviously, G_{ub} is nothing else than the symmetric normal form of $(\bar{\Gamma},\bar{f})$. (For the definition of the symmetric normal form, see 5.8). This fact is the basis of the following theorem.

<u>11.2 Theorem 11 (on local and global stability)</u>: Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a simultaneity game (Γ,f) and let b be a behavior strategy of player 1 for $\hat{\Gamma}$. Then b is a direct ESS of $\hat{\Gamma}$, if and only if b is an LSS of $\hat{\Gamma}$.

<u>Proof</u>: As has been pointed out in 9.9 a direct ESS of $\hat{\Gamma}$ always is an LSS of $\hat{\Gamma}$. It remains to show that in view of the simultaneity game property of (Γ,f), an LSS of $\hat{\Gamma}$ is a direct ESS of $\hat{\Gamma}$. Let b be an LSS of $\hat{\Gamma}$. Then b is pervasive in view of (i) in 9.9. Therefore, condition (c) of theorem 9 is satisfied. Condition (a) of theorem 3 follows by (ii) in 9.9. Condition (b) of theorem 3 follows by (iii) in view of the agreement between local games

and symmetric normal forms of symmetric b-elements discussed in 11.1. Theorem 3 permits the conclusion that b is a direct ESS of $\hat{\Gamma}$.

11.3 Essential ESS: Let $G = (\Pi, E)$ and $G_+ = (\Pi, E_+)$ be two symmetric bimatrix games which differ only with respect to their payoff functions E and E_+. Define

$$(102) \qquad |E - E_+| = \max_{\gamma,\varphi\in\Pi} |E(\gamma,\varphi) - E_+(\gamma,\varphi)|$$

We call $|E-E_+|$ the distance between G and G_+. Let q and r be two mixed strategies for $G = (\Pi, E)$. Define

$$(103) \qquad |q-r| = \max_{\pi\in\Pi} |q(\pi) - r(\pi)|$$

We call $|q-r|$ the distance between q and r. A mixed strategy q for $G = (\Pi, E)$ is an essential ESS of $G = (\Pi, E)$ if q is an ESS of G and in addition to this satisfies the following condition: For every $\varepsilon > 0$ we can find a $\delta>0$ such that every symmetric bimatrix game $G_+ = (\Pi, E_+)$ whose distance from G_+ is smaller than δ, has an ESS q_+ whose distance from q is smaller than ε.

Wu Wen-tsün and Jiang Jia-he have introduced the notion of an essential equilibrium point (Wu-Wen-tsün and Jiang Jia-he 1962). The definition of an essential ESS is analogous to their definition.

11.4 Regularity: A symmetric equilibrium strategy q of a symmetric bimatrix game $G = (\Pi, E)$ is called regular if it assigns a positive probability $q(\pi)$ to every pure best reply π to q in G. An ESS of G is called regular if it is a regular symmetric equilibrium strategy of G.

In the literature the term "regular" applied to equilibrium points has a different sense (Harsanyi 1973, van Damme 1983). Our use of the word "regular" corresponds to van Damme's use of the word "quasi-strong". It has been shown that an equilibrium point of a bimatrix game is regular in the sense of Harsanyi and van Damme, if and only if it is essential and quasi-strong (Jansen 1981, van Damme 1983, theorem 3.4.5). It will be shown in lemma 9 that a regular ESS (in our sense) is always essential. The equilibrium point connected to a regular ESS is also regular in the sense of Harsanyi and van Damme. This justifies our language use.

<u>11.5 Haigh's criterion</u>: Let $G = (\Pi, E)$ be a symmetric bimatrix game. The <u>carrier</u> of a mixed strategy r for G is the set of all pure strategies $\pi \in \Pi$ with $r(\pi) > 0$. Let q be a regular symmetric equilibrium strategy for G. The fact that q is regular can be expressed by saying that all pure best replies to q are in the carrier of q. Let $\pi_1, \ldots \pi_n$ be the pure strategies in the carrier of q. Define

$$a_{ij} = E(\pi_i, \pi_j) \tag{104}$$

for $i,j = 1,\ldots,n$. The payoffs a_{ij} form an nxn-matrix

$$A = (a_{ij})_{nxn} \tag{105}$$

We call this matrix the <u>carrier matrix</u> of q. The carrier matrix is not determined by q alone but also by the numbering of the pure strategies in the carrier. The definition must be understood relative to a fixed numbering.

Let R be the set of all mixed strategies for G which assign positive probabilities to the pure strategies in the carrier of q only. Obviously, R can be described as the set of all best replies to q in G. For any $r \in R$ let r_i be the probability $r(\pi_i)$ assigned to π_i by r. We can think of r as a column vector.

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \tag{106}$$

The same symbol r will be used for the strategy r and for this column vector. No confusion can arise from this notational convention. In the same way q also represents the n-dimensional column vector whose components q_i are the probabilities assigned to π_i by q.

We use the upper index T to indicate transposition. For any two strategies r and s in R the expected payoff in G can be written as follows:

(107) $$E(r,s) = r^T As$$

We now examine the conditions under which q is an ESS of G. The second condition (b) in 2.11 is satisfied for q, if and only if for every $r \in R$ with $r \neq q$ we have:

(108) $$q^T Ar > r^T Ar$$

If there is only one pure strategy in the carrier of q, then this is trivially true since in this case q is the only best reply to q in G. Therefore, in the following we shall assume n>1. The fact that r is a best reply to q in G can be expressed as follows:

(109) $$q^T Aq = r^T Aq$$

Subtraction of (106) from (105) yields:

(110) $$q^T A(r-q) > r^T A(r-q)$$

Therefore, the second condition (b) in 2.11 is satisfied for q, if and only if for every $r \in R$ with $r \neq q$ we have:

(111) $$(r-q)^T A(r-q) < 0$$

The components of r-q sum up to zero. Let S be the set of all vectors

(112) $$s = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix}$$

with

(113) $$\sum_{i=1}^{n} s_i = 0$$

which are different from the n-dimensional zero vector. Since q_i is positive for i=1,...,n for every s∈S we can find an r∈R and a number λ>0 such that

(114) $$r-q = \lambda s$$

Therefore, the second condition (b) in 2.11 is satisfied for q, if and only if for every s∈S we have:

(115) $$s^T A s < 0$$

Haigh's criterion is a condition on A which guarantees that (115) holds for every s∈S (Haigh 1975). In order to derive this criterion, it is useful to look at the left hand side of (115) as a quadratic form in the first n-1 components of r-q. For every s∈S, let $\hat{s}$ be the (n-1)-dimensional vector:

(116) $$\hat{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{n-1} \end{bmatrix}$$

Let D be the following n×(n-1)-matrix:

(117) $$D = (d_{ij})_{n\times(n-1)} = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \\ -1 & \dots & -1 \end{bmatrix}$$

with

(118) $$d_{ij} = \begin{cases} 1 & \text{for } i=j < n \\ -1 & \text{for } i=n \\ 0 & \text{else} \end{cases}$$

Obviously we have

(119) $$s = D\hat{s}$$

Therefore the left hand side of (115) can be rewritten as follows:

$$s^T A s = \hat{s}^T D^T A D \hat{s} \tag{120}$$

We say that Haigh's criterion is satisfied for q if the (n-1)x(n-1)-matrix $D^T AD$ is negative quasi-definite. (120) shows that the second condition (b) holds for q, if and only if Haigh's condition is satisfied. In order to extend this statement to the trival case of only one pure strategy in the carrier of q, we shall count Haigh's criterion as always satisfied in this case.

Whether Haigh's criterion is satisfied or not does not depend on the numbering of the pure strategies in the carrier. Therefore, it is not necessary to refer to a particular numbering if we speak of Haigh's criterion.

The result obtained above is expressed by the first part of the following lemma.

11.6 Lemma 9 (on Haigh's criterion): A regular symmetric equilibrium strategy q of a symmetric bimatrix game $G = (\Pi, E)$ is an ESS of G, if and only if it satisfies Haigh's criterion. Moreover, a regular ESS of G is an essential ESS of G.

Proof: The first part of the lemma has been shown in 11.5. It remains to prove the second part. Let q be a regular ESS of G.

If the carrier of q has only one element, then (q,q) is a strong equilibrium point of G and also a strong equilibrium point of every game $G_+ = (\Pi, E_+)$ sufficiently near to G. Therefore, in this case the assertion is true. In the following we shall assume that there are at least two elements in the carrier of q. We shall use the notation of 11.5. Define:

(121) $$y = q^T A q$$

y is the equilibrium payoff connected to q. The components $q_1, \ldots, q_n$ of q together with y can be looked upon as n+1 unknowns in a system of n+1 linear equations:

(122) $$a_{i1}q_1 + \ldots + a_{in}q_n - y = 0$$

for i = 1, ...,n

(123) $$q_1 + \ldots + q_n = 1$$

We shall first show that this system has only one solution. Suppose that $q_1, \ldots, q_n, y$ is not the only solution. The set of all solution vectors of a linear equation system forms a linear subspace in the space of the unknowns. Therefore, a different solution $\bar{q}_1, \ldots, \bar{q}_n, \bar{y}$ can be found arbitrarily near to $q_1, \ldots, q_n, y$. If $\pi \in \Pi$ is not in the carrier of q, then $E(\pi, q) < E(q, q)$. In view of the continuity of E we can find an $\varepsilon > 0$ such that $E(\pi, \bar{q}) < E(\bar{q}, \bar{q})$ holds for every $\bar{q}$ with $|\bar{q} - q| < \varepsilon$ and every $\pi \in \Pi$ outside the carrier of q. Let ε be a number of this kind and let $\bar{q}_1, \ldots, \bar{q}_n, \bar{y}$ be a solution of the system such that $\bar{q}_i$ is positive for i = 1,...,n and $|\bar{q}_i - q_i| < \varepsilon$ holds for i = 1,...,n. In view of (120) the $\bar{q}_i$ form a strategy $\bar{q}$ in the carrier of q. Moreover, the choice of ε guarantees that all best replies to $\bar{q}$ in G are in the carrier of q. In view of (122) all pure strategies in the common carrier of q and $\bar{q}$ yield the same payoff $\bar{y}$ against $\bar{q}$. Therefore, all these pure strategies are best replies to $\bar{q}$. It follows that $\bar{q}$ is a symmetric equilibrium strategy of G. Since q and $\bar{q}$ have the same carrier, both q and $\bar{q}$ are best replies to q :

(124) $$E(q, q) = E(\bar{q}, q)$$

This shows that $\bar{q}$ is an alternative best reply to q which violates the second condition (b) in 2.11. Since we know by the first part of the lemma that q is an ESS we can conclude that $q_1,\ldots,q_n,y$ is the only solution of the system formed by (122) and (123).

