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Bargaining with Incomplete Information
An Axiomatic Approach

by

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Abstract

Within this paper we consider a model of Nash bargaining with incomplete information. In particular, we focus on fee games, which are a natural generalization of side payment games in the context of incomplete information. For a specific class of fee games we provide two axiomatic approaches in order to establish the Expected Contract Value, which is a generalized version of the Nash bargaining solution.

SECTION 1

Introduction

Bargaining with incomplete information, fee games

Suppose two players are entitled to divide a unit of money (or less) among themselves. However, when registering the contract (i.e. a vector $x \in \mathbb{R}^2_+$, $x_1 + x_2 \leq 1$) with the court, they have to pay a fee which is proportional to the total amount $x_1 + x_2$. Before the agreement can be registered, a coin (with faces α and β) is thrown according to which the fee is computed as follows

player 1 pays : $\frac{1}{10}$ of the total with probability p_α

$\frac{7}{10}$ of the total with probability p_β

player 2 pays always 1/10 of the total.

Player 1 will be informed about the result α or β of the random move while player 2 will not.

Given this situation, which kind of contract should the players register?

More generally speaking let us imagine that various Nash bargaining situations occur with certain (a priori) probabilities of realization. The players may observe certain random variables (their private information) according to which they may compute conditional probabilities of realization ("Mixtures" of certain bargaining situations). Given these observations, how can they proceed to reach an agreement?

This is the general model.

Definition 1.1: A *cooperative game with incomplete information* (a *CII−Game*) is given by

(1) $$\Gamma = (I, T, p; \underline{X}, \underline{x}, U)$$

with the following ingredients:

$I = \{1,...,n\}$ is the set of *players*. For $i \in I$ the finite set T^i represents player i's "*types*", thus $t \in T = T^1 \times ... \times T^n$ reflects a collection of types for all players or a "*state of nature*"; t_i can be seen as i's "*private information*" concerning t.

p is a *probability* on T; the "*distribution of types*" we imagine that there is some abstract probability space $(\Omega, \underline{F}, P)$ and a random variable $\tau : \Omega \to T$ with *distribution* p; τ "chooses the types". Next

$$\bar{X} = \{x \in \mathbb{R}^n \mid e\,x \leq 1\} \tag{2}$$

is the set of ("primitive") collective decisions, contracts or *parameters*, we always use

$$e = (1,\ldots,1) \in \mathbb{R}^n_+ . \tag{3}$$

If players fail to reach an agreement then $\underline{x} = 0$, the *status quo* parameter occurs. Next

$$\partial\bar{X} = \{x \in \mathbb{R}^n \mid ex = 1\} \tag{4}$$

represents the Pareto efficient (P.E.) frontier of $\bar{X}$. Throughout this paper, $\bar{X}$ and $\underline{x} = 0$ are fixed and will not change (unless with n...). Finally,

$$U^t_i : \bar{X} \to \mathbb{R} \tag{5}$$

is player i's utility if t prevails; we want U^t_i (at least) to be continuous, strictly monotone in x_i and quasi concave, satisfying $U^t_i(0) = 0$ ($i \in I$, $t_i \in T^i$) such that

$$V^t = \{U^t(x) \mid x \in \bar{X}\} \subsetneq \mathbb{R}^n \tag{6}$$

is a closed, convex, and comprehensive subset of $\mathbb{R}^n$ with nonempty compact positive part

$$V^t_+ := V^t \cap \mathbb{R}^n_+ . \tag{7}$$

V^t or V^t_+ and $\underline{u}^t = U^t(\underline{x}) = U^t(0) = 0$ constitute a Nash bargaining problem $(0, V^t)$ for the grand coalition. Similarly, we could talk about smaller coalitions, but as the bargaining problem is our only concern, we shall never mention them. Thus, as the status quo point is fixed to be 0 in utility space, any closed convex and comprehensive set $V^0 \subsetneq \mathbb{R}^n$ with nonempty V^0_+ (and hence $0 \in V^0$) constitutes a *bargaining problem* – the Nash solution of which we denote by $\nu(V^0)$ (or $\nu(V^0_+)$).

Back to our CII–Game Γ, we are in this case also interested in the "bargaining" situation only (solely the grand coalition can cooperate). There are two stories related as to how Γ should be "played".

The tentative or "primitive" story is obvious: Chance chooses some $\omega \in \Omega$ and $I \in I$ observes τ_I, hence can compute $P(\tau_{-i} = t_{-i} \mid \tau_i = t_i)$ with $t_{-i} = (t_k)_{k \neq i}$. Given this personal information, players may agree upon some $x \in \bar{X}$ resulting in a utility $U^{\tau(\omega)}_i(x)$ for $i \in I$.

However, players may want to make use of their observations by announcing them and contracting in dependence of the announcements.

Assuming that there is no way of verifying the type of any other player, we are lead to consider Bayesian incentive compatible mechanisms (cf. [4] [5] [11]).

Definition 1.2:

1. A *mechanism* is a mapping
$$\mu : T \to \bar{X}$$

2. μ is *Bayesian incentive compatible* (BIC) if
$$E(U^\tau_i \circ \mu^\tau \mid \tau_i = t_i) \geq E\,(U^\tau_i \circ \mu^{\tau_{-i}, s_i} \mid \tau_i = t_i) \tag{8}$$
holds true for every $i \in I$ and $t_i, s_i \in T^i$.

3. μ is (*in mediis*) *individually rational* (IR) if
$$E(U^\tau_i \circ \mu^\tau \mid \tau_i = t_i) \geq 0 \tag{9}$$
holds true for every $i \in I$ and $t_i \in T^i$.

4.
$$\mathfrak{M} = \mathfrak{M}(\Gamma) = \{\mu \mid \mu \text{ is BIC and IR}\} \tag{10}$$
denotes the set of "feasible" mechanisms players will bargain about.

Thus, given a vector of announcements $t \in T$, $\mu(t) \in \bar{X}$ will be executed; (8) ensures that no player has an incentive for misrepresenting his type, provided everyone else reports the truth. (9) expresses the fact that no player would like to agree to a mechanism at which, when observing his type, he expects to receive less than by not contracting at all.

In the light of the existence of (BIC) mechanisms, we would now like to change the "primitive" story concerning the way the game is to be played. The "final" story proceeds as follows.

Given Γ, players $i \in I$ may bargain about mechanisms. As they anticipate that they will have private information (on which the result executed via a mechanism depends) they will only consider i.r. *in mediis* mechanisms.

The mechanism agreed upon is then registered with a *referee* or *court* who is capable of enforcing it. This court will only accept *BIC and I.R.* mechanisms. Thereafter the chance move takes place. Next, players announce their private observations (i.e. their types) to the court, who finally – on the basis of all reported observations – executes the mechanism and allots the actual payoff to each player.

Thus, we now prefer the idea that mechanisms are agreed upon – and registered – *in advance* (i.e., before the chance move takes place). Note that the power of the referee or *court* by which binding agreements are executed at last, has to be assumed to be much more comprehensive – for it might turn out that a player is forced to accept non–individually rational outcomes *ex post*. Thus, we may as well imagine that it is the court that imposes restrictions (8) and (9) on mechanisms: this institution prefers no one to cry foul when observing his true type and also to receive truthfull reports. Thereafter the court (and not the players) is informed about the true types on the basis of which it has to enforce the result of the mechanism employed.

As a result of this view, we assume that players find themselves in various "states of nature" which result in different utilities – we do not think of the "types" as of "players" – in marked difference to the first contribution within this field, see HARSANYI–SELTEN [2]. Consequently, e.g. affine (linear) transformations of utilities will take place with respect to players (not types) – thus involve all types of players simultaneously.

Clearly, some basic questions of utility theory should be discussed for to have a sound interpretation of the model. As cooperative Game Theory started out by discussing TU–games it seems natural to ask this question first: what is a "side–payment" or "TU" situation in our present framework. Can we, for a beginning, start out with the

naive idea that players have a universal scale of utility which in particular refers to the parameters $x \in \bar{X}$ as a vehicle of exchange ("money")?

If so, some requirements should be fulfilled. Suppose that in an *ex post* situation $t = 0$ $U^0 : \mathbb{R}^n \to \mathbb{R}^n$ maps $\bar{X}$ onto some feasible set V^0 of utilities with *side–payment character*, say

(11) $$V^0 = \{u \in \mathbb{R}^n \mid eu \leq c\} =: V_{<c>} \subsetneq \mathbb{R}^n$$

Then, as all values involved are "monetarian", we would like to have

(12) U^0 is linear.

Next, we assume that for $x \in \bar{X}$, x_i represents the coordinate "relevant for $i \in I$", thus we should have

(13) $U_i : \bar{X} \to \mathbb{R}$ is strictly monotone in x_i.

And finally, any transfer "in money" in $\bar{X}$ which leaves the total amount of money (the net transfer) unchanged should result in an unchanged total utility of all individuals involved, i.e.,

(14) If $y \in \mathbb{R}^n$ and $ey = 0$, then

$$\sum_{\substack{i \in I \\ y_i \neq 0}} U_i^0(x+y) = \sum_{\substack{i \in I \\ y_i \neq 0}} U_i^0(x) .$$

These are strong requirements and it is not hard to see that they result in a limited class of "admissible" utility functions. Indeed, we have (see [12]).

Theorem 1.3: If $U^0 : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying (11), (12), and (13), then there is $C^0 > 0$ and $b^0 \in \mathbb{R}^n$ such that

(15) $$U^0(x) = C(x - (ex)\, b^0) \qquad (x \in \mathbb{R}^n).$$

As C represents some universal scaling of measurement, we shall restrict ourselves to the case that $C = 1$. Then b^0 is interpreted as a "*fee schedule*" or "*tariff of tazes*" : if players agree upon $x \in \bar{X}$ they will have to pay a fee proportional towards the total amount ex which also depends individually on b_i^0 for player i. Note that in the notation

of (11) it follows that $c = 1 - eb^0$. We shall restrict ourselves to the case $0 \leq eb^0 \leq 1$. This motivates

Definition 1.4: Let Γ be a CII–game. Γ is said to be a *fee–game* ("in the narrow sense") if, for any $t \in T$, there is $b^t \in \mathbb{R}^n$, $0 \leq eb^t \leq 1$ such that

(16) $$U^t(x) = x - (ex)\, b^t \qquad (x \in \bar{\underline{X}})$$

holds true.

Example 1.5: Let $n = 2$ and

(17) $$T = \{\alpha,\beta\} \times \{*\}\,,$$

thus player 1 has two types and player 2 just one – hence player 1 is fully informed once he observes α or β. ("Incomplete information on one side".) Define

(18) $$b^{(\alpha,*)} = (\tfrac{1}{10}, \tfrac{1}{10}), \qquad b^{(\beta,*)} = (\tfrac{7}{10}, \tfrac{1}{10})$$

and let U^t be given by (16). Then the fee game Γ given via Definition 1.4 is the precise model of the situation we were discussing at the beginning of this section.

Before we continue the discussion of fee–games, let us tentatively return to an "*ex–post*"–situation as in (11) – (15) and in Theorem 1.3.

For $\lambda \in \mathbb{R}^n_{+,}$ with $e\lambda = n$ and $0 \leq c \leq 1$,

(19) $$V_{<\lambda,c>} := \{u \in \mathbb{R}^n \mid \lambda u \leq c\}$$

describes a bargaining problem in which side–payments with constant rate of utility transfer λ takes place; $\lambda = e = (1,\ldots,1)$ (see(3)) leads to

(20) $$V_{<c>} := V_{<e,c>} = \{u \in \mathbb{R}^n \mid eu \leq c\}.$$

Next, for $b^0 \in \mathbb{R}^n_{+,}$ $eb^0 \leq 1$ let

(21) $$U_{<\lambda,b^0>}(x) := \tfrac{1}{\lambda} \circledast (x - (ex)\, b^0)$$

(with $\frac{1}{\lambda} \circledast u := (\frac{u_1}{\lambda_1},\ldots,\frac{u_n}{\lambda_n})$), then clearly

(22) $$U_{<\lambda,b^0>} : \bar{\underline{X}} \to V_{<\lambda,1-eb^0>}$$

is a bijective mapping transferring in particular the Pareto–surface $\partial\bar{\underline{X}}$ of $\bar{\underline{X}}$ onto the one of $V = V_{<\lambda,1-eb^0>}$, which we denote suitably by ∂V. Of course, we write $U_{<b^0>}$ for $U_{<e,b^0>}$. $U_{<\lambda,b^0>}$ is the "canonical parametrization" of $V_{<\lambda,c>}$ with $c = 1-eb^0$. Thus, we imagine that sidepayment problems (including rescaled ones) are always conceived as resulting from fee–situations in which players contract about "money" and pay individually towards the contract total. The choice of b^0 (given c) allows for n–1 "degrees of freedom" which, with full information does not bear too much relevance but is crucial once incomplete information prevails.

