INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 260

The Difference Between Common Knowledge of Formulas and Sets: Part II

by

Robert S. Simon

August 1996

University of Bielefeld

33501 Bielefeld, Germany

The Difference Between Common Knowledge of Formulas and Sets: Part II

Abstract

This article concerns the interactive modal propositional calculus, using the multi-agent epistemic logic $S5$. Let there be at least two agents and let Ω be the space of maximally consistent sets of formulas. When is the member of the meet partition of the partitions of Ω generated by the knowledge of the agents determined by the set of formulas held in common knowledge? In part II, this question is investigated for infinitely generated sets of formulas held in common knowledge.

This research was supported by the German Society for the Encouragement of Scientific Research (Deutsche Forschungsgemeinschaft.)

1 Introduction

In part I we investigated finitely generated sets of formulas held in common knowledge, and established some relationships between the size of cells, the partial order by inclusion of the sets of formulas that can be held in common knowledge, and the number of the cells sharing the same sets of formulas in common knowledge. Now we investigate these and similar properties when we do not assume that the set of formulas held in common knowledge is finitely generated.

Lemmas 1, 2, and 3, Theorems 1 and 2, Propositions 1, 2, and 3, and Corollaries 1 and 2 belong to Part I.

If $S \in \mathcal{CK}$ and $F(S)$ is a singleton then we say that S is "centered," and we will also say that the single cell member of $F(S)$ is centered.

Our most important results are the following:

* a constructive proof of Corollary 2,
* with three agents there is an example of a maximal but uncentered member of $\mathcal{CK}$ (proved using Theorem 3,)
* Theorem 4: as long as $n \geq 2$ and $|X| \geq 2$ there are uncountably many cells with finite fan-out that are dense in $\Omega(X, \underline{n})$,
* Theorem 5: if a cell C is centered and does not have finite fan-out then there exists a Kripke structure μ that is mapped by ϕ_∞^μ injectively but not surjectively into C,
* Theorem 6: even with only two agents there is an $S \in \mathcal{CK}$ that is not centered but every member of $F(S)$ has finite fan-out.

We define the "depth" of a formula inductively on the structure of the formulas. If $x \in X$, then depth $(x) := 0$. If $f = \neg g$ then depth $(f) :=$ depth (g); if $f = g \wedge h$ then depth $(f):=$ max (depth (g) , depth (h)); and if $f = k_j(g)$ then depth $(f):=$ depth (g) $+1$.

Due to a change in emphasis, we must introduce more from the "worlds" approach to the structures Ω_i.

Fix $0 \leq i < \infty$ and $w \in \Omega_i$, and for every $j \in \underline{n}$ let $w \in \overline{F}_i^j \in \overline{\mathcal{F}}_i^j$. If $M_i^1, \ldots, M_i^n$ are subsets of $\overline{F}_i^1, \ldots, \overline{F}_i^n$, respectively, such that

1) $w \in M_i^j$ for every j, and
2) for every $B \in \mathcal{F}_{i-1}$ $\quad \overline{F}_i^j \cap \pi_i(B) \neq \emptyset$ implies that $M_i^j \cap \pi_i(B) \neq \emptyset$,
then there is a unique world $v \in \Omega_{i+1}$ such that $\pi_i \circ \pi_{i+1}^{-1}(v) = w$ and for every $u \in \Omega_i$ $\quad \neg k_j \neg(f(u)) \in v$ if and only if $u \in M_i^j$ (See page 387 of [Fa-Ha-Va].) For any $i \geq 0$ and $v \in \Omega_k$ with $k > i$ we define $M_i^j(v) = \{u \in \Omega_i \mid \neg k_j \neg(f(u)) \in v\}$. Notice that if $v \in \Omega_{i+1}$, $v \in F \in \overline{\mathcal{F}}_{i+1}^j$ and $\pi_i \circ \pi_{i+1}^{-1}(v) \in F^* \in \overline{\mathcal{F}}_i^j$ then $M_i^j(v) = \pi_i \circ \pi_{i+1}^{-1}(F) \subseteq F^*$.

We define an "information set" of $\Omega = \Omega(X, \underline{n})$ to be any member of $\mathcal{F}_\infty^j$ for any $j \in \underline{n}$.