The fact that $q_1,\ldots,q_n,y$ is the only solution of the system has the consequence that the matrix of the system is non-singular. If $G_+ = (\Pi,E_+)$ is sufficiently near to $G = (\Pi,E)$, then the matrix $A_+ = (a_{+ij})$ with

$$(125) \qquad a_{+ij} = E_+(\pi_i,\pi_j)$$

will still be non-singular and the system formed by (122) and (123) with coefficients a_{+ij} instead of a_{ij} will still have a unique solution $q_{+1},\ldots,q_{+n},y_+$. Moreover, this unique solution depends continuously on the payoffs in E_+. Therefore, if G_+ is sufficiently near to G, then $E_+(\pi,q_+)<E_+(q_+,q_+)$ holds for all pure strategies π not in the carrier of q. In a similar fashion as for $\bar{q}$ in the first part of the proof we can conclude that q_+ is a symmetric equilibrium strategy of G_+.

An (nxn)-matrix M is negative quasidefinite,if and only if the so-called north-west subdeterminants $\Delta_1,\ldots,\Delta_n$ of $M + M^T$ have alternating signs, beginning with a negative sign for Δ_1 (see for example Beckmann and Künzi 1973). This shows that Haigh's criterion is satisfied for the symmetric equilibrium strategy q_+ of G_+ described above, if G_+ is sufficiently near to G; it follows that q_+ is an ESS of G_+. We can conclude that q is an essential ESS of G.

<u>11.7 Remark</u>: If q is a regular ESS of a symmetric bimatrix game $G = (\Pi,E)$, then for every $\varepsilon > 0$ a $\delta > 0$ can be found such that a game $G_+ = (\Pi,E_+)$ whose distance from G is smaller than δ, has a regular ESS q_+ with $|q_+-q|<\varepsilon$. This statement is a consequence of the proof of lemma 9. The ESS q_+ of a game G_+ sufficiently near to G which has been constructed there, satisfies Haigh's criterion. Therefore, the second part of the lemma can be applied to q_+. It is

also clear that the trivial case of only one pure strategy in the carrier of q and q_+ poses no difficulties.

<u>11.8 Example of an inessential ESS</u>: Figure 15 shows a class of symmetric bimatrix games G_ε. The pure strategy π_1 is an ESS of the game G_0 with $\varepsilon = 0$. It can be seen immediately that for $\varepsilon > 0$ a mixed strategy q is a symmetric equilibrium strategy of G_ε, if and only if it satisfies the following condition:

$$q(\pi_1) = \frac{1}{1+\varepsilon} \tag{126}$$

The complementary probability $\varepsilon/(1+\varepsilon)$ can be distributed in any way on the two remaining pure strategies π_2 and π_3. Let q and $\bar{q}$ be two different mixed strategies which assign $1/(1+\varepsilon)$ to π_1. Obviously, both q and $\bar{q}$ yield the same payoff 1 against $\bar{q}$ in G_ε. Therefore $\bar{q}$ violates the second condition as am alternative best reply to q. We can conclude that none of the games G_ε with $\varepsilon>0$ has an ESS. It follows that π_1 fails to be an essential ESS of G_0.

	π_1	π_2	π_3
π_1	1 / 1	1 / $1+\varepsilon$	1 / $1+\varepsilon$
π_2	$1+\varepsilon$ / 1	0 / 0	0 / 0
π_3	$1+\varepsilon$ / 1	0 / 0	0 / 0

<u>Figure 15:</u> A class of symmetric bimatrix games G_ε. The pure strategy π_1 is an ESS of G_0; however, this ESS is not essential since for $\varepsilon > 0$ the game G_ε does not have an ESS.

Lemma 9 cannot be applied to π_1 in G_0 since π_1 is not regular. According to the language use introduced in 11.4 Haigh's criterion is trivially satisfied for π_1 in G_0. However, in cases where a symmetric equilibrium strategy q is not regular one should maybe look at the set of all pure best replies to q in G instead of the carrier of q. We call the set of all pure best replies to q in G the extended carrier of q. A matrix of the extended carrier of q can be defined analogously to the carrier matrix in 11.5. In the case of G_0 in figure 15 the matrix $\bar{A}$ of the extended carrier does not satisfy Haigh's criterion.
One obtains:

$$(127) \qquad D^T\bar{A}D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

This matrix is not negative quasi-definite.

It may be true that an ESS of a symmetric bimatrix game is essential if and only if Haigh's criterion applied to the extended carrier is satisfied. This question will not be pursued here.

11.9 Lemma 10(composition of strategy and payoff perturbance): Let r be a regular ESS of a symmetric bimatrix game $G=(\Pi,E)$. Then for every $\varepsilon>0$ a number $\delta > 0$ can be found such that every perturbed game $\hat{G}_+ = (\Pi,E_+,\eta)$ with $|\eta|<\delta$ and $|E_+-E|<\delta$ has an ESS $\hat{r}_+$ with $|\hat{r}_+-r|<\varepsilon$.

Proof: Let $\hat{G}_+ = (\Pi,E_+,\eta)$ be a perturbed game of $G_+ = (\Pi,E_+)$. For every pure strategy $\pi\in\Pi$ let q^π be that mixed strategy for $\hat{G}_+$ which concentrates as much probability on π as possible. We call q^π the extreme strategy of $\hat{G}$ with the intended choice π (see 8.5). We construct a symmetric bimatrix game $G_{++} = (\Pi,E_{++})$ which will be called the equivalent unperturbed game of $\hat{G}_+$. The payoff function E_{++} of G_{++} is defined as follows:

$$(128) \qquad E_{++}(\pi,\gamma) = E_+(q^\pi,q^\gamma)$$

for every $\pi\in\Pi$ and every $\gamma\in\Pi$. For every mixed stragey s in G_{++} we define a mixed strategy $\hat{s}$ for $\hat{G}_+$ which <u>corresponds</u> to s in $\hat{G}_+$:

$$(129) \qquad \hat{s}(\pi) = \eta_\pi + (1 - \sum_{\varphi\in\Pi} \eta_\varphi)s(\pi)$$

for every $\pi\in\Pi$. Suppose that a player in G_+ plays each of the extreme strategies q^π with its probability $s(\pi)$. Obviously, this is the same as playing each $\pi\in\Pi$ with probability $\hat{s}(\pi)$. Therefore, the following relationship between E_+ and E_{++} holds, whenever $\hat{s}$ and $\hat{t}$ correspond to s and t, respectively in $\hat{G}_+$:

$$(130) \qquad E_+(\hat{s},\hat{t}) = E_{++}(s,t)$$

A one-to-one mapping of the mixed strategies in G_+ onto the mixed strategies in $\hat{G}$ is defined by (129). Moreover, this mapping conserves payoffs in the sense of (130). This permits the conclusion that a strategy $\hat{r}_+$ which in $\hat{G}_+$ corresponds to an ESS r_{++} of G_{++} is an ESS of $\hat{G}_+$

Since r is a regular ESS of G, it follows by remark 11.7 that for every $\varepsilon>0$ a number $\delta>0$ can be found such that a game $G_+ = (\Pi,E_+)$ with $|E_+-E| < \delta$ has a regular ESS r_+ with $|r_+ - r| < \varepsilon/3$. If δ is chosen sufficiently small then for $|\eta| < \delta$ the equivalent unperturbed game $G_{++} = (\Pi,E_+)$ of $\hat{G}_+ = (\Pi,E_+,\eta)$ has an ESS r_{++} with $|r_{++}-r_+| < \varepsilon/3$; this follows by lemma 9 and the regularity of r_+. Moreover δ can be chosen sufficiently small to secure that $|\hat{s}-s| < \varepsilon/3$ holds if $\hat{s}$ corresponds to s in $\hat{G}_+$. Let $\hat{r}_+$ be the ESS of $\hat{G}_+$ which corresponds to an ESS r_{++} of G_{++} in $\hat{G}_+$ with $|r_{++}-r_+| < \varepsilon/3$. Then we have $|r-\hat{r}_+| < \varepsilon$, if δ is chosen sufficiently small. This shows that the assertion of the lemma is true.

<u>11.10 Theorem 12 (sufficient local conditions for a limit ESS)</u>: Let (Γ,f) be a simultaneity game and let b be a behavior strategy of player 1 for Γ. Then b is a limit ESS of (Γ,f), if the following conditions (a) and (b) are

satisfied for the b-elements of (Γ,f).

(a) If $\bar{\Gamma}$ is an essential asymmetric b-element of (Γ,f) then the strategies $\bar{b}$ and $\bar{b}'$ induced by b and f(b), respectively on $\bar{\Gamma}$ form a strong equilibrium point $(\bar{b},\bar{b}')$ of $\bar{\Gamma}$.

(b) If $(\bar{\Gamma},\bar{f})$ is an essential symmetric b-element of (Γ,f), then the strategy $\bar{b}$ induced by b on $\bar{\Gamma}$ is a regular ESS of the symmetric normal form of $(\bar{\Gamma},\bar{f})$.