It is technically preferable to restrict the discussion on "fee–schedules" b^0 such that $eb^0 \leq 1$ holds true. To further restrict the rescaling λ to the case that $e\lambda = n$ is no additional loss of generality.

"Canonical representation" of NTU–games is at length discussed in [12]; we will not dwell further on the subject.

However, we want to state

Definition 1.6: A CII–game Γ is a fee–game "in the wider sense" if there exist vectors $b^t \in \mathbb{R}^n$, $0 \leq eb^t \leq 1$, $(t \in T)$ and $\lambda \in \mathbb{R}^n_{+,}$ $e\lambda = n$, such that

(23) $$U^t = U_{<\lambda,b^t>} \qquad (t \in T)$$

holds true. In this case we write

(24) $$\Gamma = \Gamma_{<\lambda,b>}$$

with $b = (b^t)_{t\in T}$ and, of course,

$$\Gamma_{<b>} := \Gamma_{<e,b>}\,.$$

Some further notation will be explained again in the context of some *ex post* situation reflected by some $b^0 \in \mathbb{R}^n_{+,}$ $0 \leq eb^0 \leq 1$.

Consider the case $n = 2$. Figure 1.1 shows how we should view the geometric situation between "parameters/money" ($\bar{\underline{X}}$) and "utility" ($V^0 = V_{<e-eb^0>}$).

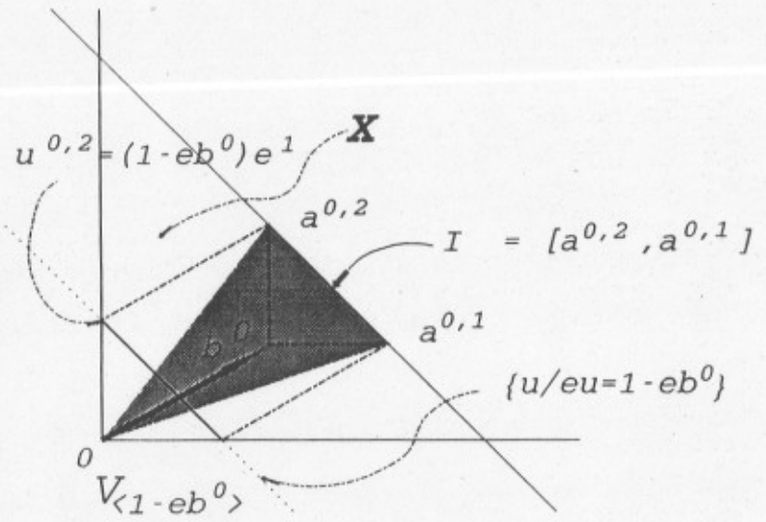

Figure 1.1
Canonical representation of a sidepayment bargaining problem

The mapping

(25) $$U^0 = U_{<b^0>},\ U^0(x) = x - (ex)b^0 \quad (x \in \bar{\underline{X}}),$$

throws the parameters (elements of $\bar{\underline{X}}$)

(26) $$a^{0,i} := b^0 + (1 - eb^0)\, e^i \qquad (i = 1,2)$$

onto the utility–vectors $u^{0,i} := (1 - eb^0)\, e^i$, the P.E. extremals of $V_{<1-eb^0>}$. Thus, the interval

(27) $$I^0 = [a^{0,2}, a^{0,1}] = \text{convex hull } (\{a^{0,2}, a^{0,1}\})$$

contains exactly the P.E. and individually rational (*ex post*) parameters. Simultaneously the triangle (convex hull of...) $[0, a^{0,2}, a^{0,1}]$ (which is bijectively mapped on $[0, u^{0,2}, u^{0,1}]$, consists of the individually rational parameters, among which e.g. $[0, a^{0,2}]$ amounts to those parameters resulting in zero–utility for player 1.

Thus, players following the NASH bargaining–solution concept would just agree on the midpoint $x^0 = \frac{1}{2}(a^{0,1}, a^{0,2})$ of I^0 this way dividing the surplus utils equally.

The ordering we impose on $\partial\bar{\underline{X}}$ is the one imposed by player 1's utility; in this context we may sometimes write $x < y$ if $x_1 < y_1$; thus e.g. $a^{0,2} < a^{0,1}$ holds true.

While this is all quite trivial with full information, the situation looks different when players have private information. The geometric situation is at best discussed in the simplest case where incomplete information is restricted to one player.

To this end we introduce

Definition 1.7:

1. Γ is a game with *incomplete information one one side*, if $n = 2$ and $|T^2| = 1$, thus

(28) $$T = T^1 \times \{*\}.$$

2. $\mathcal{G}^1 := \{\Gamma \mid \Gamma \text{ is a fee game (in the narrow sense) with incomplete information on one side and } |T^1| = 2\}$.

For $\Gamma \in \mathcal{G}^1$ we write

(29) $$T = \{\alpha, \beta\} \times \{*\}.$$

For games with incomplete information on one side we may omit the index $*$, thus in particular for $\Gamma \in \mathcal{G}^1$ we write

(30) $$U^\alpha := U^{(\alpha,*)},\ U^\beta := U^{(\beta,*)}$$

etc. Hence, $\Gamma \in \mathcal{G}^1$ is essentially described by the two "fee vectors" b^α, $b^\beta \in \mathbb{R}^2$, we shall always assume

(31) $$b_1^\alpha < b_1^\beta \qquad (\Gamma \in \mathcal{G}^1),$$

meaning, that state α is preferable to state β for the informed player 1, as he pays less fees.

If $\Gamma \in \mathcal{G}^1$, then a mechanism $\mu : T \to \bar{\underline{X}}$ is tantamount to a pair $\mu = (\mu^\alpha, \mu^\beta) \in \bar{\underline{X}} \times \bar{\underline{X}}$. If we view the two copies of Figure 1.1 referring to α and β in a joint sketch, vectors b^α and b^β result in intervals I^α, I^β and further quantities ($a^{\alpha,1}$, $a^{\beta,1}$ etc.) as depicted in Figure 1.2.

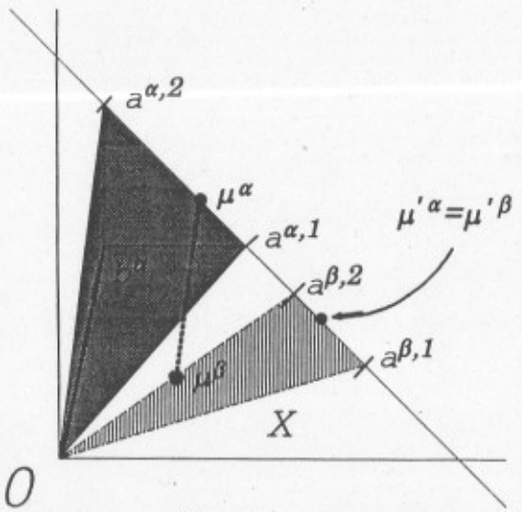

Figure 1.2
The paradigma of a fee–game

In this context, what can be said about BIC–mechanisms $\mu \in \mathfrak{M}$ that, in addition are efficient?

Essentially, we are interested in *ex ante Pareto–efficient* mechanisms, i.e., those that cannot be improved upon simultaneously for all players by another mechanism (w.r. to expectation *ex ante*). Clearly, even if a mechanism is efficient, it can (and will) result in payoffs that are not efficient in every *ex post* situation, i.e., we will *not* find that

(32) $\qquad \mu^t \in \partial\underline{\hat{X}} \qquad (t \in T)$

holds true (or, equally, $U^t(\mu^t) \in \partial V^t$ $(t \in T)$, in view of (13)).

To describe the situation, let us first give

Definition 1.8: $\mu \in \mathfrak{M}$ is (*ex ante*) *Pareto–efficient* if there is no $\hat{\mu} \in \mathfrak{M}$ such that

(33) $\qquad E(U_i^\tau \circ \mu^\tau) \geq E(U_i^\tau \circ \hat{\mu}^\tau)$

holds true for $i \in I$ with strict inequality for at least one $i_0 \in I$. μ is *globally efficient* if (32) holds true.

Next, we cite the relevant results from [11], see also [12] and [10].

Theorem 1.9: (See Theorem 2.8, Remark 2.9 of [11])

1. Let $n = 2$. Given I, T and U (or $(b^t)_{t\in T}$) (and, of course, $\underline{x} = 0$ and $\bar{\underline{X}}$), there is an *open and dense set of distributions* p such that for every fee game $\Gamma = (I,T;$ $p; \underline{x},\bar{\underline{X}},U)$ the following holds true: whenever μ is globally efficient, then μ is constant (i.e. $\mu^t = x \in \bar{\underline{X}}$ $(t \in T)$ for some $x \in \bar{\underline{X}}$).

2. If $|T^2| = 1$, then the above statement holds true for *all* distributions instead of almost all.

Theorem 1.10: (see Theorem 3.4 of [11])

Let $\Gamma \in \mathfrak{G}^1$ (thus, $b_1^\alpha < b_1^\beta$) and $0 < eb^\alpha, eb^\beta < 1$. Let $\mu \in \mathfrak{M}$ be *ex ante Pareto efficient* and *non–constant*. Then, if

(34) $\qquad E(U_2^\tau \circ \mu^\tau) > 0,$

it follows that

(34) $\qquad E(U_1^\tau \circ \mu^\tau) \mid \tau_1 = \beta) = U_1^\beta(\mu^\beta) = 0,$

(35) $\qquad U_1^\alpha(\mu^\alpha) = U_1^\alpha(\mu^\beta).$

Thus, regarding the class $\mathfrak{G}^1$ in particular, if mechanisms are *ex ante* P.E., they may be globally efficient and constant. Or else player 1, in his worse situation receives a zero–payoff if player 2 receives anything at all ((34) and (35)). Moreover, the IC–constraints are binding (i.e. (35)).

Inspecting Figure 1.2 once more we observe that this essentially characterizes two types of mechanisms as represented by $\mu = (\mu^\alpha, \mu^\beta)$ and $\mu' = (\mu'^\alpha, \mu'^\beta)$. (Player 2's i.r. constraint must also be taken into consideration!)

Thus in $\mathfrak{G}^1$, the class $\mathfrak{M}$ is structurally quite tractable. In the light of these results we shall attempt to provide an axiomatic treatment of the (suitably generalized) Nash bargaining–solution, thus as well providing an answer for the question raised at the beginning of this section.

The following development seems to be natural. In the "primitive" scenario the discussion centers around "parameters" $x \in \bar{X}$, the possible results are reflected by

(36) $$V^{\bar{X}} := \{E\, U^\tau(x) \mid x \in \bar{X}\}$$

which, by the way, equals $V_{<1-eb^\tau>}$ in case of a fee game in the narrow sense. In the final version, mechanisms $\mu \in \mathfrak{M}$ are at stake thus, as bargaining takes place *ex ante*, we have to consider

(37) $$V^{\mathfrak{M}} := \{E\, U^\tau \circ \mu^\tau \mid \mu \in \mathfrak{M}\}$$

a convex, compact polyhedron satisfying

(38) $$V^{\mathfrak{M}} \subsetneq V^{\bar{X}}_{+},$$

(Lemma 2.6 of [11]). In SECTION 2, we will discuss the structure of $\mathfrak{M}$ and $V^{\mathfrak{M}}$ for "generic cases" of $\mathfrak{G}^1$; these we call "scenarios of the world".

As we want to generalize the NASH bargaining solution, we have to provide the framework for the IIA–axiom, i.e., the *appropriate* versions of extensions of games (hyperplane games in the traditional setup). This is done in SECTION 3.

In SECTION 4 we finally collect all pieces and provide an axiomatic characterization (two–fold) of a generalized Nash solution, the "expected contract value".

SECTION 2
Scenarios of the world –
Divina commedia

Within this section, we discuss some members of the class $\mathfrak{G}^1$ of CII–games with incomplete information on one side. The examples we are listing do not provide an extensive description – some border–cases will be left out.

However, the case treated in Example 2.1 is rather "generic" and the ones discussed in 2.2 and 2.3 are important for axiomatization purposes as discussed in the subsequent sections. We feel that those alternatives that are being left out, are "not relevant" – and apart from that distinguishing too many detailed cases and providing proofs accordingly always tends to result in a tedious presentation, not necessarily offering a clear view.