2 A special observing agent

Before going into specific examples, let us develop a perspective on special sets of the form $\mathbf{Ck}(T) \subseteq \Omega(X, \underline{n+1})$ for $n \geq 2$.

For any $n \geq 2$, we define a "consistent filter of partitions" of $\Omega = \Omega(X, \underline{n})$ to be a sequence of partitions $(\mathcal{P}_0, \mathcal{P}_1, \ldots)$ of Ω such that
1) for every $0 \leq i < \infty$ $\mathcal{P}_i$ is equal to or coarser than $\mathcal{F}_i$,
2) for every $0 < i < \infty$ $\mathcal{P}_i$ is equal to or finer than $\mathcal{P}_{i-1}$, and
3) for every $0 < i < \infty$ if $P_i \in \mathcal{P}_i$, $P_{i-1} \in \mathcal{P}_{i-1}$ and $P_i \subseteq P_{i-1}$ then $P_i \cap B \neq \emptyset$ for every $B \in \mathcal{F}_{i-1}$ with $B \subseteq P_{i-1}$.

For any consistent filter of partitions $\mathcal{B} = (\mathcal{P}_i \mid 0 \leq i < \infty)$ of $\Omega(X, \underline{n})$ define a Kripke structure

$$\mu(\mathcal{B}) = (\Omega(X, \underline{n}); \mathcal{F}_\infty^1, \ldots, \mathcal{F}_\infty^n, \mathcal{P}_\infty; X; \overline{\psi})$$

where the partition $\mathcal{P}_\infty$ for the $n+1$st agent is the limit of the partitions $\mathcal{P}_i$, meaning that z and z' share the same member of $\mathcal{P}_\infty$ if and only if they share the same member of $\mathcal{P}_i$ for every $i < \infty$, and $\overline{\psi}$ and the $\mathcal{F}_\infty^j$ are the same used to define the Kripke structure $\Omega(X, \underline{n})$. For every $i < \infty$ and $w \in \Omega_i$ define $P_i(w)$ to be the member of $\mathcal{P}_i$ containing $\pi_i^{-1}(w)$ and for any $z \in \Omega$ define $P_i(z)$ to be member of $\mathcal{P}_i$ containing z. For every $0 \leq i < \infty$ define a formula $h(\mathcal{P}_i) \subseteq \mathcal{L}(X, \underline{n+1})$ of depth $i+1$ by

$$h(\mathcal{P}_i) := \bigwedge_{w \in \Omega_i} \Big(f(w) \Rightarrow \Big(\bigwedge_{v \in \pi_i(P_i(w))} \neg k_{n+1} \neg f(v) \bigwedge_{v \notin \pi_i(P_i(w))} k_{n+1} \neg f(v) \Big) \Big).$$

Next, define the set of formulas $T(\mathcal{B}) := \{h(\mathcal{P}_0), h(\mathcal{P}_1), \ldots\}$.

Theorem 3: The Kripke structures $\mathbf{Ck}(T(\mathcal{B}))$, as a closed subset of $\Omega(X, \underline{n+1})$, and $\mu(\mathcal{B})$ are isomorphic with fixed ground set and agents using the bijection $\Gamma : \mathbf{Ck}(T(\mathcal{B})) \to \Omega(X, \underline{n})$ defined by

$$\Gamma(z) := \{f \in \mathcal{L}(X, \underline{n}) \mid f \in z\} = z \cap \mathcal{L}(X, \underline{n}).$$

Furthermore, the map Γ induces a homeomorphism between $\Omega(X, \underline{n})$ and $\mathbf{Ck}(T(\mathcal{B}))$, and the inverse of Γ is $\phi_\infty^{\mu(\mathcal{B})}$.