<u>Corrolary</u>: Under the assumptions of the theorem an $\varepsilon > 0$ can be found for every b with (a) and (b) such that b is a limit equilibrium point of every test sequence $\hat{\Gamma}^1,\hat{\Gamma}^2, \ldots$ for (Γ,f) with $\hat{\Gamma}^k = (\Gamma,f,\eta^k)$ which has the following properties:

(i) $|\eta^k| < \varepsilon$ for $k = 1,2, \ldots$

(ii) For every $k = 1,2, \ldots$ the minimum probabilities η_c^k assigned by η^k to choices c of players 1 or 2 are always positive.

<u>Proof</u>: We shall prove the corrolary since the corrolary implies the theorem. We shall use induction on the number of essential b-elements of (Γ,f). If the game has no essential b-element, then player 1 has only one behavior strategy and the assertion is trivially true. Assume that the assertion holds for symmetric extensive 2-person games with up to n essential b-elements. Let b be a behavior strategy of player 1 which satisfies (a) and (b). Further assume that (Γ,f) has n + 1 essential b-elements.

It is clear that the main b-truncation of (Γ,f) is symmetric. Let $(\tilde{\Gamma},\tilde{f})$ be the main b-truncation of (Γ,f). We shall first consider the case that $(\tilde{\Gamma},\tilde{f})$ is essential. The result will be used in the proof of the assertion for the case that $(\tilde{\Gamma},\tilde{f})$ is not essential.

Assume that the main b-truncation $(\tilde{\Gamma},\tilde{f})$ of (Γ,f) is essential. Obviously, in this case fewer than n+1 essential b-elements belong to a maximal subgame of (Γ,f). The corrolary can be applied to the maximal subgames of (Γ,f).

Let $\hat{\Gamma}^1\hat{\Gamma}^2,\ldots$ be a test sequence with $\Gamma^k = (\Gamma,f,\eta^k)$ such that all minimum probabilities prescribed by the η^k are positive. For every maximal subgame Γ_y of Γ let η_y^k be the perturbance induced on Γ_y by η^k and let b_y and b_y' be the strategies induced by b_y and $f(b_y)$, respectively on Γ_y.

Consider an asymmetric subgame Γ_y of (Γ,f) and the sequence $\hat{\Gamma}_y^1,\hat{\Gamma}_y^2,\ldots$ of perturbed games of Γ_y with $\hat{\Gamma}_y^k = (\Gamma_y,\eta_y^k)$. For every $k = 1,2,\ldots$ let b_y^k and $b_y'^k$ be those strategies of players 1 and 2 for $\hat{\Gamma}_y^k$ which prescribe the extreme local strategies with the intended choice prescribed by b_y and b_y', respectively. If ε is sufficiently small and $|\eta^k| < \varepsilon$ holds for $k = 1,2,\ldots$ then $(b_y^k,b_y'^k)$ is a strong equilibrium point of $\hat{\Gamma}_y^k$; this can be seen with the help of (a) and lemma 6.

Now consider a symmetric subgame (Γ_y,f_y) of (Γ,f). The corrolary can be applied to (Γ_y,f_y). If ε is sufficiently small and $|\eta^k| < \varepsilon$ holds for $k = 1,2,\ldots$, then $\hat{\Gamma}_y^1,\hat{\Gamma}_y^2,\ldots$ with $\hat{\Gamma}_y^k = (\Gamma_y,f_y,\eta_y^k)$ is a test sequence for (Γ_y,f_y) which has b_y as a limit ESS. For every $\hat{\Gamma}_y^k$ in the sequence we can find an ESS b_y^k such that b_y^k converges to b_y for $k \to \infty$. For $k = 1,2,\ldots$ define $b_y'^k = f_y(b_y^k)$.

For $k = 1,2,\ldots$ let $(\tilde{\Gamma}^k,\tilde{f})$ be defined as follows: with the exception of the payoffs at decomposition endpoints $\tilde{\Gamma}^k$ agrees with $\tilde{\Gamma}$; at the origin y of a maximal subgame the payoffs $\tilde{h}(y)$ and $\tilde{h}'(y)$ of players 1 and 2 are the expected payoffs $E_y(b_y^k,b_y'^k)$ and $E_y'(b_y^k,b_y'^k)$ of players 1 and 2, respectively in Γ_y for the strategy pair $(b_y^k,b_y'^k)$ defined above.

Let $\tilde{G} = (\Gamma,\tilde{E})$ be the symmetric normal form of $(\tilde{\Gamma},\tilde{f})$ and and for $k = 1,2,\ldots$ let $\tilde{G}^k = (\Pi,\tilde{E}^k)$ be the symmetric normal form of $(\tilde{\Gamma}^k,\tilde{f})$. In view of (b) the strategy $\tilde{b}$ induced by b on $(\tilde{\Gamma},\tilde{f})$ is a regular ESS of $\tilde{G}$. Let $\tilde{\eta}^k$ be the perturbance induced by η^k. It follows by lemma 10 that for sufficiently small ε and for $|\eta^k| < \varepsilon$ for k = 1,2, ... we can find an ESS $\tilde{b}^k$ for each of the perturbed games $(\tilde{\Gamma}^k,\tilde{f},\tilde{\eta}^k)$ such that the sequence $\tilde{b}^1,\tilde{b}^2,\ldots$ converges to $\tilde{b}$.

Assume $|\eta^k| < \varepsilon$ for k = 1,2, ... with an ε sufficiently small to permit the construction of all the sequences $b_y^1, b_y^2, \ldots$ for symmetric and asymmetric maximal subgames and of $\tilde{b}^1,\tilde{b}^2,\ldots$. Since Γ is finite, an ε of this kind can be found. For k = 1,2, ... let b^k be that strategy of player 1 which agrees with the pertinent b_y^k on every maximal subgame and with $\tilde{b}^k$ on the main truncation. The construction guarantees that the conditions (i), (ii) and (iii) in definition 9.9 of an LSS are satisfied for b^k with respect to $\hat{\Gamma}^k$. It follows by theorem 10 that b^k is an ESS of $\hat{\Gamma}^k$. Therefore, b is a limit ESS of the test sequence $\hat{\Gamma}^1,\hat{\Gamma}^2,\ldots$. This completes the induction step as far as games (Γ,f) with an essential main b-truncation are concerned.

We continue to assume that (Γ,f) has n+1 essential b-elements but we now do not exclude the case of a main b-truncation $(\tilde{\Gamma},\tilde{f})$ which fails to be essential. Let m be the number of inessential b-elements of (Γ,f). We shall use induction on m. We already know that the assertion holds for m = 0, since in this case $(\tilde{\Gamma},\tilde{f})$ is essential. Assume the assertion holds for up to m-1 inessential b-elements, where m is positive. We have to show that the assertion holds for m inessential b-elements in (Γ,f) in the case that the main b-truncation $(\tilde{\Gamma},\tilde{f})$ is inessential. In this case the maximal subgames of (Γ,f) have at most m-1 inessential b-elements and at most n+1 essential b-elements. The assertion of the corrolary can be applied to the maximal subgames. A construction analogous to that used in the case of an essential b-main truncation can be employed here, too, in order to show that the assertion holds for (Γ,f). The construction is even easier than there, since $(\tilde{\Gamma},\tilde{f})$ is inessential. It is not necessary to describe the construction in detail. It is now clear that the theorem and the corrolary hold.

11.11 Comment: The corrolary shows that the sufficient conditions (a) and (b) of theorem 12 guarantee an additional robustness property. An analogous property for equilibrium points has been introduced by Okada under the name of "strict perfectness" (Okada 1981). The robustness property permits us to say that it does not matter which slight mistakes are how much more probable than others provided that all mistake probabilities are sufficiently small and positive.

It may be true that the regularity of the induced ESS is not really needed in condition (b). The theorem may still be true with "essential" instead of "regular" in condition (b). This conjecture is connected to the conjecture expressed at the end of 11.8. However, a sharper version of theorem 12 would probably require much longer proofs.

12. A many period model with ritual fights and escalated conflicts

The results obtained in sections 10 and 11 can be used as tools for the analysis of special models which take the form of a simultaneity game. Even if the game structure is quite complicated one may be able to describe the properties of a limit ESS in sufficient detail to answer theoretically important questions. This will be demonstrated for a many period model with ritual fights and escalated conflicts.