Example 2.1: ("The profane world")

This case is actually the one typically of highest interest, since it is of some generality. We assume

(1) $$b_1^\alpha < b_1^\beta,\ b_2^\alpha > b_2^\beta$$

and

(2) $$p_\alpha\,(1\text{–}eb^\alpha) > 1 - b_1^\beta - Eb_2^\tau\,,\ b_1^2 < 1 - Eb_2^\tau\,.$$

This is readily translated into

(3) $$a^{\alpha,2} < a^{\beta,2}\,,\ a^{\alpha,1} < a^{\beta,1}$$

and

(4) $$p_\alpha\,(1\text{–}eb^\alpha) > a_2^{\beta,2} - Eb_2^\tau\,,$$
$$a^{\beta,2} < Ea^{\tau,1}\,.$$

While condition (3) is at once interpreted to represent a particular arrangement of I^α and I^β in $\partial\bar{X}$ (see Fig. 2.1) condition (4) will serve to exhibit a crucial extremepoint of $\partial V^{\mathfrak{M}}$.

Indeed, by Theorems 1.9 and 1.10 we know that two classes of *ex ante* P.E. and BIC Mechanisms $\mu = (\mu^\alpha, \mu^\beta)$ may occur, namely those with $\mu^\alpha \neq \mu^\beta$ and the constant ones.

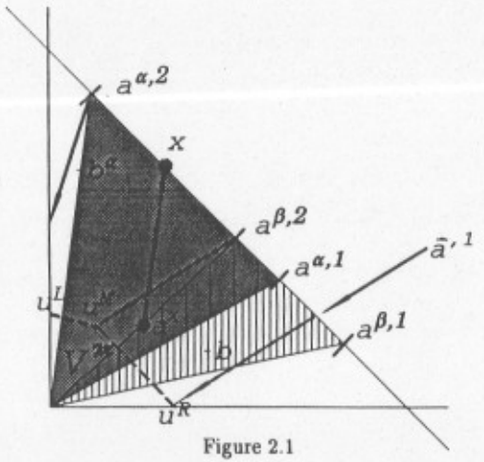

Figure 2.1
The profane world

And clearly, constant mechanisms $\mu = (x,x) \in \mathfrak{M}$ will occur if and only if $x \in [a^{\beta 2}, Ea^{\tau}]$, for to the left or to the right of this interval, the i.r. condition for player 1 or player 2 respectively is violated. This explains the second part of (4).

Next, if $\mu = (\mu^{\alpha}; \mu^{\beta})$ and $\mu^{\alpha} \neq \mu^{\beta}$, then by Theorem 1.10 we have $U_1^{\beta}(\mu^{\beta}) = 0$, and $U_1^{\alpha}(\mu^{\beta}) = U_1^{\alpha}(\mu^{\alpha})$. That is, μ^{β} is located on the intersection of the interval $[0,a^{\beta 2}]$ and the straight line through μ^{α} that is parallel to $[0, a^{\alpha 2}]$. (For, if $x \in [0, a^{\alpha 2}]$, then $U_1^{\alpha}(x) = 0$, thus on the straight line parallel the utility of player 1 is constant and equal to the one of μ^{α}). cf. Fig. 2.1 again.

From these observations the extreme (and *ex ante* P.E.) mechanisms arise as follows:

To the left we have

(5) $$\mu^L := (a^{\alpha 2}; 0),$$

in the middle we find

(6) $$\mu^M := (a^{\beta 2}; a^{\beta 2}),$$

and to the right arises

(7) $$\mu^R := (\bar{a}^1; \bar{a}^1)$$

with $\bar{a} = E\, a^{\tau}$.

Accordingly, any $x \in [\mu^M, \mu^R]$ yields a constant $\mu = (x;x) \in \mathfrak{M}$. And any $x \in [\mu^L, \mu^M)$, by choosing a^x as the unique point on $[0, a^{\beta,2}]$ such that $[a^x, x]$ is parallel to $[0, a^{\alpha,2}]$ and putting

(8) $$\mu = \mu^x := (x, a^x).$$

gives rise to a nonconstant, *ex ante* efficient $\mu \in \mathfrak{M}$.

The extremals of $V^{\mathfrak{M}}$ are now obtained by computing the expectations, we find

(9)
$$\begin{aligned} u^L &= EU^{\tau} \circ \mu^{L\tau} = p_{\alpha} U^{\alpha}(a^{\alpha,2}) \\ &= p_{\alpha}(0, 1-eb^{\alpha}) = (0, p_{\alpha}(1-eb^{\alpha})) \\ &= u^{\alpha,2}, \end{aligned}$$

next

(10)
$$\begin{aligned} u^M &= EU^{\tau} \circ \mu^{M\tau} = a^{\beta,2} - Eb^{\tau} = a^{\beta,2} - \bar{b} \\ &= \bar{u}(a^{\beta,2}) = \bar{u}^{\beta,2}, \end{aligned}$$

and finally

(11)
$$\begin{aligned} u^R &= EU^{\tau} \circ \mu^{R\tau} = \bar{a}^1 - Eb^{\tau} = \bar{a}^1 - \bar{b} \\ &= (1 - E\, eb^{t\tau}, 0) = \bar{u}^1, \end{aligned}$$

that is $u^R = \bar{u}^1$ is the right endpoint of $\bar{I} = EI^{\tau}$ transformed via $\bar{U}$ into utility–space.

Note that (10) and (11) explain condition (4): otherwise, u^L is Pareto–dominated.

Note also that the right hand interval of $\partial V^{\mathfrak{M}}$ is located on

(12) $$\{ u \mid eu = 1 - E\, eb^{\tau} \},$$

since $1 - E\, eb^{\tau} = 1 - e\bar{b}$ is the utility obtained in expectation from constant mechanism.

Example 2.2: ("The world of truth")

Consider the case that player 1 pays the same percentage as fees in both states of the world α and β, more precisely assume that

(13) $$b_1^{\alpha} = b_1^{\beta},\ b_2^{\alpha} \leq b_2^{\beta}$$

holds true (the second inequality w.l.o.g.). This amounts to

(14) $$a^{\alpha,2} = a^{\beta,2},\ a^{\alpha 1} \geq a^{\beta 2}$$

(see Fig.2.2). Now, all *ex ante* P.E. mechanisms $\mu \in \mathfrak{M}$ are constant, i.e., they are given by

(15) $$\{\mu = (x,x) \mid x \in [a^{\alpha,2}, \bar{a}^1]\}$$

with $\bar{a}^1 = Ea^{\tau 1}$.

Obviously $V^{\mathfrak{M}}$ has "T.–U.–character", as

(16) $$\begin{aligned} V^{\mathfrak{M}} &= EU^{\tau}(x) \mid x \in [a^{\alpha,2}, \bar{a}^1]\} \\ &= \{x - \bar{b} \mid eu \leq 1 - eb\} \cap \mathbb{R}^2_+ \\ &= V_{<e,1-eb>} \end{aligned}$$

In particular, if we have an equation everywhere in (13), i.e., if

(17) $$b^{\alpha} = b^{\beta}$$

holds true, den $V^{\alpha} = V^{\beta} = V^{\mathfrak{M}}$ holds true and Γ is in an obvious way "canonically isomorphic" to a bargaining situation with *complete information*. We refer to this case accordingly.

Clearly, there is no incentive for player 1 to misrepresent this type in any "world of truth".

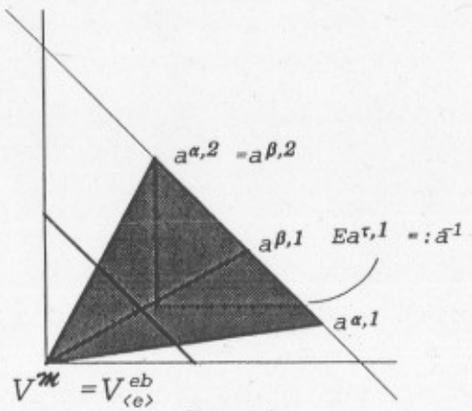

Figure 2.2
The world of truth

Also, if complete information prevails, then there is an "obvious" or "canonical" extension of the Nash bargaining solution. For

(18) $$\bar{x} = \bar{x}^{\alpha} = \bar{x}^{\beta} = b^{\alpha} + \frac{e}{2}(1 - eb^{\alpha})$$

clearly yields a constant mechanism $\bar{\mu} = (\bar{x}, \bar{x}) \in \mathfrak{M}$ such that

(19) $$U^{\alpha}(\bar{x}) = U^{\beta}(\bar{x}) = EU^{\tau} \circ \bar{\mu}^{\tau} = \nu(V^{\alpha}) = \nu(V^{\beta}) = \nu(V^{\mathfrak{M}}).$$

In the general world of truth it is not *ex ante* clear how to proceed – however, there is a unique constant mechanism μ yielding $EU^{\tau} \circ \mu^{\tau} = \nu(V^{\mathfrak{M}})$.

Example 2.3: ("Dante's world")

In this situation we consider Γ such that

(20) $$b^{\alpha}_1 < 1 - b^{\alpha}_2 < b^{\beta}_1 = 1 - b^{\beta}_2$$

holds true. Since $eb^{\beta} = 1$, this amounts to

$$a^{\alpha 2} < a^{\alpha 1} < a^{\beta 1} = a^{\beta 2}.$$

The *ex ante* P.E. elements of $\mathfrak{M}$ are easily characterized. For *any* $x \in I^{\alpha}$, choose

(21) $$t_x := \frac{|x - a^{\alpha,2}|}{|a^{\beta,1} - a^{\alpha,2}|}$$

such that

(22) $$x = t_x\, a^{\beta,1} + (1 - t_x)\, a^{\alpha,2}$$

is a consequence; then put

(23) $$a^x := t_x\, a^{\beta,1},\ \mu^x := (x, a^x) \qquad (x \in I^{\alpha}).$$

(cf. Fig.2.3). Thus

(24) $$\{\mu^x \mid x \in I^{\alpha}\}$$

yields "essentially" all *ex ante* P.E. mechanisms of $\mathfrak{M}$. Compare Example 2.1, a^x is constructed analogously. But as $eb^{\beta} = 1$, μ^{β} is not uniquely determined by μ^{α} (Theorem 1.10 fails) hence, for any $0 \leq t \leq 1$ the mechanisms $(x, t\, a^x)$ are also BIC, IR and PE – and result in the same utilities as the μ^x.

In situation β you have no hope of gaining anything, hence you are in hell ("Lasciate ogni speranza, voi ch'entrate...").

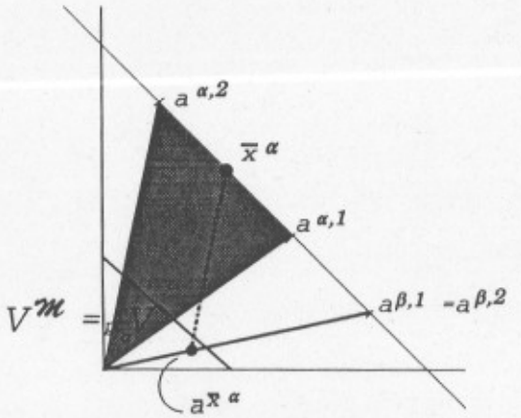

Figure 2.3
Dante's World

On the other hand, any $x \in I^{\alpha}$ and hence any utility $u \in V_e^{b^{\alpha}}$ can be organized with ease: by taking care for the appropriate incentives in hell, player 1 will always tell the truth, when he finds himself in heaven (in α...). Thus, there is truth in heaven (as in the case of complete information on earth).

As for the description of $V^{\mathfrak{M}}$, observe that $\partial V^{\mathfrak{M}}$ is the straight line

$$\begin{aligned} \partial V^{\mathfrak{M}} &= \{p_\alpha U^\alpha(x) + \underbrace{p_\beta \; U^\beta(a^x)}_{0} \mid x \in I^\alpha\} \\ &= \{p_\alpha(x - b^\alpha) \mid x \in I^\alpha\} \\ &= p_\alpha \, \partial V^\alpha . \end{aligned}$$

Thus

(25) $$V^{\mathfrak{M}} = p_\alpha V^\alpha ,$$

which again looks like a side–payment game.