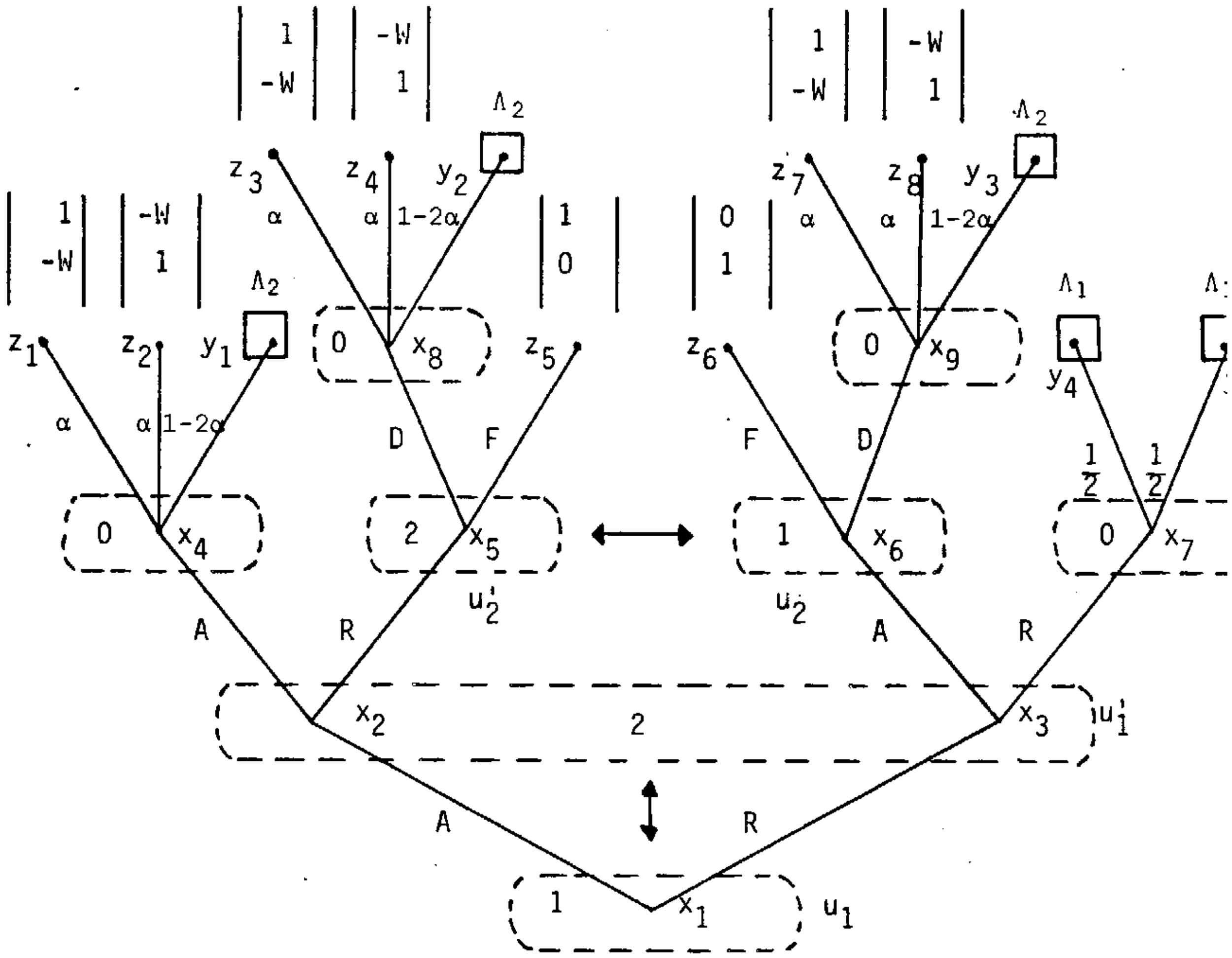
In the model two animals are in conflict over a resource like a territory, a female, etc.. They may engage in ritual fights or serious fights. Ritual fight is modelled as a random mechanism which permits the conventional determination of a "winner" and a "looser", and does not directly influence payoffs.

A contestant who wants to engage in a ritual fight, but finds himself seriously attacked, may either defend himself or flee. In the latter case the attacker gets the resource. If there is a serious fight, both face a risk of receiving a serious wound like in the hawk dove game explained in 2.10. However, the probability of being wounded may be small in one period of serious fight; after each period of serious fight, both players have to decide whether they want to continue the fight or to flee. In this way a serious fight may end without serious consequences.

It will be shown that a limit ESS must be of one of two types. In one case referred to by the name "attacker advantage", the asymmetry created by a unilateral attack leads to a subgame equilibrium which favors the attacker; he gets the resource and the other animal flees. In the second case, named "defender advantage", the attacked animal gets the resource if he defends himself and the fight ends in a draw. However, this does not necessarily mean that a unilaterally attacked animal actually will defend himself. Therefore, two subcases must be distinguished. In the first subcase a unilaterally attacked animal flees, since the defender advantage of getting the resource after a draw is not great enough in comparison to the risk of being wounded. We may refer to this subcase as "ineffective defender advantage". In the second subcase the defender advantage is strong enough to make it worthwhile for a unilaterally attacked animal to defend himself. This subcase can be called "peaceful" since it completely excludes serious fights; unilateral attacks are effectively deterred by the willingness to defend. It is interesting that this kind of deterrence cannot work, unless the risk of being wounded within one round of serious fight is not too high.

<u>12.1 Explanation of the model</u>: The model takes the form of a simultaneity game. The extensive game can be thought of as composed of many copies of two <u>building blocks</u> Λ_1 and Λ_2 shown in figures 16 and 17. The game is played over T periods 1,...,T with $T \geq 2$. The game structure for period 1 is that of Λ_1. First, both players have to decide independently whether they want to choose A (serious attack) or R (ritual fight). If both choose A, a round of serious fighting results whose outcome is decided by the random choice at x_4. With probability α player 2 receives a serious wound and suffers a fitness loss of W and player 1 gets the resource whose fitness value is normed to 1. With the same probability α the opposite result occurs. With probability $1-2\alpha$ the round of fight results in a draw and the second period begins at y_1. This vertex y_1 is a <u>connecting point</u>.

Connecting points are graphically indicated by little enclosing squares. The name of the building block used for continuation is shown above this little square. Pictorially speaking, the origin of a copy of this building block is glued to the connecting point. A new period begins after a connecting point, as long as period T has not yet been reached. Of course, the game is not continued



Symbols:

A serious attack

R ritual fight

D defend

F flee

W fitness loss of wound

α probability being wounded

[■] connecting points (for t=T endpoints with zero payoffs)

Assumptions on the parameters

$W > 1$

$0 < \alpha < 1$

Figure 16: The building block Λ_1

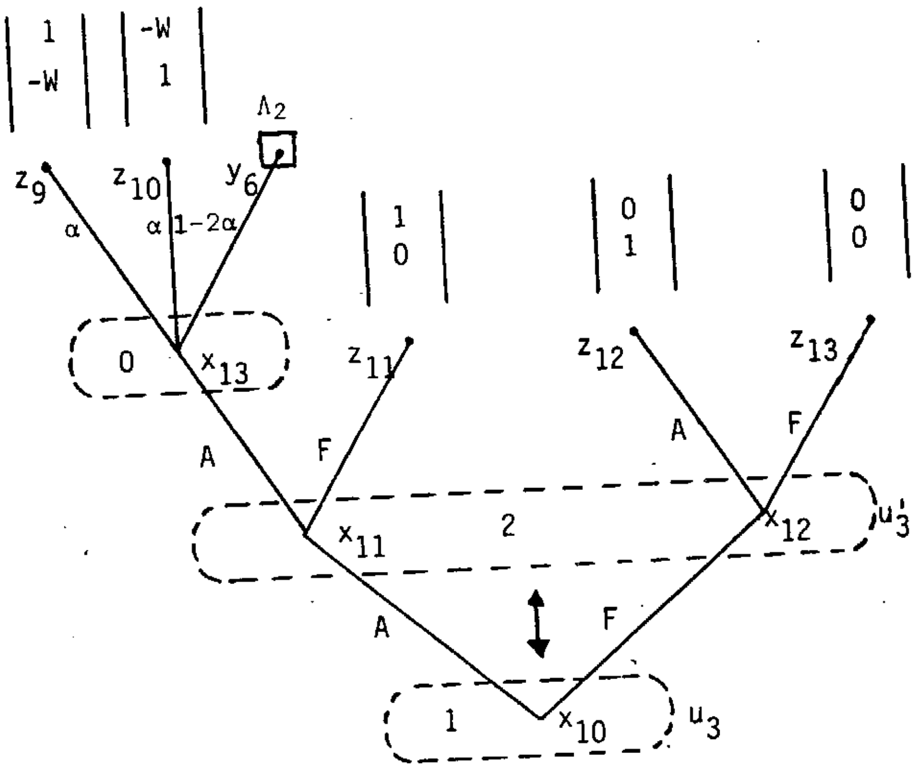


Figure 17: The building block Λ_2. For the meaning of symbols see figure 16.

beyond period T. At the end of period T connecting points become endpoints with zero payoffs.

If player 1 selects A and player 2 chooses R, then at x_5 in Λ_1 player 2 has to decide between D (defend) and F (flee In the case of D a serious fight takes place with the same array of consequences as after a double choice of A. In the case of F player 1 receives the resource whose value is 1 and player 2 receives zero payoffs.

The same situation with reversed roles results from a choice of R by player 1 and of A by player 2.

If both players choose R, then a round of ritual fighting takes place. Ritual fighting is not modelled in detail but simply as a random mechanism whose outcome clearly distinguishes both players. This is expressed by the distinguishing random choise at x_7. From there two connecting points y_4 and y_5 are reached with equal probabilities. Who is regarded as the "winner" or "looser" of a round of ritual fight is not determined by the rules of the game but by the conventions embodied in an ESS.

At y_4 and y_5 the game continues with copies of Λ_1. We do not exclude the possibility of several rounds of ritual fight. However, once serious fighting has begun, it cannot stop unless one of the animals decides to flee or receives a serious wound; in the latter case the wounded animal has no choice and must flee. These assumptions underly the structure of building block Λ_2 where the players have to decide between A (serious attack) or F (flee). If both choose A a new round of serious fight takes place with the same array of consequences as in Λ_1. If one player attacks and the other flees, the attacker receives the resource. We do not exclude the possibility that both of them flee and none of them receives the resource.

It is now clear how in principle a graphical representation of the full extensive game can be constructed. One begins with a copy of Λ_1 for period 1. One continues with copies of Λ_1 at y_4 and y_5 and with copies of Λ_2 at y_1, y_2 and y_3; thereby one builds up period 2. As long as the end of period T has not yet been reached, new copies of building blocks are glued to the connecting points at the end of period t in order to build up period t+1. Finally, at the end of period T the connecting points become endpoints with zero payoffs for both players.

The natural symmetry of the game is based on the convention that choices are mapped to choices with the same name. The symmetric images of information sets are indicated by double arrows in figure 16 and 17. However, these arrows describe the natural symmetry only for those copies of building blocks which are not preceded by an asymmetric pair of choices like (A,R) or (R,A) in a copy of Λ_1 or by a distinguishing random choice at a copy of x_7. Such copies are called symmetric; other copies are asymmetric.

The copy of Λ_1 for the first period is the only symmetric copy of Λ_1. The only symmetric copies of Λ_2 are those which follow an unbroken sequence of pairs (A,A) of choices in previous periods.

The naming convention for choices together with the double arrows in figures 16 and 17, understood in the way described above, completely determine the natural symmetry of the game. It can be seen immediately that the origin of a symmetric building block copy is the origin of a symmetric subgame and that the origin of an asymmetric building block copy is the origin of an asymmetric subgame.

A parameter triple (α, W, T) will be called admissable if T is an integer with $T \geq 2$ and if in addition to this

(131) $0 < \alpha < 1/2$

and

(132) $W > 1$

hold for α and W. For every admissable parameter triple (α, W, T) the model generates a simultaneity game (Γ,f). The structure of this game is sufficiently clear from the explanations given above. It seems to be unnecessary to add a description by a more precise set theoretical formalism. It can be seen easily without such a formalism that the game (Γ,f) generated by the model for an admissable parameter combination (α, W, T) is a simultaneity game in the sense of definition 11.1.

12.2 Preview of the analysis: We shall first rely on necessary conditions which must be satisfied by a limit ESS according to theorem 10. It will be shown that every limit ESS must be of one of two types named "attacker advantage" and "defender advantage" which have been mentioned already in the introduction to this section. There, it has been explained already that the defender advantage may be ineffective in the sense that it does not provide sufficient incentive to choose D in period 1. Whether this is the case or not depends on the following constant g, called defense gain:

(133) $g = 1-\alpha-\alpha W$

In a defender advantage limit ESS a player who selects D in period 1 can expect to win the resource with probability α by inflicting a serious wound and with probability $1-2\alpha$ after a draw;with probability α he himself will be wounded. The defense gain g is nothing else than the local payoff for D in period 1 under a defender advantage limit ESS.

For $g > 0$ a defender advantage limit ESS is peaceful in the sense explained in the introduction of this section. It will turn out that for $g < 0$ a defender advantage ESS differs from an attacker advantage ESS only in unreached parts of the game.

Some questions concerning the borderline case $g = 0$ will be left unanswered.

In the course of the analysis the necessary conditions of theorem 10 will be examined for various subgames und truncations. A Λ_1-subgame is a subgame which starts with a copy of Λ_1 and a Λ_2-subgame is a subgame whichs starts with a copy of Λ_2. First we shall look at the last period. The next step will be the analysis of the symmetric Λ_2-subgames. It will be shown that payoffs of these subgames are zero, if a limit ESS is played. Then we shall prove that the necessary conditions for a limit ESS exclude serious fighting in asymmetric Λ_2-subgames. Already at the beginning of such subgames one player must attack and the other must flee.

The analysis of the asymmetric Λ_1-subgames will reveal that in these subgames, too, the necessary conditions of theorem 10 exclude serious fighting; moreover, one of both players must receive the resource in the end.

On the basis of the results on Λ_1-subgames and Λ_2-subgames, it will be possible to investigate the first period. It will be shown that the necessary conditions for a limit ESS permit only two patterns of first period behavior. One of these patterns is produced by a defender advantage limit ESS with $g > 0$ and the other one is characteristic for all other cases with $g \neq 0$.

After the exploitation of the necessary conditions of theorem 10, the existence of an attacker advantage limit ESS and a defender advantage limit ESS will be shown for every admissable parameter triple with $g \neq 0$ by the construction of specific strategies of both types. It will not be difficult to check the sufficient conditions of theorem 12.

In sections 12.3 and 12.8 we shall always assume that b is a limit ESS of the game (Γ,f) generated by the model for an admissable parameter triple (α,W,T).

<u>12.3 The last period</u>: A subgame which starts with the beginning of the last period will be called a <u>last period subgame</u>. Obviously, a last period subgame is either a copy of Λ_1 or a copy of Λ_2 with zero payoffs for both players at the connecting points.

It is convenient to introduce the following constant a:

(134) $\quad a = \alpha W - \alpha$

In a last period subgame the expected payoff of each of both players at vertices corresponding to x_4, x_7, x_8 and x_9 is $-a$. The constant a can be interpreted as the expected loss conncected to a fight with zero payoffs after a draw. In view of $W > 1$ the constant a is always positive.

Consider a last period subgame which is a Λ_1-subgame. In view of $a > 0$ it is clear that F is the only best reply

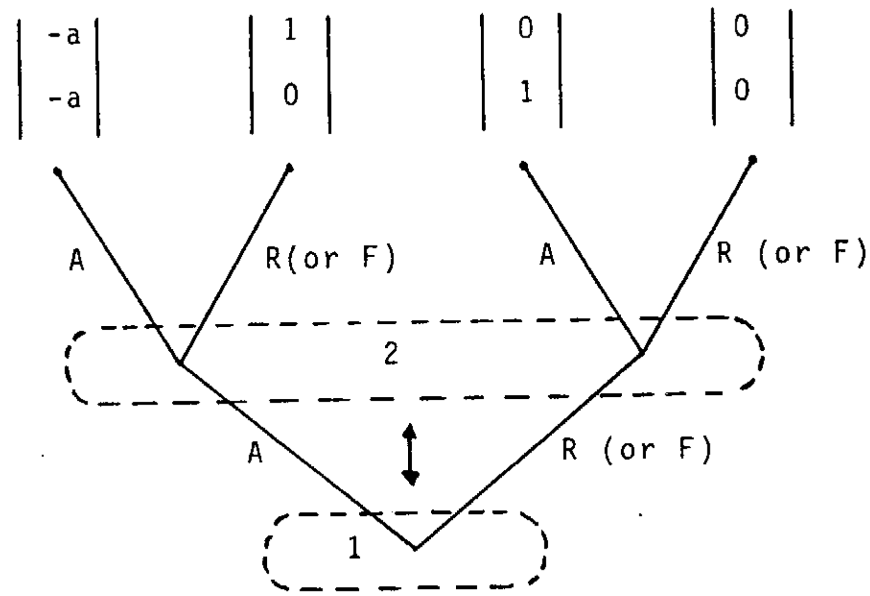


Figure 18: The common structure of all main b-truncations of last period subgames.

	A	R (or F)
A	-a, -a	1, 0
R (or F)	0, 1	0, 0

Figure 19: Normal form of the game of figure 18.

at the vertices corresponding to x_5 and x_6. It follows by (a) in theorem 10 that b prescribes F at these vertices. It follows that the main b-truncation of a last period subgame of this kind looks like the game of figure 18.

It can be seen immediately that the main b-truncation of a last period subgame derived from a copy of Λ_2 also looks like the game of figure 18.

With the help of figure 19 it can be seen easily that the following is true: If a last period subgame is symmetric, then its main b-truncation has exactly one symmetric equilibrium strategy which assigns the following probability q_A to choice A:

$$q_A = \frac{1}{1+a} \qquad (135)$$

In view of (d) in theorem 10 the local strategies prescribed by b in a symmetric last period subgame select A with probability q_A at information sets corresponding to u_1 and u_3 in figures 16 and 17. If in figure 19 both players play the symmetric equilibrium strategy, then both of them receive zero payoffs.

In view of (a) in theorem 10 a pure strategy equilibrium point is induced by (b,f(b)) on an asymmetric last period subgame. One of the players must choose A and the other has to choose R in the case of a Λ_2-subgame. This results in payoffs of 1 for the player who chooses A and in zero payoffs for the other player.

The results obtained for the last period are summarized by the table in figure 20.

12.4 Symmetric Λ_2-subgames: Consider a symmetric Λ_2-subgame beginning in period T-n. We shall show by induction on n that the main b-truncation of a subgame of this kind has the structure of the game of figure 18. As we have seen

Type of last period subgame	main decision 1)		second decision in Λ_1 2)	subgame payoffs	
	player 1	player 2		player 1	player 2
asymmetric Λ_1-subgame	A or R	R A	F	1 0	0 1
asymmetric Λ_2-subgame	A or F	F A	-	1 0	0 1
symmetric Λ_2-subgame	$q_A = \frac{1}{1+a}$ 3)		-	0	0

Figure 20: Strategies induced by a limit ESS and its symmetric image on last period subgames and corresponding payoffs. (Last period subgames cannot be symmetric Λ_1-subgames.)

1) Decisions at information sets corresponding to u_1 and u_1' in Λ_1 and to u_3 and u_3' in Λ_2
2) Decisions at information set corresponding to u_2 and u_2' in Λ_1
3) Probability for A

in 11.3 this is true for n = 0. Suppose that the assertion holds for n. Then in a symmetric Λ_2-subgame beginning in period T-n the expected payoffs for both players are zero if b and f(b) are played. Therefore, the expected payoffs for (F,F) in the main b-truncation of a symmetric Λ_2-subgame beginning in period T-n-1 are zero for both players. This shows that the assertion is true. It follows by (d) in theorem 10 that b and f(b) induce local strategies on a main b-truncation of a symmetric Λ_2-subgame which select A with probability q_A. The results are summarized by table 21.

decisions		payoffs	
player 1	player 2	player 1	player 2
$q_A = \frac{1}{1+a}$		0	0

Figure 21: Strategies induced on main b-truncations of symmetric Λ_2-subgames by a limit ESS b and its symmetric image f(b) and corresponding payoffs.

12.5 Asymmetric Λ_2-subgames: Consider an asymmetric Λ_2-subgame Γ_y. In view of (a) a pure strategy equilibrium point (b_y, b'_y) is induced on Γ_y by (b,f(b)). In the last period one of the players must choose F in (b_y, b'_y). This follows by the table in figure 20. If in some earlier period one of the players chooses F in (b_y, b'_y), then the other one must choose A in this period. Consider that player who is the first to choose F in (b_y, b'_y). If this choice of F did not occur already in the first period of Γ_y, then he could improve his payoff by choosing F in the first period of Γ_y. In this way, he would avoid unnecessary losses incurred by serious fighting. It follows that (b_y, b'_y) must prescribe A to one player and F to the other player already at the beginning of Γ_y. This results in subgame payoffs of 1 for the player who chooses A at the beginning of Γ_y; the other one receives zero. The results are summarized by the table in figure 22.

decisions		payoffs	
player 1	player 2	player 1	player 2
A	F	1	0
or			
F	A	0	1

Figure 22: Strategies induced on main (b,f(b))-truncations of asymmetric Λ_2-subgames by a limit ESS b and its symmetric image f(b) and corresponding payoffs.