In this situation the (vague) question of a generalized Nash–solution can also be answered "canonically". For let

(27) $$\bar{x}^\alpha = b^\alpha + \frac{e}{2}(1 - eb^\alpha)$$

be the midpoint of I^α; the "induced" mechanism

(28) $$\bar{\mu} = (\bar{x}^\alpha, a^{\bar{x}^\alpha})$$

yields payoffs

(29) $$U^\alpha(\bar{\mu}^\alpha) = \bar{u}^\alpha = \bar{x}^\alpha - b^\alpha = \nu(V^\alpha)$$

and

(3) $$U^\beta(\mu^\beta) = 0 = \nu(V^\beta).$$

Hence

(31) $$\begin{aligned} EU^\tau \circ \bar{\mu}^\tau &= p_\alpha U^\alpha(\bar{\mu}^\alpha) = p_\alpha \nu(V^\alpha) \\ &= \nu(p_\alpha V^\alpha) = \nu(V^{\mathfrak{M}}) \\ &= \bar{u} = E\bar{u}^\alpha . \end{aligned}$$

So $\nu(V^\alpha)$ and $\nu(V^\beta)$ can be implemented incentive compatibly and yield the ex–ante expected Nash–solution $\nu(V^{\mathfrak{M}})$. In the simple world of Dante, everyone receives justice.

SECTION 3

Creating additional alternatives – The extension of games

The axiomatization of the NASH–solution hinges on the IIA–axiom and on the appropriate construction of hyperplane–games "supporting" general "convex games" (i.e. feasible sets).

If we view $V^{\mathfrak{M}}$ as depicted for the "profane world" of Example 2.1, then it becomes clear that the presence of BIC mechanisms renders the feasible set to lose its side payment or TU–character. So the construction of "appropriate" hyperplane games "supporting" $V^{\mathfrak{M}}$ all of sudden is an open problem which we do not encounter in the side payment context with full information. (And fee games are supposed to be the analogue to side payment games.)

Of course it is quite simple to construct side payment games with complete information (say, in the sense of Example 2.3) "supporting" $V^{\mathfrak{M}}$. But this approach can hardly be called "appropriate" – the information structure is totally different and an IIA–axiom constructed accordingly would be forcible and unappealing.

This section discusses the natural way of constructing "supporting hyperplane games", "extensions", or "irrelevant alternatives". The problem is to do this by a procedure which leaves the information structure – and the incentives – unchanged.

Essentially, we have to perform this task for the profane world (i.e. Example 2.3) only. Thus we discuss the effect of changing the fee schedule slightly and keeping the type of mechanisms that (eventually) implements the appropriate version of the NASH–solution.

Lemma 3.1: Let $\Gamma = \Gamma_{<b>}$ be a profane world. Define for $\epsilon \in \mathbb{R}^2$

(1)
$$h^{\epsilon} := \frac{\epsilon_1}{p_\alpha}(1,-1) - \frac{e\epsilon}{p_\beta}(0,1);$$
$$b^{\beta,\epsilon} := b^{\beta} + h^{\epsilon}\ ;\ b^{(\epsilon)} := (b^{\alpha}, b^{\beta,\epsilon}).$$

Then, for sufficiently small $\epsilon \in \mathbb{R}^2$, it follows that $\Gamma_{<b^{(\epsilon)}>}$ is a profane world such that the (P.E.) extreme points of $V^{\mathfrak{M},\epsilon}$ are given as follows

(2)
$$u^{L,\epsilon} = u^{L}$$
$$u^{M,\epsilon} = u^{M} + \epsilon$$
$$u^{R,\epsilon} = u^{R} + (e\epsilon,0)\,.$$

Figure 3.1 shows the desired result of changing b in $b^{(\epsilon)}$; it is seen that *ex post / in mediis* additional utilities in $V^{\mathfrak{M},\epsilon}$ are created. By a proper choice of ϵ, it can be established that u^{L}, u^{M}, and $u^{M,\epsilon}$ are collinear.

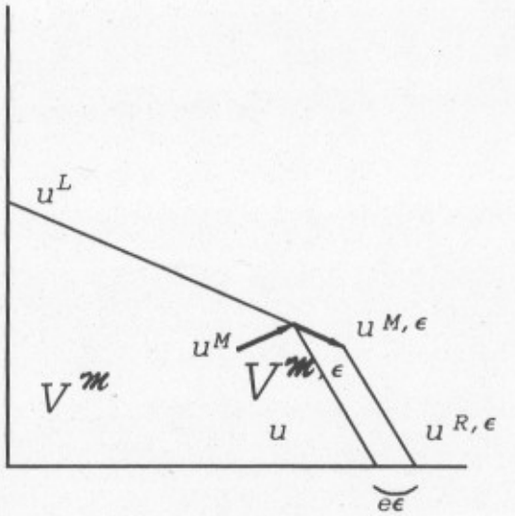

Figure 3.1
Extending $V^{\mathfrak{M}}$

Proof: This requires just a few computations.

Observe that

(3)
$$eb^{\beta,\epsilon} = eb^{\beta} - \frac{e\epsilon}{p_\beta}\ ;\ EB^{\gamma,\epsilon} = Eb^{\gamma} + p_\beta h^{\epsilon}$$

is obvious, hence we have

(4) $$eEb^{\tau,\epsilon} = Eeb^{\tau,\epsilon} = Eb^{\tau} - e\epsilon,$$

that is the total amount of fees to be saved in expectation is $e\epsilon$. Now recalling $a^{\beta,2} = (b_1^{\beta}, 1-b_1^{\beta})$ (see SEC.1, (26)) we conclude

(5) $$a^{\beta,2,\epsilon} = a^{\beta,2} + (h_1^{\epsilon}, -h_1^{\epsilon}) = a^{\beta,2} + \frac{\epsilon_1}{p_\alpha}(1,-1).$$

Using these data it is now straightforward to directly compute the (P.E.–) extremals of $V^{\mathfrak{M},\epsilon}$ in accordance with the ones of $V^{\mathfrak{M}}$ as follows.

First of all, the left extremepoint (cf. SEC.2, formula (9)), i.e., $u^L = p_\alpha(0,1-eb^\alpha)$ is not disturbed at all by the ϵ–change in b^β, hence

$$u^{L,\epsilon} = u^L$$

is obvious. Next as $u^R = (1-Eeb^{\tau},0)$ (in view of (11) in SEC.2), we use (4) and come up with

$$u^{R,\epsilon} = u^R + (e\epsilon, 0).$$

(Thus in $\mu^{R,\epsilon}$ all savings go directly to player 1!)

Finally, recalling $u^M = a^{\beta,2} - Eb^{\tau}$ (as in (10) of SEC.2), we employ (5) and (3), obtaining

$$u^{M,\epsilon} = u^M + \frac{\epsilon_1}{p_\alpha}(1,-1) - p_\beta h^\epsilon$$

where the last terms after some consideration indeed collaps to ϵ, q.e.d.

Corollary 3.2: (The position of $I^{\beta,\epsilon}$ and $V^{\beta,\epsilon}$)

Let $\Gamma = \Gamma_{<b>}$ be a profane world and let, for sufficiently small $\epsilon \in \mathbb{R}^2$ the profane world $\Gamma_{<b^{(\epsilon)}}$ be defined via Lemma 3.1. Assume that

(6) $$\epsilon_1 > 0 > \epsilon_2\,,\; e\epsilon > 0$$

holds true. Then

1. The endpoints of I^β behave as follows:

$$a^{\beta,1,\epsilon} = a^{\beta,1} + (\frac{\epsilon_1}{p_\alpha} + \frac{e\epsilon}{p_\beta})(1,-1)$$

(7) $$a^{\beta,2,\epsilon} = a^{\beta,2} + \frac{\epsilon_1}{p_\alpha}(1,-1)$$

$$b^{\beta,\epsilon} = b^\beta + h^\epsilon \qquad \text{(cf. (1))}$$

2. The distances in I^β behave linearly in ϵ:

(8) $$|\, a^{\beta,1,\epsilon} - a^{\beta,2,\epsilon} \,| = |\, a^{\beta,1} - a^{\beta,2} \,| + \frac{2e\epsilon}{p_\beta}$$

(9) $$|\, a^{\beta,2,\epsilon} - \bar{a}^{1,\epsilon} \,| = |\, a^{\beta,2} - \bar{a}^{1} \,| + 2\epsilon_2$$

3. In addition, we have

(10) $$V^{\beta,\epsilon} \supseteq V^\beta\,.$$

This is the obvious consequence: if the vector ϵ points in direction of $u^M - u^L$ (the P.E. extreme points of $V^{\mathfrak{M}}$) then, in view of (2) in Lemma 3.1, we will have $V^{\mathfrak{M},\epsilon} \supseteq V^{\mathfrak{M}}$ such that u^L, u^M, and $u^{M,\epsilon}$ are collinear. In this setup, equations (7) show that the Interval I^β (and the triangle $[a^{\beta,1,\epsilon}, a^{\beta,2,\epsilon}, b^\beta]$, compare Fig.1.1) move to the south–east. Thus, it is increasingly more difficult to obtain constant mechanisms in $\mathfrak{M}$ – resulting in a larger interval $[u^{L,\epsilon}, u^{M,\epsilon}]$ (see Fig.3.1).

Of course it is important that simultaneously (10) holds true: there are more opportunities for player 1 in state β (and for player 2 in state *), but the results of states α and β are more and more diverging as ϵ increases towards the south–east. (8) and (9) are also important details: while $I^{\beta,\epsilon}$ increases in length, $\bar{a}^{1,\epsilon}$ moves towards the boundary point $a^{\beta,2,\epsilon}$ ($\epsilon_2 < 0$!) – and once both points coincide, there will be no constant mechanisms in $\mathfrak{M}$. At this instant, $V^{\mathfrak{M},\epsilon}$ looks like a hyperplane game.

We will take up this topic again in the next Theorem. First of all, let us give some hints towards the

Proof: (of Corollary 3.2)

As to the first statement, this follows from the definition of $a^{\beta,1} = (1-b_2^\beta, b_2^\beta)$ (cf. SEC.1, (26)) and of $a^{\beta,2}$ as well as from (1) in Lemma 3.1.

Consider the second statement. Use (7) to compute

(11) $$a^{\beta,1,\epsilon} - a^{\beta,2,\epsilon} = a^{\beta,1} - a^{\beta,2} + \frac{e\epsilon}{p_\alpha}(1,-1);$$

and as $a^{\beta,1} - a^{\beta,2}$ points in direction of $(1,-1)$ (which has norm 2), (8) follows at once.

In a similar fashion we check (9), for $\bar{a}^{1,\epsilon} = Ea^{\tau,1,\epsilon}$ can be computed as well by employing (7); thus we find

(12) $$a^{\beta,2,\epsilon} - \bar{a}^{1,\epsilon} = a^{\beta,1} - \bar{a}^{1} - \epsilon_2(1,-1).$$

Since $\epsilon_2 < 0$ and $a^{\beta,1} - \bar{a}^1$ points in direction of $(-1,1)$, we see indeed that (9) holds true.

Finally our third statement, i.e., (10) is of course a consequence of (3), since the total amount of fees in state β decreases.

Theorem 3.3: (The extended game – the crucial type)

Let $\Gamma = \Gamma_{<b>} = (I,T,p;\ \underline{X},\underline{x};\ U_{<b>})$ be a profane world. Then there exists $\hat{\Gamma} = \Gamma_{<\hat{b}>} = (I,T,p;\ \underline{\hat{X}},\underline{x};\ U_{<\hat{b}>})$ with the following properties:

1. $\partial V^{\hat{\mathfrak{M}}}$ is a straight line
2. $V^{\mathfrak{M}} \subsetneq \hat{V}^{\mathfrak{M}}$
3. $V^\alpha = \hat{V}^\alpha$, $V^\beta \subsetneq \hat{V}^\beta$
4. For every $\mu \in \mathfrak{M}$ which is non–constant and *ex–ante* P.E., there is $\hat{\mu} \in \hat{\mathfrak{M}}$ which is as well non–constant and ex–*ante P.E.* such that the following holds true.

(13) $$\mu^\alpha = \hat{\mu}^\alpha,\ U^\alpha(\mu^\alpha) = \hat{U}^\alpha(\hat{\mu}^\alpha).$$

(14) $$U^\beta(\mu^\beta) = \hat{U}^\beta(\hat{\mu}^\beta)$$

(15) $$EU^\tau \circ \mu^\tau = E\hat{U}^\tau \circ \hat{\mu}^\tau.$$

Thus, for any profane world we can find a game of hyperplane–type with additional alternatives such that all utilities available from efficient non–constant $\mathfrak{M}$–mechanisms may be obtained efficiently within the framework of the extension.

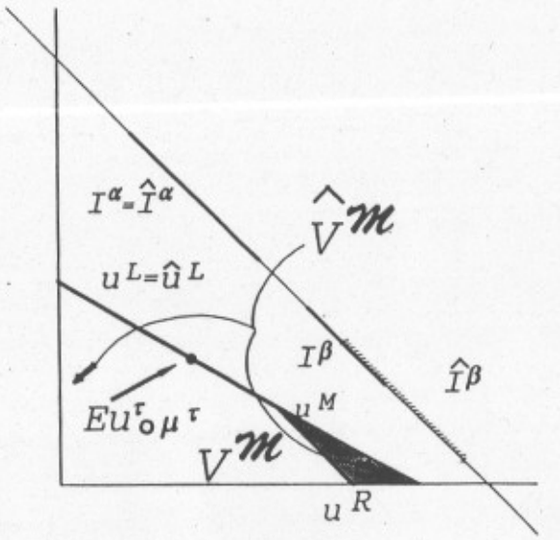

Figure 3.2
Extending a profane world crucially

Figure 3.2 shows the situation (see also Figure 3.1). Note that we claim not only the extension of $V^{\mathfrak{M}}$ but also the one of V^α and V^β (in fact $U^\alpha = \hat{U}^\alpha$, state α does not change at all!).

Proof:

Going back to Lemma 3.1 we may choose $\epsilon = \epsilon_0(u^M - u^L)$ for small $\epsilon_0 > 0$; the resulting $V^{\mathfrak{M},\epsilon}$ will contain $V^{\mathfrak{M}}$ and the interval $[u^L, u^M]$ will be part of $\partial V^{\mathfrak{M},\epsilon}$; this interval is constituted exactly by those mechanisms in $\mathfrak{M}$ that are P.E. and non–constant. These claims follow from Lemma 3.1 and Corollary 3.2, see also Fig.3.1.

By gradually increasing ϵ_0, the extremals $u^{M,\epsilon} = u^M + \epsilon$ and $u^R = u^R + (e\epsilon,0)$ approach each other so that $\partial V^{\mathfrak{M},\epsilon}$ obtains "eventually" the character of a straight line.

Indeed, observe that $\epsilon_1 > 0 > \epsilon_2$, $e\epsilon > 0$ follows from the fact that Γ is a profane world, thus Corollary 3.2 is indeed applicable! Moreover, since

$$u^M - u^R = (a_{\alpha 1} - Eb^\gamma) - (Ea^{\gamma 1} - Eb^\gamma) = a^{\alpha 1} - \bar{a}^1$$

(see Remark 2.1 and Fig.2.1), implies that

(16) $$|u^M - u^R| = |a^{\alpha,1} - \bar{a}^1| \leq |a^{\beta 2} - \bar{a}^1|$$

(see again Fig.2.1) and thus

(17) $$|a^{\beta,2,\epsilon} - \bar{a}^{1,\epsilon}| \to 0$$

for increasing ϵ_0 by (9) of Corollary 3.2 implies

(18) $$|u^{M,\epsilon} - u^{R,\epsilon}| \to 0.$$

This shows that for a suitable choice of $\hat{\epsilon}_0$ (or $\hat{\epsilon}$) we have indeed that our first two statements are satisfied by $\hat{V}^{\mathfrak{M}} = V^{\mathfrak{M},\hat{\epsilon}}$.

Now, to 3.: As α is not touched by the increase in ϵ, $V^\alpha = \hat{V}^\alpha$ is obvious. Again, $V^\beta \subseteq \hat{V}^\beta$ follows from (10) of Corollary 3.2, i.e., essentially from (3).

Thus it remains to verify 4.: To this end, fix $\mu = (\mu^\alpha, \mu^\beta) \in \mathfrak{M}$, P.E. and nonconstant.

Note that $\hat{I}^\beta$ moves towards the south–east, hence μ^β which satisfies $U_1^\alpha(\mu^\beta) = 0$ (Theorem 1.10) is not individually rational for player 1 in state β. However, given μ^α we can find $\hat{\mu}^\beta$ uniquely such that $(\mu^\alpha, \hat{\mu}^\beta)$ form an *ex ante* P.E. and nonconstant mechanism in $\hat{\mathfrak{M}}$. This verbal description is of course depicted in Figure 3.3. (Note that $\hat{\mu}^\beta$ appears to constitute less utility for player 2 – but as less fees are paid in state β, the utility is actually the same.

Formally: Since $\hat{V}^{\mathfrak{M}} \supseteq V^{\mathfrak{M}}$, there exists $\hat{\mu} \in \hat{\mathfrak{M}}$ such that

(19) $$E\hat{U}^\gamma \circ \hat{\mu}^\gamma = EU^\gamma \circ \mu^\gamma$$

holds true, $\hat{\mu}$ must be non constant. Hence we have $U_1^\beta(\mu^\beta) = \hat{U}_1^\beta(\hat{\mu}^\beta) = 0$ and (19) implies

(20) $$p_\alpha \hat{U}_1^\alpha(\hat{\mu}^\alpha) = E\hat{U}_1^\gamma \circ \hat{\mu}^\gamma = EU_1^\gamma \circ \mu^\gamma = p_\alpha U_1^\alpha(\mu^\alpha).$$

Now, since

(21) $$\hat{U}_1^\alpha(x) = U_1^\alpha(x) = x_1 - b_1^\alpha$$

holds true for any $x \in \partial\underline{\hat{X}}$ we conclude from (20) that

(22) $$\hat{\mu}_1^\alpha = \mu_1^\alpha$$

must necessarily follow. But as $\hat{\mu}^\alpha, \mu^\alpha \in \partial\underline{\hat{X}}$ it follows at once that

(23) $$\hat{\mu}^\alpha = \mu^\alpha$$

is also true.

Now again by inserting (23) and (20) into (19) we obtain

$$\hat{U}(\hat{\mu}^\beta) = U(\mu^\beta);$$

thus finally our theorem is proved.

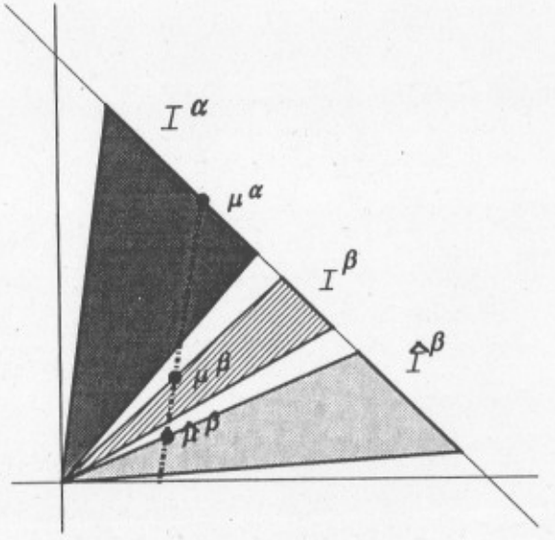

Figure 3.3
The construction of $\hat{\mu}$

The development within this section served to construct a type of "hyperplane extension" for a given profane world $\Gamma_{<b>}$. This extension has the conspicuous property that the interval $[u^L, u^M]$ generated by $\Gamma_{<b>}$ is maintained to be efficient: $\partial\hat{V}^{\mathfrak{M}}$ consists of a hyperplane that is tangent to $V^{\mathfrak{M}}$ in all points of $[u^L, u^M]$. (See Fig.3.2)

The next step consists in a similar construction. However, the hyperplane to be constructed shall touch $V^{\mathfrak{M}}$ only in $u^{(M)}$ – but with all normal vectors that are feasible for $\partial V^{\mathfrak{M}}$ in $u^{(M)}$.

Lemma 3.4: Let $\Gamma = \Gamma_{<b>}$ be a profane world. Define for $\epsilon \in \mathbb{R}^2$

$$b^{\alpha,\epsilon} := b^\alpha - \epsilon \tag{24}$$

$$b^{(\epsilon)} := (b^{\alpha,\epsilon}, b^\beta)$$

Then, for ϵ sufficiently small, $\Gamma^\epsilon = \Gamma_{<b^{(\epsilon)}>}$ is a profane world and the following relations hold true.

$$\begin{aligned} u^{L,\epsilon} &= u^L + (0, p_\alpha e\epsilon) \\ u^{M,\epsilon} &= u^M + p_\alpha \epsilon \\ u^{R,\epsilon} &= u^R + (p_\alpha e\epsilon, 0). \end{aligned} \tag{25}$$

Proof:

Compute

$$eb^{\alpha,\epsilon} = eb^\alpha - e\epsilon;\ Eb^{\tau,\epsilon} = Eb^\tau - p_\alpha \epsilon;\ eEb^{\tau,\epsilon} = eb^\tau - p_\alpha e\epsilon\ . \tag{26}$$

Then (25) follows from $u^L = (0, p_\alpha(1-eb^\alpha))$, $u^M = a^{\beta 2} - Eb^\tau$, and $u^R = (1-eEb^\tau, 0)$.

Lemma 3.5: ("Raising the slope")

Let $\Gamma = \Gamma_{<b>}$ be a profane world. Assume that $0 > \delta_0 > -1$ is the slope of $[u^L, u^M]$. Then there is $\eta > 0$ – depending on δ_0 and u_2^M only – and $\tilde{\Gamma} = \Gamma_{<\tilde{b}>}$ (all ingredients being the same except $U_{<\tilde{b}>}$ – cf. Lemma 3.3) with the following properties

$$\begin{aligned} &1. \quad \tilde{u}^L = u^L + \eta e^2 \\ &2. \quad u^M \in (\tilde{u}^L, \tilde{u}^M) \\ &3. \quad V^{\mathfrak{M}} \subsetneq \tilde{V}^{\mathfrak{M}} \\ &4. \quad V^\alpha \subsetneq \tilde{V}^\alpha,\ V^\beta = \tilde{V}^\beta\ . \end{aligned} \tag{27}$$

(See Fig. 3.4.)

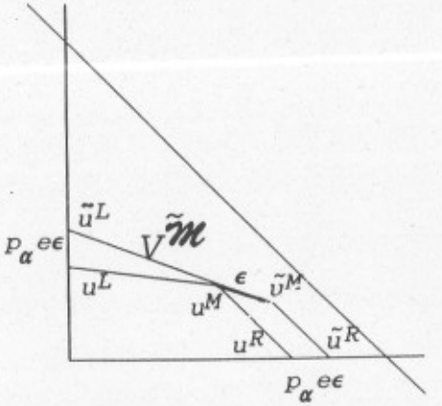

Figure 3.4
The Intermediate Step

Proof:

For *small* $\eta > 0$ the slope δ of $[u^L + \eta e^2, u^M]$ satisfies

$$0 > \delta_0 > \delta > -1. \tag{28}$$

Therefore we may choose η such that

$$0 > \epsilon_2 := \frac{\eta}{p_\alpha} \frac{\delta}{1+\delta} > -u_2^M \tag{29}$$

holds true. Put $\epsilon_1 := \frac{\epsilon_2}{\delta} > 0$, then clearly

$$p_\alpha e\epsilon = \eta\ . \tag{3}$$

Now, define $\tilde{\Gamma} = \Gamma_{<b^\epsilon>}$ via Lemma 3.4. Then, in view of (25) we have

$$\begin{aligned} \tilde{u}^L = u^{L,\epsilon} &= u^L + (0, p_\alpha e\epsilon) \\ &= u^L(0, \eta)\ , \\ \tilde{u}^M = u^{M,\epsilon} &= u^M + \epsilon \in \mathbb{R}^2_+ \end{aligned}$$

(because of (29)).

Hence, the slope of $[\tilde{u}^L, u^M]$ and the one of $[u^M, \tilde{u}^M]$ is all the same, namely δ. That is $\tilde{u}^L$, u^M, $\tilde{u}^M$ are collinear, this proves the first three statements in (27). The fourth statement clearly follows from (24).

Theorem 3.6: (The extended game – the corner type)

Let $\Gamma = \Gamma_{<b>}$ be a profane world then, for any slope δ (i.e. a real number) exceeding –1 and bounded by the slope of $[u^L, u^M]$ there exists a profane world $\Gamma_{<\hat{b}>}$ (all ingredients being the same except $U_{<\hat{b}>}$ – cf. Theorem 3.3) with the following properties.

(27)

1. $\partial\hat{V}^{\mathfrak{M}}$ is a straight line which has slope δ
2. $V^{\mathfrak{M}} \subsetneq \hat{V}^{\mathfrak{M}}$
3. $V^\alpha \subsetneq \hat{V}^\alpha$, $V^\beta \subsetneq \hat{V}^\beta$
4. Given the mechanism μ^M (which yields u^M) there is $\hat{\mu} \in \hat{\mathfrak{M}}$ (*ex ante P.E.* and nonconstant) such that the following holds true.