<u>12.6 Asymmetric Λ_1-subgames</u>: Asymmetric Λ_1-subgames always start at a vertex corresponding to a connecting point y_4 in a copy of Λ_1. Let Γ_y be an asymmetric Λ_1-subgame which starts in a period before the last period. We shall use the notation $\bar{x}_k$ for the vertex which corresponds to x_k in the starting period of Γ_y. Similarily $\bar{z}_k, \bar{u}_k$ and $\bar{u}'_k$ denote that endpoint or information set of Γ_y which corresponds to z_k, u_k and u'_k, respectively, in the starting period of Γ_y.

In order to have a convenient way of speaking we shall introduce the "starting period truncation" of Γ_y. Let M be the multisubgame of Γ_y containing the subgames at $\bar{x}_4$, $\bar{x}_8$, $\bar{x}_9$ and $\bar{x}_7$. Moreover, let b_y and b'_y be the strategies induced by b and f(b) on Γ_y. The <u>starting period truncation</u> $\bar{\Gamma}_y$ of Γ_y is the (b_y,b'_y)-truncation of Γ_y with respect to M.

Let $\bar{\Gamma}_4,\bar{\Gamma}_8,\bar{\Gamma}_9$ and $\bar{\Gamma}_7$ be the subgames at $\bar{x}_4,\bar{x}_8,\bar{x}_9$ and $\bar{x}_7$ of Γ_y. In view of the table in figure 22 it is clear that in the asymmetric Λ_2-subgames of $\bar{\Gamma}_4,\bar{\Gamma}_8,\bar{\Gamma}_9$ and $\bar{\Gamma}_7$ one of both players receives 1 and the other one receives 0, if b and f(b) are played. This has the consequence that in the starting period truncation $\bar{\Gamma}_y$ at each of the endpoints $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$ one of the players receives -a and the other receives g.

a can be interpreted as the expected loss for a serious fight if 0 is the payoff after a draw and g as the expected gain for a serious fight if 1 is the payoff after a draw.

Figure 23 shows all possible structures of a starting period truncation $\bar{\Gamma}_y$ of an asymmetric Λ_1-subgame Γ_y. The payoffs β_1 and β_2 remain unspecified.

It follows by (b) in theorem 10 that a pure strategy equilibrium point $(\bar{b}_y,\bar{b}'_y)$ is induced on $\bar{\Gamma}_y$ by (b,f(b)).

The equilibrium payoffs connected to $(\bar{b}_y, \bar{b}'_y)$ cannot be negative since each of both players can enforce zero by choosing R and F. Therefore, the equilibrium play generated by $(\bar{b}_y, \bar{b}'_y)$ cannot end in one of the endpoints $\bar{x}_4, \bar{x}_8$ or $\bar{x}_9$ since there, one of both players receives a payoff of -a. Consequently, the play generated by $(\bar{b}_y, \bar{b}'_y)$ leads to $\bar{z}_5, \bar{z}_6$ or $\bar{x}_7$.

We shall continue the investigation of asymmetric Λ_1-subgames in 12.7. The results obtained up to now are summarized by figure 24.

12.7 The payoffs β_1 and β_2: The payoffs β_1 and β_2 in figure 23 are averages of equilibrium payoffs for two asym-

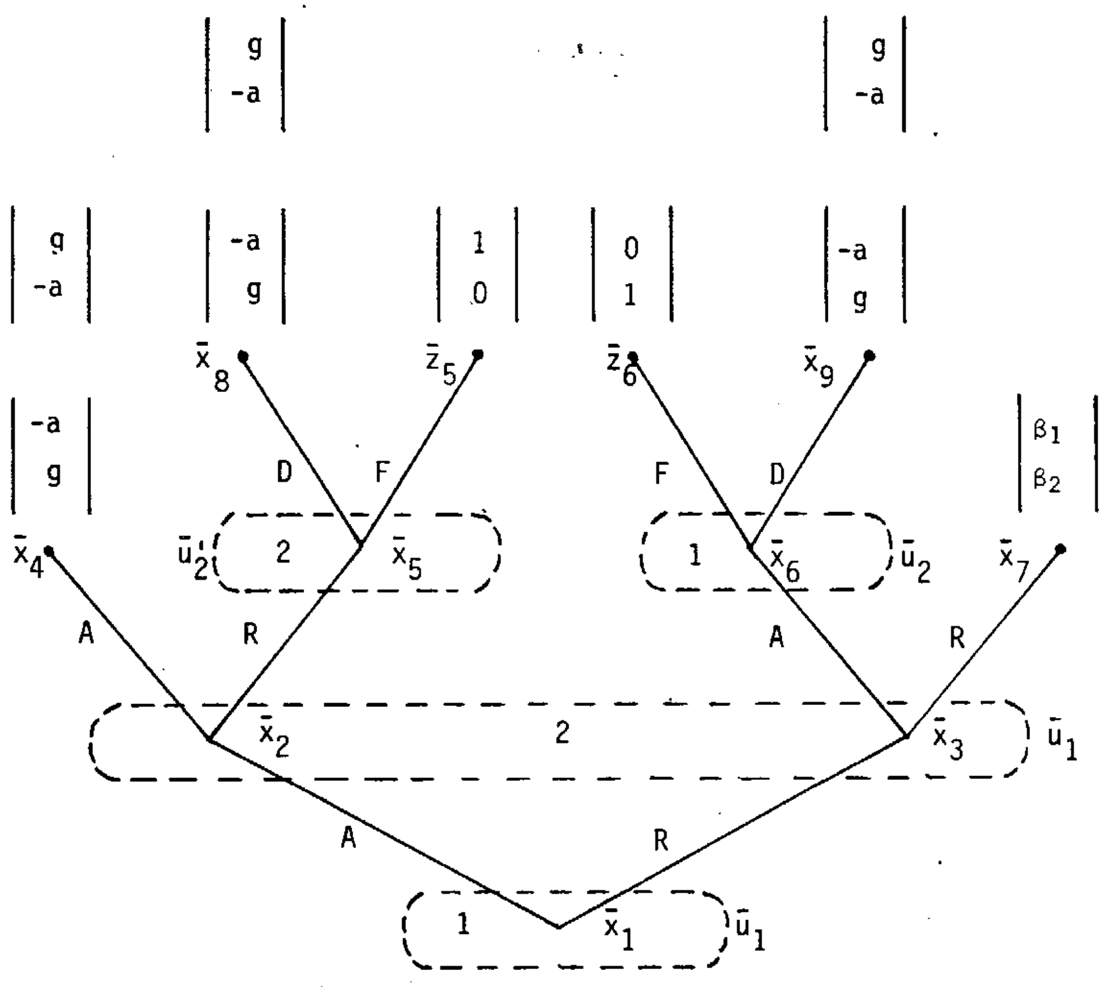


Figure 23: Possible structures of a starting period truncation of an asymmetric Λ_1-subgame. Both of the possible payoff vectors are shown above $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$.

metric Λ_1-subgames. In these subgames each of both players can enforce zero, simply by always choosing R and F. Therefore β_1 and β_2 are nonnegative:

(136) $\beta_1 \geq 0$

(137) $\beta_2 \geq 0$

main decision 1)		second decision 2)		payoff	
player 1	player 2	player 1	player 2	player 1	player 2
A	R		F	1	0
or					
R	A	F		0	1
or					
R	R			β_1	β_2

Figure 24: Possible equilibrium plays induced on starting period truncations of asymmetric Λ_1-subgames by a limit ESS and its symmetric image and corresponding payoffs.

1) decisions at information sets corresponding to u_1 and u_1'

2) decisions at information sets corresponding to u_2 and u_2'

We shall show that in addition to this we have:

(138) $\beta_1 + \beta_2 = 1$

Consider the strategy pair (b_y, b'_y) induced on an asymmetric Λ_1-subgame Γ_y by $(b,f(b))$. Let z be an endpoint of Γ_y whose realization probability $\gamma_y(z,b_y,b'_y)$ in Γ_y under b_y and b'_y is positive. We shall show that the payoffs at z are either $h(z) = 1$ and $h'(z) = 0$ or $h(z) = 0$ and $h'(z) = 1$. In view of figure 20 this is the case, if Γ_y is a last period subgame. If Γ_y starts earlier, then it follows by figure 24 that z has this property if it belongs to the starting period; otherwise it belongs to a shorter Λ_1-subgame. A simple induction argument shows that also in this case z has the asserted property. (138) is an immediate consequence of this.

<u>12.8 The first period</u>: For the sake of simplicity the same names of vertices and information sets as in Λ_1 are used for the copy of Λ_1 which constitutes the first period of (Γ,f).

Let (Γ_4,f_4), Γ_8,Γ_9, and (Γ_7,f_7) be the subgames of (Γ,f) at x_4,x_8,x_9 and x_7, respectively. Let M be the multisubgame containing these four subgames. Since the two asymmetric subgames Γ_8 and Γ_9 in M are symmetric images of each other, M is symmetric. The <u>first period truncation</u> $(\bar{\Gamma},\bar{f})$ is the b-truncation of (Γ,f) with respect to M.

As we shall see the first period truncation either has the form of figure 25 or that of figure 26. In view of figure 21 the payoffs for the strategies induced by b and f(b) in the symmetric Λ_2-subgame at the connecting point y_1 immediately following x_4 are zero for both players. Therefore, in figures 25 and 26 both players receive -a at x_4.