(28) $$U_1^\alpha(\mu^\alpha) = \hat{U}_1^\alpha(\mu^\alpha),\ U_1^\beta(\mu^\beta) = \hat{U}_1^\beta(\hat{\mu}^\beta)$$

(29) $$EU^\gamma \circ \mu^\gamma = E\hat{U}^\gamma \circ \mu^\gamma.$$

Proof:

1. STEP: Let us show that we may construct $\Gamma_{<\hat{b}>}$ satisfying 1., 2., and 3. such that

$$u^M \in \partial\hat{V}^{\mathfrak{M}}$$

is satisfied. As we have been very detailed so far, we feel it is justified to provide just a verbal argument and not to go through the ϵpsilontics.

To this end consider again Figure 3.4.

If it so happens, that the desired slope δ is just provided by ϵ_2/ϵ_1, then $[\tilde{u}^L, \tilde{u}^M]$ has already slope δ. We may now apply Theorem 3.3 with respect to $\Gamma_{<\hat{b}>}$ and construct an appropriate $\hat{\Gamma}$ accordingly. Since the slope of $\partial\hat{V}^{\mathfrak{M}}$ is the same as the one of $[\tilde{u}^L, \tilde{u}^M]$ we are already done since $u^M \in \partial\tilde{V}^{\mathfrak{M}}$ will also be an element of $\partial\hat{V}^{\mathfrak{M}}$ – inspect Theorem 3.3!

And in fact, Lemma 3.5 shows, that this procedure works for small slopes (i.e. δ close to the one provided by $[u^L, u^M]$ and small η accordingly).

There seems to be a problem in our reasoning in Lemma 3.6 when η increases once ϵ_2 increases in absolute value as to violate (29). This will occur if η is large, and hence $e\epsilon$ has to increase while ϵ_2 increases (in absolute value) thus forcing the vector $u^M + \epsilon$ to touch the u_1–axis.

At this moment, $\partial\tilde{V}^{\mathfrak{M}}$ is constituted by nonconstant mechanisms and $\tilde{u}^M$ and $\tilde{u}^R$ coincide.

However, all that happens is that $\Gamma_{<\hat{b}>}$ ceases to be a profane world in the sense of Example 2.1. Instead, we obtain a situation where $Ea^{\gamma 1} = \bar{a}^1$ is no longer contained in I^β and hence $\mu^M = (a^{\beta,2}, a^{\beta 2})$ is no longer *in mediis (= ex ante)* i.r. for player 2.

Nevertheless we may continue with enlarging η, thus enlarging ϵ and obtaining a $\Gamma_{<\hat{b}>}$ such that $\partial\tilde{V}^{\mathfrak{M}}$ has the desired slope δ and still continues to contain u^M. In fact, this amounts to moving I^α "to the left" sufficiently much.

This finishes the first step of our proof. I.e., we have established 1., 2., and 3.

As to the remaining part, we proceed as in Theorem 3.3.

2. STEP: Indeed, as $u^M \in \partial\hat{V}^{\mathfrak{M}}$, pick $\hat{\mu} \in \hat{\mathfrak{M}}$ which is P.E. and satisfies

(30) $$EU^\gamma \circ \mu^\gamma = E\,\hat{U}^\gamma \circ \hat{\mu}^\gamma$$

where $\mu = \mu^M = (a^{\beta,2}, a^{\beta 2})$ stems from $\Gamma_{<b>}$, the original profane world.

Necessarily $\hat{\mu}$ has to be non–constant and hence (Theorem 1.10) we have $\hat{U}_1^\beta(\hat{\mu}^\beta) = 0$.

Since

(31) $$U_1^\beta(\mu^\beta) = \hat{U}_1^\beta(\hat{\mu}^\beta) = 0$$

we obtain in view of (30)

$$p_\alpha\, U_1^\alpha(\mu^\alpha) = p_\alpha\, \hat{U}_1^\alpha(\hat{\mu}^\alpha) \qquad \text{q.e.d.}$$

Theorem 3.7: (The extended game – the trivial case)

Let $\Gamma = \Gamma_{<b>}$ be a profane world. Then there exists a world of truth $\hat{\Gamma} = \Gamma_{<\hat{b}>}$ (with the same data except $U_{<\hat{b}>}$) such that the following holds true.

1. $\partial \hat{V}^{\mathfrak{M}}$ is a straight line
2. $V^{\mathfrak{M}} \subseteq \hat{V}^{\mathfrak{M}}$
3. $V^{\alpha} = \hat{V}^{\alpha}, \; V^{\beta} = \hat{V}^{\beta}$
4. For every constant mechanism $\mu \in \mathfrak{M}$ which is *ex ante* P.E. there exists an *ex ante* P.E. and constant mechanism $\hat{\mu} \in \hat{\mathfrak{M}}$ such that

(32) $$E\, U^{\tau} \circ \mu^{\tau} = E\hat{U}^{\tau} \circ \hat{\mu}^{\tau}$$

holds true.

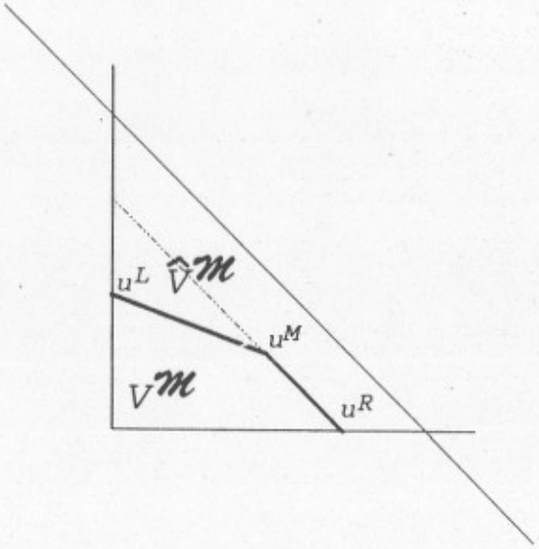

Figure 3.5
The trivial extension

Proof:

Choose $\hat{b}^{\alpha} = b^{\alpha}$ and $\hat{b}^{\beta}$ such that $\hat{b}^{\beta}_{\tau} = b^{\alpha}_{\tau}$ and $e\hat{b}^{\beta} = eb^{\beta}$.

This results in

(33) $$\hat{a}^{\alpha,2} = \hat{a}^{\beta,2},$$

– thus $\hat{\Gamma}$ is a world of truth – as well as in

(34) $$eb^{\alpha} = e\hat{b}^{\alpha}, \; eb^{\beta} = e\hat{b}^{\beta}$$

– thus fees in total do not change (cf. Example 2.2). Statements 1. and 3. are now obvious while statement 2. follows from the fact that u^M as well as u^R are located on $\{u \mid eu = 1-\bar{b}\}$ (see Example 2.1), the positive part of which is actually $\partial \hat{V}^{\mathfrak{M}}$ (see Example 2.2).

This rather trivial situation is illustrated in Figure 3.5.

SECTION 4
The First Axiomatic Approach: IIA

We will now attempt to axiomatize (a version of) the NASH value (NASH [9]) on a class of CII–games. This class is very restricted: fee–games with incomplete information on one side and two types of the informed player only. Note, however, that Definition 1.7 assumes that player 1 is the informed one and that fee games are defined in the narrow sense. Now, since we want to speak of symmetry and linear transformation of utility, we shall deal with fee games in the wider sense (Definition 1.6) and admit that $T = T^1 \times T^2$, $T^1 = \{*\}$, $T^{3-i} = \{\alpha,\beta\}$ holds true for $i = 1,2$. This class is denoted by $\mathfrak{G}^{12}$.

We shall first of all shortly discuss the operations on CII–games: permutation of players and rescaling. Then the "appropriate" way of phrasing the axioms has to be discussed. Finally, it turns out that these axioms uniquely define a solution.

Recall that by Theorem 1.10 and Corollary 3.5 of [11], Pareto efficient utilities of $V^{\mathfrak{M}}$ are uniquely implemented by a mechanism in $\mathfrak{M}$. Thus, the NASH–value $\nu\ (V^{\mathfrak{M}})$ corresponds uniquely to an (*ex ante* P.E.) mechanism in $\mathfrak{M}$, this mechanisms is denoted by $\chi^{(\nu)}(\Gamma)$. The mapping (bargaining solution) $\chi^{(\nu)}$ as defined on $\mathfrak{G}^{12}$ is the one to be axiomatized.

By reasons explained in [11] and [12] as well later in SEC.5, χ is called the "*expected contract value*".

Now let us first treat operations on CII–games.

A permutation $\pi : I \to I$ on the set of individuals induces various actions which we as well denote by the letter π. These actions are defined for the following objects.

1. *Types:* For $t \in T$, $\pi(t)$ is given by

(1) $$\pi(t)_i = t_{\pi^{-1}(i)}, \qquad (i \in I)$$

hence πT is described by $(\pi T)^i = T^{\pi^{-1}(i)} \qquad (i \in I)$.

2. *Distributions:* For probabilities p on T, define

(2) $$\pi p = p \circ \pi^{-1}$$

as usual as a distribution on πT.

3. *Vectors* (and *subsets*) of $\mathbb{R}^n$: For $x \in \mathbb{R}^n$ we write πx where

(3) $$(\pi x)_i := x_{\pi^{-1}(i)},$$

similarly $\pi A = \{\pi x \mid x \in A\}$ for $A \subseteq \mathbb{R}^n$ ("permutation of axis). *Note that* $\pi \underline{\underline{X}} = \underline{\underline{X}}$ *and* $\pi \underline{x} = \underline{x}$ *since* $\underline{x} = 0$.

4. *Utilities:* If $U : T \times \underline{\underline{X}} \to \mathbb{R}^n$ describes the utilities of a CII–game Γ, then $\pi U : \pi T \times \underline{\underline{X}} \to \mathbb{R}^n$ is given by

(4) $$(\pi U)(t',x') := \pi U(\pi^{-1}(t'), \pi^{-1}(x'))$$
$$= U_{\pi^{-1}(\cdot)}(\pi^{-1}(t'), \pi^{-1}(x')) \quad (t' \in \pi T, x' \in \bar{\underline{X}})$$

5. *CII–Games:* Now clearly, for any $\Gamma = (I,T,p; \bar{\underline{X}},\underline{x}; U)$ we define

(5) $$\pi\Gamma := (I, \pi T, \pi p; \bar{\underline{X}}, \underline{x}; \pi U)$$

which employs (1), (2), (3), and (4).

6. *Mechanisms:* If $\mu : T \to \bar{\underline{X}}$ is a mechanism, then $\pi\mu : \pi T \to \bar{\underline{X}}$ is given by

(6) $$(\pi\mu)(t') := \pi\mu(\pi^{-1}(t')) \qquad (t' \in \pi T).$$

It makes sense to denote by Π the set of permutations of I.

Similarly if $\lambda \in \mathbb{R}^n_+$, and $e\lambda = n$, then λU is defined by $\frac{1}{\lambda} \bullet U$ with

(7) $$(\lambda U)^t_i = (\tfrac{1}{\lambda} \bullet U)^t_i = \frac{U^t_i}{\lambda_i}$$

and $\lambda\Gamma = (I,T,p; \bar{\underline{X}},\underline{x}, \lambda U)$ represents the transformed game. Let $\Lambda = \{\lambda \in \mathbb{R}^n_+ \mid e\lambda = n\}$.

Definition 4.1:

1. Let $\mathfrak{G}$ be a class of games such that for any $\Gamma \in \mathfrak{G}$, $\pi \in \Pi$ and $\lambda \in \Lambda$ it follows that $\pi\Gamma \in \mathfrak{G}$ and $\lambda\Gamma \in \mathfrak{G}$. Then $\mathfrak{G}$ is called an *invariant* class.

2. A mapping

(8) $$\chi : \mathfrak{G} \to \bigcup \{\bar{X}^T \mid \Gamma = (\ldots, T, \ldots) \text{ for some } \Gamma \in \mathfrak{G}\}$$

is a *bargaining solution* if the following holds true.

(9)
1. $\chi(\Gamma) \in \mathfrak{M}(\Gamma)$ $\quad (\Gamma \in \mathfrak{G})$
2. $\chi(\Gamma)$ is *ex ante* P.E. in $\mathfrak{M}(\Gamma)$ $\quad (\Gamma \in \mathfrak{G})$
3. $\chi(\pi\Gamma) = \pi\chi(\Gamma)$ $\quad (\Gamma \in \mathfrak{G}, \pi \in \Pi)$
4. $\chi(\lambda\Gamma) = \lambda\chi(\Gamma)$ $\quad (\Gamma \in \mathfrak{G}, \lambda \in \Lambda)$

Definition 4.2: Let Γ, $\hat{\Gamma}$ be CII–games such that I, T, and p are identical. Then $\hat{\Gamma}$ is called an *extension* of Γ if

(10)
1. $V^{\mathfrak{M}} \subseteq \hat{V}^{\mathfrak{M}}$
2. $E(V^\tau \mid \tau_i = t_i) \subseteq E(\hat{V}^\tau \mid \tau_i = t_i)$

holds true for $i \in I$, $t_i \in T^i$.