In the same way as for the starting period truncation in figure 23 it can be seen that at x_8 and x_9 one player receives g and the other receives -a; moreover, in view of

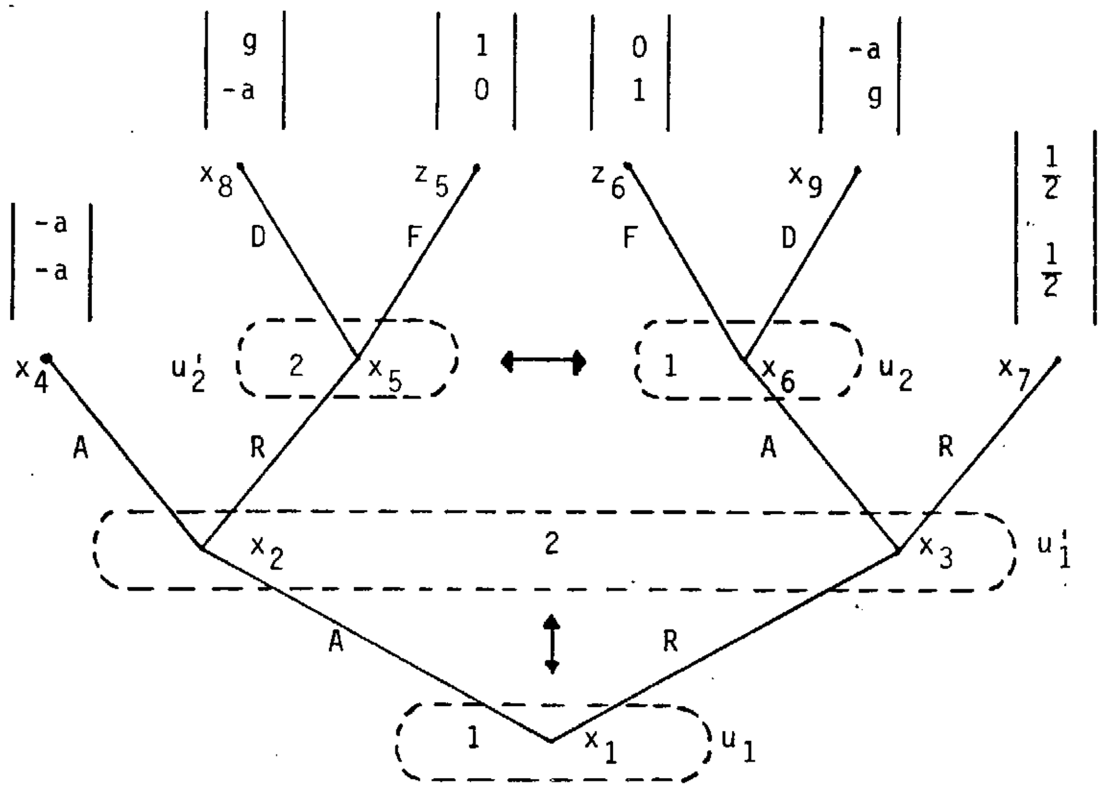


Figure 25: Structure of the first period truncation in the case of an attacker advantage limit ESS.

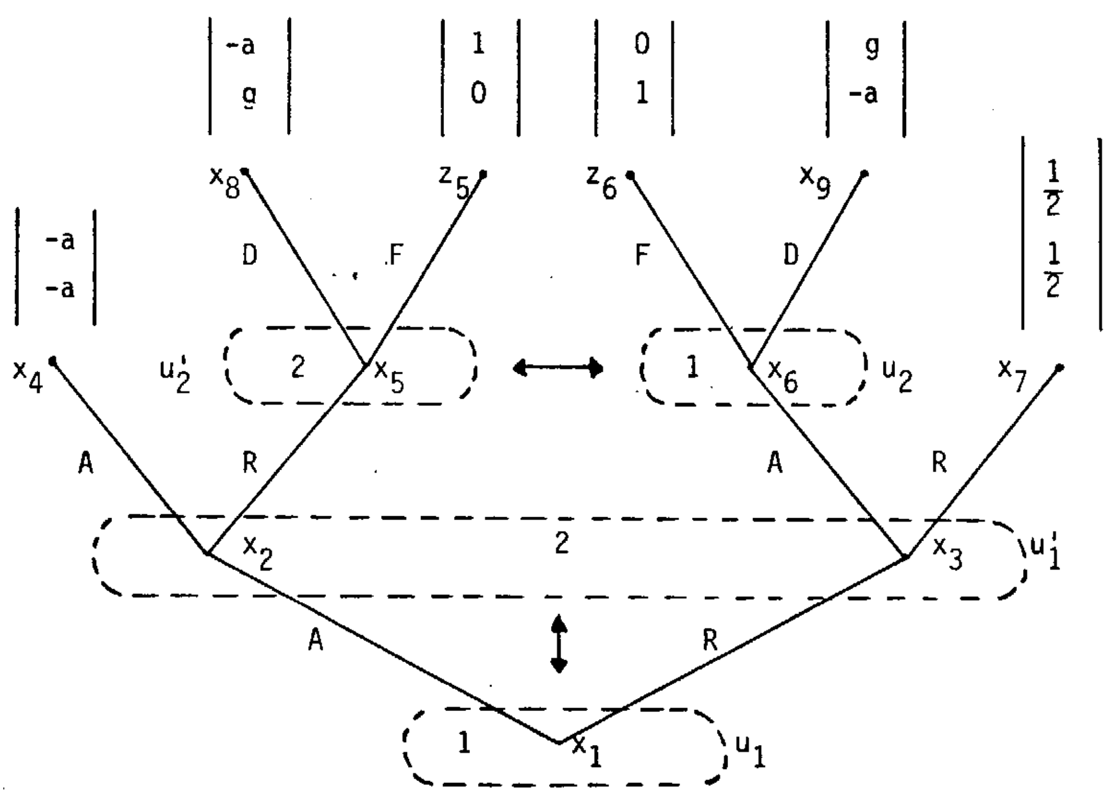


Figure 26: Structure of the first period truncation in the case of a defender advantage limit ESS.

$\Gamma_9 = f(\Gamma_8)$ one player's payoff at x_8 is the other player's payoff at x_9. There are only two possibilities with respect to the payoffs at x_8 and x_9 which are shown by figure 25 and 26.

The subgame (Γ_7, f_7) begins with a random choice leading with equal probabilities to two asymmetric Λ_1-subgames Γ_4 and Γ_5 which are symmetric images of each other; this has the consequence that both players have the same payoffs at x_7. Moreover, it is clear from figure 24 and (138) that the payoffs of both players must sum up to 1. Therefore, each of both players receives 1/2 at x_7.

We call b an attacker advantage limit ESS, if the first period truncation has the form of figure 25 and a defender advantage limit ESS, if it has the form of figure 26. The names are suggested by the interpretation of the model. The difference concerns the convention in the two subgames reached by a unilateral attack in period 1 followed by a defense and a draw in the resulting fight. There, in the attacker advantage case the attacker gets the resource and the defender flees and in the defender advantage case the defender gets the resource and the attacker flees.

Statement (b) in theorem 10 can be applied to the subgames of the first period truncation at x_5 and x_6. Equilibrium points in pure strategies must be induced there by (b,f(b)). This has the consequence that b and f(b) must prescribe F at u_2 and u_2' in figure 25.

It is necessary to make a case distinction between $g < 0$ and $g > 0$ in the case of figure 26. We shall omit the treatment of the borderline case $g = 0$. For $g < 0$ the situation is essentially the same as in figure 25. The choice F must be prescribed by b and f(b). For $g > 0$ the choice D must be prescribed by b and f(b) at u_2 and u_2' in figure 26.

	A	R
A	$-a$, $-a$	1, 0
R	0, 1	$\frac{1}{2}$, $\frac{1}{2}$

Figure 27: Symmetric normal form of the main b-truncation in the case of an attacker advantage and in the case of a defender advantage with $g < 0$.

	A	R
A	$-a$, $-a$	$-a$, g
R	g, $-a$	$\frac{1}{2}$, $\frac{1}{2}$

Figure 28: Symmetric normal form of the main b-truncation in the case of a defender advantage with $g > 0$.

type of limit ESS	main decision [1]	second decision [2]	individual payoffs
attacker advantage	$q(A) = \frac{1}{1+2a}$ [3]	F	$\frac{1}{1+2a}$
defender advantage for $g < 0$			
defender advantage for $g > 0$	R	D	$\frac{1}{2}$

Figure 29: Strategies induced on the first period truncation by a limit ESS and corresponding payoff.

1) decision at u_1
2) decision at u_2
3) probability of A

The main b-truncation of (Γ,f) is nothing else than the main $\bar{b}$-truncation of the first period truncation $(\bar{\Gamma},\bar{f})$ where $\bar{b}$ is the strategy induced by b on $\bar{\Gamma}$. From what has been said on the decisions at u_2 and u_2' in figures 25 and 26, it can be seen immediately how the main b-truncation of (Γ,f) looks like. The two cases which can arise with respect to the symmetric normal form of the main b-truncation are shown by figures 27 and 28.

It follows by (d) in theorem 10 that a symmetric equilibrium strategy is induced by b on the main b-truncation. The induced equilibrium strategies can be examined in the symmetric normal form of the main b-truncation.

The game of figure 27 has exactly one symmetric equilibrium strategy q which assigns the following probability q(A) to A.

$$q(A) = \frac{1}{1 + 2a} \tag{139}$$

Obviously, A and R fail to be symmetric equilibrium strategies of the game of figure 27. It can be checked easily that (139) describes the only mixed symmetric equilibrium strategy.

The game of figure 28 also has exactly one symmetric equilibrium strategy, namely R. In view of $g > 0$ it is clear That R is not only a strong best reply to R but to all mixed strategies.

The results obtained for the first period are summarized by figure 29.