The second requirement speaks of the NTU– or sidepayment–games a player views given his private information, clearly we mean

$$E(V^\tau \mid \tau_i = t_i) = \{E(U^\tau(x) \mid \tau_i = t_i) \mid x \in \bar{X}\}.$$

Of course in case of incomplete information on one side, this amounts to viewing V^α and V^β for player 1 and

$$V^{\bar{X}} = \{EU^\tau(x) \mid x \in \bar{X}\}$$

for player 2. Equivalently for fee–games, this would then be expressed by $eb^\alpha \geq \hat{eb}^\alpha$, $eb^\beta \geq \hat{eb}^\beta$.

Definition 4.3: ("The IIA axiom")

A bargaining solution χ, defined on some (invariant) class $\mathfrak{G}$ satisfies the IIA–axiom if, for any Γ, $\hat{\Gamma} \in \mathfrak{G}$ such that $\hat{\Gamma}$ is an extension of Γ, the following holds true:

1. If $\hat{\mu} = \chi(\hat{\Gamma})$ is non–constant and there is a non–constant $\mu \in \mathfrak{M}$ such that
$$E(U_i^\tau \circ \mu^\tau \mid \tau_i = t_i) = E(\hat{U}_i^\tau \circ \hat{\mu}^\tau \mid \tau_i = t_i) \qquad (i \in I,\ t_i \in T^i)$$
then $\mu = \chi(\Gamma)$.

2. If $\hat{\mu} = \chi(\hat{\Gamma})$ is constant and there is a constant $\mu \in \mathfrak{M}$ such that
$$E(U^\tau \circ \mu^\tau) = E(\hat{U}^\tau \circ \mu^\tau)$$
then $\mu = \chi(\Gamma)$.

Remark 4.4:

1. In what follows we shall only deal with the class $\bar{\mathfrak{G}}^{1,2}$ of fee–games with incomplete information on one side where the informed player has two types. Within this framework we know by Corollary 3.5 of [11], *that any* $u \in \partial V^{\mathfrak{M}}$ *is uniquely obtained by some* $\mu \in \mathfrak{M}$ – apart from Dante's world (cf. Example 2.3) where μ^β (the result in hell) is not unique (but unimportant, after all). In this sense we will slightly abuse the term "unique" – meaning "up to some anomalities in Dante's hell".

2. Within this class $\mathfrak{G}^1$, we could actually strengthen the requirement of $\hat{\Gamma}$ being an extension of Γ, i.e., Definition 4.2. Indeed, as we proved more in SEC.3 than we are actually going to use the stronger requirement would be that $\hat{\Gamma}$ is an extension of Γ if
$$1'. \quad V^{\mathfrak{M}} \subseteq V^{\mathfrak{M}'}$$
and
$$2'. \quad b^\alpha = \hat{b}^\alpha,\ eb^\beta \leq \hat{eb}^\beta$$
holds true. Of course, this version is specifically adopted to fee–games in the narrow sense while the one offered by (10) can be regarded to be very general.

Theorem 4.5: There is a unique bargaining solution on $\bar{\mathfrak{G}}^{1,2}$ which satisfies the IIA axiom. This solution is obtained by the unique mechanism $\chi^{(\nu)}(\Gamma)$ which for $\Gamma \in \bar{\mathfrak{G}}^{1,2}$ implements the Nash solution of $V^{\mathfrak{M}}$, i.e., satisfies

(11) $$EU^\tau \circ \chi^{(\nu)\tau}(\Gamma) = \nu(V^{\mathfrak{M}(\Gamma)}) \qquad (\Gamma \in \mathfrak{G}^1).$$

Proof:

1. STEP: Let Γ be a world of truth (Remark 2.2) (so in particular: a game with incomplete information). In this case, the only mechanisms yielding utilities in $\partial V^{\mathfrak{M}}$ are constant ones. Since the axioms imposed upon χ by Definition 4.1 and 4.3 are equivalent to the axioms for the Nash–value if we focus on utility space, $EU^{\tau} \circ \chi^{\tau}(\Gamma)$ has to be the midpoint of $\partial V^{\mathfrak{M}}$ – hence $\chi(\Gamma) = \chi^{\nu}(\Gamma)$.

2. STEP: Next consider a game Γ that admits of *no* constant *ex ante* P.E. mechanisms. Dante's world is of such nature (Example 2.3) but also the game $\hat{\Gamma}$ providing the extension in Theorem 3.3 provides an example.

By applying a linear transformation of utility we may as well assume that

(12) $$V^{\mathfrak{M}} = \{u \in \mathbb{R}^2_{\ddagger} \mid eu \leq 1\} = V_{<1>}$$

holds true.

Now, let us compute the expected payoff of any χ that satisfies the axioms (in particular the symmetry requirement (9).3). To this end, we have to employ a random variable $\tau' : \Omega \to \pi T$ with distribution πp. But, if $\tau : \Omega \to T$ has distribution p and τ' is *defined* via $\tau' = \pi \circ \tau$, then τ' indeed does have distribution $\pi p = p \circ \pi^{-1}$.

Hence the desired expected payoff is

(13) $$E(\pi U)^{\tau'} \circ \chi^{\tau'}(\pi\Gamma).$$

Now, as χ is symmetric, we have, for $t' \in \pi T$

(14) $$\chi^{t'}(\pi\Gamma) = \pi\chi(\Gamma)(t') = \pi\chi^{\pi^{-1}(t')}(\Gamma)$$

(see (6)), thus

(15) $$\chi^{\tau'}(\pi\Gamma) = \pi\chi^{\pi^{-1}\circ\tau'}(\Gamma) = \pi\chi^{\tau}(\Gamma).$$

Similarly (4) implies

(16) $$\pi U^{\tau'}(x') = \pi U^{\tau}(\pi^{-1}(x'))$$

for $x' \in \underline{X}$: Now, plugging (15) and (16) into (13) we find

(17)
$$\begin{aligned} &E(\pi U)^{\tau'} \circ \chi^{\tau'}(\pi\Gamma) \\ &\quad = E(\pi U)^{\tau'} \circ \pi\chi^{\tau}(\Gamma) \\ &\quad = E\pi U^{\tau}(\pi^{-1}(\pi\chi^{\tau}(\Gamma))) \\ &\quad = E\pi\, U^{\tau} \circ \chi^{\tau}(\Gamma) \\ &\quad = \pi EU^{\tau} \circ \chi^{\tau}(\Gamma)\,. \end{aligned}$$

Also, it is seen that

$$V^{\mathfrak{M}}(\pi\Gamma) = V^{\mathfrak{M}}(\Gamma) = \pi V^{\mathfrak{M}}(\Gamma)$$

holds true. Therefore *the IIA–axiom* yields

$$\pi EU^{\tau} \circ \chi^{\tau}(\Gamma) = EU^{\tau} \circ \chi^{\tau}(\Gamma).$$

That is, $EU^{\tau} \circ \chi^{\tau}(\Gamma)$ has to be the midpoint of $\partial V^{\mathfrak{M}}$ and this settles the case we are discussing in the second step, as $\chi(\Gamma)$ is uniquely defined and equal to $\chi^{(\nu)}(\Gamma)$ by Corollary 3.5 of [11].

3. STEP: Finally, let $\Gamma = \Gamma_{<b>}$ be a profane world and denote by $\bar{u} = \nu(V^{\mathfrak{M}})$ the Nash–payoff of $V^{\mathfrak{M}}$. Use either one of Theorems 3.3, 3.6, or 3.7 to construct $\hat{\Gamma}$ such that $\hat{V}^{\mathfrak{M}} \supseteq V^{\mathfrak{M}}$ and $\bar{u} \in \partial\hat{V}^{\mathfrak{M}}$ is as well the Nash–solution of $\hat{V}^{\mathfrak{M}}$.

By the previous two steps we know that $\chi(\hat{\Gamma}) = \chi^{(\nu)}(\hat{\Gamma})$ implements $\bar{u}$. By the IIA–axiom it follows that ($E(U^{\tau} \circ \chi^{\tau}(\Gamma) = \bar{u}$ and) $\chi(\Gamma) = \chi^{(\nu)}(\Gamma)$ (by Corollary 3.5 of [11]).

4. STEP: We have now essentially completed the proof of uniqueness, though not the full variety of all worlds has been treated exhaustively (see SEC.1, SEC.2.).

Existence of a bargainign solution does not constitute a problem since it is not hard to prove that $\chi^{(\nu)}$ indeed satisfies the axioms,

q.e.d.

SECTION 5

The second axiomatic approach: The expected contract

The symmetry axiom or the covariance with permutation, as reflected by (9), 3. of Definition 4.1 is open to some criticisms. For instance, $V^{\mathfrak{M}}$ may look very symmetric without reflecting the essential differences in information (and mechanisms): compare Dante's world and the world of truth. On the other hand, the elements Γ of $\mathfrak{G}^1$ are games with incomplete information on one side – a very non–symmetric situation in many cases. The IIA–axiom (Definition 4.3) reflects this asymmetric standing suitably. If we exchange the players names, then χ should react accordingly – however, the construction in the two first steps of Theorem 4.5 uses symmetry in a heavy way.

In any case we want to offer a second axiomatization which is not based on symmetry. Instead, we shall use the axiom of "expected contract" as developed in [11] and [12]. This axiom together with a slightly weakened version of IIA will provide a second framework in which to justify χ^{ν} axiomatically.

For the purpose of this section we shall therefore drop the symmetry axiom, i.e., the covariance property with respect to permutations as expressed in the 3. requirement of (9) in Definition 4.1.

Therefore, an *invariant class* of games is now (deviating from Definition 4.1) a class that is *stable with respect to linear transformations of utility* as represented by $\lambda \in \mathbb{R}_{++}$, $e\lambda = n$. Essentially we focus on the class $\bar{\mathfrak{G}}^1$ of fee games (in the wider sense) where player 1 has full information and two types (the linear extension of $\mathfrak{G}^1$, see Definition 1.7). In this context a bargaining solution is a mapping that satisfies 1., 2., and 4. of (9) in Definition 4.1.

Nevertheless, it will turn out that symmetry prevails in th end – however, it enters the scene via a conclusion, since χ^{ν} turns out to be the result of the axiomatic approach.

Now, in order to formulate the EC–axiom, let us first turn to the expected Nash–payoff and the expected contract – and for games $\Gamma \in \mathfrak{G}^1$ only.

Definition 5.1: Let $\Gamma = \Gamma_{<b>}$ be a fee game. For any $t \in T$, let

(1) $$\bar{x}^t := \frac{1}{n} \sum_{i=1}^{n} a^{t,i}$$

be the midpoint of I^t and let

(2) $$\bar{u}^t = U^t(\bar{x}^t) = \nu(V^t)$$

be the midpoint of ∂V^t and Nash–solution of V^t. Then

(3) $$\bar{u} = E\, \bar{u}^{\tau}$$

is the *expected Nash–payoff* and

(4) $$\bar{x} = E\, \bar{x}^{\tau}$$

is the *expected contract*.

Because all utility functions U^t are linear, it is seen at once that *the expected contract implements* $\bar{u}$, i.e., that

(5) $$EU^{\tau}(\bar{x}) = EU^{\tau}(\bar{x}^{\tau}) = \bar{u}$$

holds true (see also [11]).

Definition 5.2: A bargaining solution χ on an invariant class $\mathfrak{G}$ satisfies the *expected contract axiom* (the *EC–axiom*) if, for all $\Gamma \in \mathfrak{G}$ satisfying $\bar{u} \in V^{\mathfrak{M}}$, it follows that

(EC) $$EU^{\tau} \circ \chi^{\tau}(\Gamma) = \bar{u}$$

holds true. In particular, if $\mathfrak{G} = \bar{\mathfrak{G}}^1$, this means (in view of Theorem 1.10) that, whenever $\bar{\mu} = (\bar{x}, \bar{x}) \in \mathfrak{M}$ holds true, it follow that $\chi(\Gamma) = \bar{\mu}$; thus the expected contract is chosen once it is available.

The following remark presents some motivation; see also [11] [12].

Remark 5.3:

Let us discuss some types of worlds in which the expected Nash–payoff may be implemented by means of some (unique) $\mu \in \mathfrak{M}$. In doing so we want to provide the motivation for the EC–axiom.

The first type is Dante's world. Indeed, let us return to Remark 2.3. If $\bar{x}^{\alpha}$ again denotes the midpoint of I^{α} then 2.3 shows that the mechanism

(6) $$\bar{\mu} = (\bar{x}^{\alpha}, a^{\bar{x}^{\alpha}}) \in \mathfrak{M}$$

(see SEC.2, (28) – (30) and Fig. 2.3) yields

(7) $$E(U^\tau \circ \bar{\mu}^\tau) = E\nu(V^\tau) = \nu(V^{\mathfrak{M}}).$$

Therefore, Dante's world poses no problem of motivation for introducing $\bar{\mu}$, that is, for requiring the EC–axiom: complete justice can be implemented in heaven by imposing the suitable version of punishment in hell.

However, how about the profane world?

Indeed, suppose that for some fee–game $\Gamma_{<b>}$ the expected contract $\bar{x}$ happens to constitute a constant mechanism $\bar{\mu} = (\bar{x},\bar{x}) \in \mathfrak{M}$ ($\bar{\mu}$ is always BIC but not in any case IR).

Let us first focus on the uninformed player, player 2. Originally, he was viewing

(8) $$V_+^{\bar{X}} := \{EU^\tau(x) \mid x \in \bar{X}\} \cap \mathbb{R}_+^2$$
$$= V_{<1-Eb^t>} \cap \mathbb{R}_+^2 = \{x-Eb^\tau \mid x \in \bar{X}\} \cap \mathbb{R}_+^2 = V^{\mathfrak{M}}(\bar{\Gamma}) = V^{\mathfrak{M}}(\Gamma_{<\bar{b}>})$$

since he has no private information. However, player 1 told him, that some $x \in \bar{X}$ are unacceptable for him (even if $EU^\tau(x) \in V_+^{\bar{X}}$) since *in mediis* they are not IR. Thus, player 1 wanted to make decisions dependent on his observations (sometimes, at least). But since player 2 could not convince himself to trust his opponent under all circumstances they ended up with mechanisms $\mu \in \mathfrak{M}$.

By Lemma 2.6 of [11] it turns out that $V^{\mathfrak{M}}$ is a compact polyhedron, satisfying

(9) $$V^{\mathfrak{M}} \subsetneq V_+^{\bar{X}}$$

and in most profane worlds, the inclusion is a proper one – an inconvenience, but what can you expect of the profane world.

Now, it turns out that the expected contract $\bar{x} = E\bar{x}^\tau$ happens to be in $\mathfrak{M}$, thus $\bar{u} \in V^{\mathfrak{M}}$. Clearly, this is what player 2 wanted from the beginning: the Nash–value of $V_+^{\bar{X}}$ – so he should have no objections. In fact, (8) and (9) show that, from the viewpoint of player 2, "some kind of IIA–argument" requires to agree upon $\bar{x}$.

Now to some considerations concerning player 1:

On one hand, player 1 could propose some constant mechanism other than $\bar{x}$, say $\hat{x} \neq \bar{x}$. Certainly we expect player 1 to choose $\hat{x}$, individually rational *in mediis*, therefore $\hat{u} := EU^\tau(\hat{x}) \in V^{\mathfrak{M}}$. In proposing some constant mechanism, player 1 waves the opportunity to exploit his private information *in mediis*, so his proposal amounts to some utility $\hat{u} \in V^{\mathfrak{M}}$ which differs from the midpoint $\bar{u}$ of $V^{\bar{X}}$ although $\bar{u} \in V^{\mathfrak{M}}$ holds true. It is hard to imagine anyone who favors the Nash–solution bringing forward such a proposal.

On the other hand, player 1 could bring up a non–constant mechanism, say $\mu = (\mu^\alpha,\mu^\beta)$. In a profane world with $\bar{u} \in V^{\mathfrak{M}}$, it is seen at once that

(10) $$U_1^\alpha(\mu^\alpha) < U_1^\alpha(\bar{x}),$$

and from Theorem 1.10 we know that

(11) $$U_1^\beta(\mu^\beta) = 0.$$

From this it follows that player 1 *ex ante* as well as *in mediis* is worse off at μ than at $\bar{x}$ – so why should he bring forward nonconstant mechanisms at all?

Definition 5.4:

1. Let $\Gamma, \hat{\Gamma} \in \bar{\mathfrak{G}}^1$ be such that I, T, and p are identical. Then $\hat{\Gamma}$ is called a *weak extension* of Γ if

(12)
1. $V^{\mathfrak{M}} \subsetneq \hat{V}^{\mathfrak{M}}$
2. $V^\alpha \subsetneq \hat{V}^\alpha$

holds true.

2. A bargaining solution χ on $\mathfrak{G}^1$ satisfies the *weak IIA–axiom* if, for any $\Gamma, \hat{\Gamma} \in \bar{\mathfrak{G}}^1$ such that $\hat{\Gamma}$ is a weak extension of Γ, the following holds true:
If $\hat{\mu} = \chi(\hat{\Gamma})$ is non–constant and there is a non–constant $\mu \in \mathfrak{M}$ such that

(13) $$E(U_i^\tau \circ \mu^\tau \mid \tau_i = t_i) = E(\hat{U}^\tau \circ \hat{\mu}^\tau \mid \tau_i = t_i)$$

is satisfied, then $\chi(\Gamma) = \mu$ holds true.

Theorem 5.5: There is a unique bargaining solution on $\bar{\mathfrak{G}}^1$ which satisfies the EC–axiom and the weak IIA–axiom. This solution is χ^{ν}.

Proof: Consider a profane world Γ. If the expected contract $\bar{x}$ yields $\bar{\mu} = (\bar{x}, \bar{x}) \in \mathfrak{M}$, then any χ satisfying the axioms has to yield $\chi(\Gamma) = \bar{\mu}$. This settles the analogue to the first step of the proof of 4.5.

Note that in view of Remark 5.2 any χ satisfying the axioms must yield $\chi(\Gamma) = \bar{\mu} = (\bar{x}^{\alpha}, a^{\bar{x}^{\alpha}})$ whenever Γ is Dante's world. Therefore, the second and third step can be dealt with analogously in view of the following Lemma.

Lemma 5.6: Let Γ be a profane world such that

(14) $$\nu(V^{\mathfrak{M}}) \in [u^L, u^M].$$

Then there exists a weak extension $\hat{\Gamma}$ of Γ with the following properties.

1. $\hat{\Gamma}$ is a world of Dante
2. $\hat{\bar{\mu}} = \chi(\hat{\Gamma})$ as defined by Remark 5.2 satisfies

(15) $$E\hat{U}^{\tau} \circ \hat{\bar{\mu}}^{\tau} = EU^{\tau} \circ \chi^{(\nu),\tau}(\Gamma)$$

(16) $$\hat{U}_1^{\alpha}(\hat{\bar{\mu}}^{\alpha}) = U_1^{\alpha}(\chi^{(\nu),\alpha}(\Gamma)).$$

Proof:

We will only treat the case that $\nu(\mathfrak{M}) \in [u^L, u^M)$; preferably the reader should inspect the profane world as depicted in Fig. 2.1 once again.

Now, extend the straight line $[u^L, u^M]$ this way constructing a (λ–) hyperplane–game V^0, see Fig. 5.1.

However, by scale covariance we may assume that $\lambda = (1,1)$. Define $\hat{V}^{\alpha} := \frac{1}{p_{\alpha}} V^0$ and let $\hat{b}^{\alpha}$ be such that $\hat{U}^{\alpha}$ is the canonical representation of $\hat{V}^{\alpha}$ (see SEC.1). Choose $\hat{b}^{\beta}$ with $e\hat{b}^{\beta} = 1$ such that $\hat{\Gamma}$ is Dante's world.

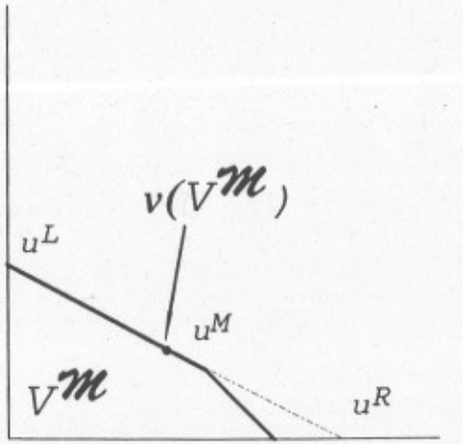

Figure 5.1
Imbedding the profane world into Dante's world

Then clearly

(17) $$\hat{V}^{\mathfrak{M}} = p_{\alpha} \hat{V}^{\alpha} = V^0$$

and hence $\hat{\Gamma}$ is indeed a weak extension of Γ.

Now, inspecting Example 2.3 and taking $\hat{\bar{\mu}} = (\hat{\bar{x}}^{\alpha}, a^{\hat{\bar{x}}^{\alpha}})$ into consideration, it is clear that $\hat{\bar{\mu}}$ implements the midpoint of $[u^L, u^0]$, i.e. $\nu(V^{\mathfrak{M}})$ – and so does $\chi^{(\nu)}(\Gamma)$ in Γ. Of course $\hat{\bar{\mu}} = \chi(\hat{\Gamma}) = \chi^{(\nu)}(\hat{\Gamma})$ and (15) is satisfied.

In order to show (16), observe that (10) means in particular

(18) $$p_{\alpha} U_1^{\alpha}(\chi^{\alpha}) + p_{\beta} U_1^{\beta}(\chi^{\beta}) = p_{\alpha} \hat{U}_1^{\alpha}(\hat{\chi}^{\alpha}) + p_{\beta} \hat{U}_1^{\beta}(\hat{\chi}^{\beta})$$

(with $\chi = \chi(\Gamma)$, $\hat{\chi} = \chi(\hat{\Gamma})$) and on both sides the right hand summands vanish – in Γ because of Theorem 1.10 and in $\hat{\Gamma}$ because β is hell... q.e.d.

Example 5.7: (= 1.5)
How should we split the dollar in Example 1.5, our introductory problem. We have $\Gamma_{<b>} \in \mathfrak{G}^1$ with $b^\alpha = (\frac{1}{10}, \frac{1}{10})$ and $b^\beta = (\frac{7}{10}, \frac{1}{10})$.

Depending on the distribution $p = (p_\alpha, p_\beta)$ $(p_\alpha + p_\beta = 1)$ we compute $\chi = \chi^{(\nu)}(\Gamma) = (\chi^\alpha, \chi^\beta)$. Also we list the *ex ante* expected utility $u = EU^\tau \circ \chi^{(\nu)\tau}(\Gamma)$.

For $0 \leq p_\alpha \leq 1/3$ we have

$$\chi^\alpha = \chi^\beta = \frac{1}{10}(8-3p_\alpha, 2+3p_\alpha)$$

(19)

$$u = \frac{1}{10}(1+3p_\alpha, 1+3p_\alpha).$$

(Note that for $0 \leq p_\alpha \leq 1/4$, Γ is not a profane world). Here we implement a constant mechanism, for $\frac{1}{4} < p_\alpha \leq 1/3$ the framework of Theorem 3.7 is appropriate.

Next for $1/3 \leq p_\alpha \leq 1/2$ we find

$$\chi^\alpha = \chi^\beta = (\frac{7}{10}, \frac{3}{10})$$

(20)

$$u = (\frac{3}{5}p_\alpha, \frac{1}{5}) = u^M.$$

Here we are implementing the central extremepoint u^M of $V^{\mathfrak{M}}$ (cf. also Theorem 3.6).

Finally, for $1/2 \leq p_\alpha \leq 1$ we have to choose nonconstant mechanisms (cf. Theorem 3.3). We find

$$\chi^\alpha = \frac{1}{5}\frac{1}{8p_\alpha - 2}(16\,p_\alpha - 1,\ 24\,p_\alpha - 9)$$

(21)

$$\chi^\beta = \frac{1}{8p_\alpha - 2}\frac{32p_\alpha - 1}{6}(\frac{7}{10}, \frac{3}{10})$$

(where $(\frac{7}{10}, \frac{3}{10}) = a^{\beta,2}$). Also

(22) $$u = \frac{2}{5}p_\alpha(\frac{6p_\alpha}{8p_\alpha - 2}, 1).$$

The solutions of HARSANYI–SELTEN [2] (see also WEIDNER [13]) MYERSON [6] [7] [8], although formulated in different context, can be transferred to this problem; for some computations as well as pros and cons see [10].

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