12.9 Discussion: The application of the necessary conditions of theorem 10 to the model has covered all admissable parameter triples with the exception of the borderline case $g = 0$. In the following we shall always assume $g \neq 0$.

It is interesting to know whether serious fights occur with positive probability or not, if a limit ESS and its symmetric image are played. This is maybe the most important question to be asked about a limit ESS. The answer to this question depends on the distinction between an attacker advantage limit ESS and a defender advantage limit ESS. It has been shown that no serious fights occur for $g > 0$ in the case of a defender advantage limit ESS and that at least one round of serious fighting occurs with probability $1/(1+2a)^2$ in all other cases.

The necessary conditions exclude serious fights after a round of ritual fight in the first period. Ritual fight always leads to a peaceful settlement where one contestant flees and the other receives the resource. However, a great variety of conventions can be used in order to determine the final owner of the resource.

The first round of ritual fighting may be decisive or many rounds may be necessary before a settlement can be reached. Obviously, there is not much interest in a complete classification of all possibilities which may arise in this respect. The probability distribution of final results does not depend on the details of the conventions surrounding settlement by ritual fight. After a ritual fight in the first period serious fights do not occur and both players have the same chance of 1/2 to receive the resource.

One may be interested to know what sequences of interactions involving no ritual fights can occur with positive probability if a limit ESS and its symmetric image are played. On the basis of the distinction between an attacker advantage limit ESS and a defender advantage limit ESS the conclusions obtained from the necessary conditions permit a complete answer to this question.

For $g > 0$ a defender advantage limit ESS excludes a choice of A in period 1. Consider an attacker advantage limit ESS or a defender advantage limit ESS with $g < 0$. Figure 21 shows what happens if a symmetric Λ_2-subgame is reached after a choice of A by both players in the first period. After a unilateral attack in the first period, the attacked animal flees and the attacker receives the resource.

A choice of D cannot be observed if a limit ESS and its symmetric image are played. Nevertheless, it is important that in the case of a defender advantage ESS with $g > 0$ a unilaterally attacked animal would respond with D. The contingent threat of defense deters an attack.

The discussion has shown that the salient features of a limit ESS are captured by the distinction between the two cases which can arise with respect to the behavior in the first period. In order to have a convenient way of speaking we shall refer to the case of an attacker advantage limit ESS or a defender advantage limit ESS with $g < 0$ as <u>partially ritualized</u> and to the case of a defender ESS with $g > 0$ as <u>completed ritualized</u>.

Complete ritualization requires a positive value of the defense gain g. Since g decreases with α and W this leads to the conclusion that complete ritualization is favored by a relatively low risk of being wounded in one round of serious fighting.

If it is easy to disengage from a serious fight, a round of serious fighting must be thought of as short. α can be expected to be small, if a round of serious fighting is short. This suggests that easiness of disengagement from serious fight favors complete ritualization.

It has not yet been shown that an attacker advantage limit ESS and a defender advantage limit ESS exist for every $g \neq 0$. This will be done in 12.10. A limit ESS in each of both classes will be constructed for every admissable parameter triple with $g \neq 0$. The conventions surrounding settlements by ritual fight will be fixed in the simplest possible way which is maybe not the most plausible one. The same is true for other conventions concerning unreached parts of the game which also need to be specified by the complete description of a limit ESS.

<u>12.10 Construction of specific examples of both types of a limit ESS</u>: For $i = 1,\dots 5$ let $\bar{y}_i$ be that vertex of the copy of Λ_1 for the first period which corresponds to y_i in figure 16. Let (Γ^1, f^1) and $\Gamma^2,\dots,\Gamma^5$ be the subgames at $\bar{y}_1,\dots,\bar{y}_5$, respectively. The conventions surrounding settlement by ritual fight and other conventions concerning unreached parts of the game will be fixed in such a way that the player who is favored by the settlement is always the same in the same asymmetric subgame Γ^i with $i = 2,\dots,5$. This permits a relatively simple description of the construction.

<table>
<tr><td rowspan="3"></td><td rowspan="3">after first period connecting point</td><td colspan="8">decisions in copies of Λ_1</td></tr>
<tr><td colspan="4">main decision 1)</td><td colspan="4">second decision 2)</td></tr>
<tr><td colspan="2">attacker advantage</td><td colspan="2">defender advantage</td><td colspan="2">attacker advantage</td><td colspan="2">defender advantage</td></tr>
<tr><td></td><td></td><td>$g < 0$</td><td>$g > 0$</td><td>$g < 0$</td><td>$g > 0$</td><td>$g < 0$</td><td>$g > 0$</td><td>$g < 0$</td><td>$g > 0$</td></tr>
<tr><td>symmetric $t = 1$</td><td>-</td><td colspan="3">$\frac{1}{1 + 2a}$ 3)</td><td>R</td><td colspan="3">F</td><td>D</td></tr>
<tr><td rowspan="2">asymmetric $t=2,...,T-1$</td><td>$\bar{y}_4$</td><td colspan="4">A</td><td>F</td><td>D</td><td>F</td><td>D</td></tr>
<tr><td>$\bar{y}_5$</td><td colspan="4">R</td><td colspan="4" rowspan="3">F</td></tr>
<tr><td rowspan="2">asymmetric $t = T$</td><td>$\bar{y}_4$</td><td colspan="4">A</td></tr>
<tr><td>$\bar{y}_5$</td><td colspan="4">R</td></tr>
</table>

<table>
<tr><td rowspan="2"></td><td rowspan="2">after first period connecting point</td><td colspan="2">decisions in copies of Λ_2</td></tr>
<tr><td>attacker advantage</td><td>defender advantage</td></tr>
<tr><td>symmetric $t=2,...,T$</td><td>$\bar{y}_1$</td><td colspan="2">$\frac{1}{1 + a}$ 3)</td></tr>
<tr><td rowspan="4">asymmetric $t=2,...,T$</td><td>$\bar{y}_2$</td><td>A</td><td>F</td></tr>
<tr><td>$\bar{y}_3$</td><td>F</td><td>A</td></tr>
<tr><td>$\bar{y}_4$</td><td>A</td><td>A</td></tr>
<tr><td>$\bar{y}_5$</td><td>F</td><td>F</td></tr>
</table>

1) decisions at information sets corresponding to u_1 in figure 16

2) decisions at information sets corresponding to u_2

3) probability of A

Figure 30: Special attacker advantage limit ESS and special defender advantage limit ESS for $g \neq 0$.

Figure 30 shows a specific attacker advantage limit ESS and a specific defender advantage ESS for every $g \neq 0$. The figure contains two separate tables for decisions in copies of Λ_1 and Λ_2. The first column indicates whether a symmetric or an asymmetric subgame starts with the copy under consideration in one of the periods listed there. The second column shows after which of the vertices $\bar{y}_1,\ldots,\bar{y}_5$ the decision has to be made.

No case distinction between $g < 0$ and $g > 0$ needs to be made with respect to the decisions in copies of Λ_2. However, the distinction is important for copies of Λ_1.

It can be seen without difficulty that the strategies shown in figure 30 satisfy the necessary conditions summarized by figures 20, 21, 22, 24 and 29. In the following we shall show that in all cases the sufficient conditions of theorem 12 are satisfied, too.

We begin with the main b-truncation of (Γ,f). It can be verified easily that the symmetric equilibrium strategy for the game of figure 28 satisfies Haigh's criterion (see 11.5). The same is true for the symmetric equilibrium strategy of the game of figure 29. Therefore, condition (a) in theorem 12 is satisfied for the main b-truncation in both cases.

Now consider the main b-truncations of the asymmetric subgames at the vertices x_5 and x_6 of the first period copy of Λ_1 (see figures 25 and 26). It can be seen that a strong equilibrium point is induced there in both cases. Condition (b) of theorem 12 is satisfied for the second decisions in period 1.

The conventions surrounding settlement by ritual fight are fixed in favor of player 1 after $\bar{y}_4$ and in favor of player 2 after $\bar{y}_5$. Consider an asymmetric Λ_1-subgame which starts in one of the periods 2,...,T-1.

The starting period truncation looks like figure 23 with the upper payoff vectors at $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$ for a subgame after $\bar{y}_4$ and with the lower payoff vectors at $\bar{x}_4$, $\bar{x}_8$ and $\bar{x}_9$ for a subgame after $\bar{y}_5$. It can be seen that in all cases strong equilibrium points are induced on the b-elements which belong to starting period truncations of Λ_1-subgames beginning before the last period. It is also clear that strong equilibrium points are induced on b-elements belonging to last period asymmetric Λ_1-subgames.

Haigh's criterion is satisfied for the symmetric equilibrium strategy of the game of figure 19. Therefore, condition (a) of theorem 12 is satisfied for the main b-truncations of symmetric Λ_2-subgames. In view of $g < 1$ it can be seen immediately that strong equilibrium points are induced on all main b-truncations of asymmetric Λ_2-subgames.

It is now clear that the sufficient conditions of theorem 12 are satisfied for the strategies described by figure 30. We can conclude that these strategies have the properties of a limit ESS.

<u>12.11 Concluding remark</u>: The analysis of the model has shown how the necessary conditions of theorem 10 and the sufficient conditions of theorem 12 can be used in order to analyse simultaneity games of considerable complexity. The results obtained are not without interest from the substantial point of view. New light is thrown on the old problem of stable conventions involving a choice between ritual fight and serious attack. The hawk-dove-game is an ingeniously simple way of posing the problem. The model examined here is necessarily much more complex since it explicitly considers a great variety of possible interactions extended over many periods.

The model examined here partially confirms the intuitions underlying the simplifying assumptions embodied in the hawk-dove-game. In fact, the main b-truncation in figure 28 has the form of a hawk-dove-game. Figure 29, however, shows a different picture. The total absence of serious fights can also be supported by stable conventions if the risk of being wounded within one period of serious fighting is sufficiently low in the sense of a positive defense gain.

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