Proof: First we show that $h(\mathcal{P}_i)$ is valid in $\mu(\mathcal{B})$ for every i. Let z be any member of $\Omega(X, \underline{n})$ and let $z \in P \in \mathcal{P}_\infty$. Due to the Stability Lemma it suffices to show that
for every $B \in \mathcal{F}_i$ with $B \subseteq P_i(z)$ it follows that $P \cap B \neq \emptyset$ and
for every $B \in \mathcal{F}_i$ with $B \cap P_i(z) = \emptyset$ it follows that $P \cap B = \emptyset$.
The latter follows from the fact that $\mathcal{P}_\infty$ is finer than or equal to $\mathcal{P}_i$. For the former, Condition 3 of the definition of consistent filters of partitions implies the existence of a nested decreasing sequence of non-empty compact sets $B_k \in \mathcal{F}_k$, $k \geq i$, with $B_i = B$ and $B_k \subseteq P_k(z)$ for every $k \geq i$. The limit $z' := \cap_{k=i}^\infty B_k$ will share the same member of $\mathcal{P}_\infty$ with z. Therefore $\phi_\infty^{\mu(\mathcal{B})}$ maps $\Omega(X, \underline{n})$ to $\mathbf{Ck}(T(\mathcal{B}))$.

Notice from the definition of $\mu(\mathcal{B})$ that if $f \in \mathcal{L}(X, \underline{n})$ then $f \in z \in \Omega(X, \underline{n})$ if and only if $f \in \phi_\infty^{\mu(\mathcal{B})}(z)$. This implies that the map $\Gamma \circ \phi_\infty^{\mu(\mathcal{B})}$ is the identity on $\Omega(X, \underline{n})$.

If $z \in \Omega(X, \underline{n})$ then there exists at least one world $v \in \Omega_i(X, \underline{n+1})$ with $f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge E(h(\mathcal{P}_{i-2})) \wedge h(\mathcal{P}_{i-1}) \in v$, namely $v = \phi_i^{\mu(\mathcal{B})}(z)$. (By $E(\cdot)$ we mean $\wedge_{j \in \underline{n+1}} k_j(\cdot)$.) Define $\Gamma_i : \mathbf{Ck}(T(\mathcal{B})) \to \Omega_i(X, \underline{n})$ by

$$\Gamma_i(y) := y \cap \mathcal{L}_i(X, \underline{n}) = \pi_i \circ \Gamma(y).$$

Furthermore, if $y \in \mathbf{Ck}(T(\mathcal{B}))$ then $f(\Gamma_i(y)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1}) \in \pi_i(y)$.

The following claim is equivalent to the claim that for every $i < \infty$ and $z \in \Omega(X, \underline{n})$ $\quad f(\phi_i^{\mu(\mathcal{B})}(z)) \Leftrightarrow (f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})) \in \mathcal{L}(X, \underline{n+1})$ is a tautology.

Claim: If $z \in \Omega(X, \underline{n})$ then $v = \phi_i^{\mu(\mathcal{B})}(z)$ is the only world in $\Omega_i(X, \underline{n+1})$ containing $f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})$.

We prove the claim by induction on i. If $i = 0$ then $\Omega_0(X, \underline{n}) = \Omega_0(X, \underline{n+1})$ and $f(\phi_0^{\mu(\mathcal{B})}(z)) = f(\pi_0(z))$. Otherwise, if $i \geq 1$, any two different worlds v

and v' in $\Omega_i(X, \underline{n+1})$ both containing $f(\pi_i(z))$ must differ concerning the containment of $k_j(f(u))$ for some $j \in \underline{n+1}$ and some $u \in \Omega_{i-1}(X, \underline{n+1})$. If $\neg(E^{i-2}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-2})) \in u \in \Omega_{i-1}(X, \underline{n+1})$, then $E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})$ being true at v and v' would mean that both v and v' would contain $k_j \neg f(u)$. So for the rest of the proof we assume for the sake of contradiction that there are two worlds v and v' in $\Omega_i(X, \underline{n+1})$ both containing $f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})$ and $k_j f(u) \in v$ and $\neg k_j f(u) \in v'$ for some $u \in \Omega_{i-1}(X, \underline{n+1})$ with $E^{i-2}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-2}) \in u$. For $k < \infty$ define $\gamma_k : \Omega_k(X, \underline{n+1}) \rightarrow \Omega_k(X, \underline{n})$ by $\gamma_k(v) := v \cap \mathcal{L}_k(X, \underline{n})$. By the induction hypothesis we must assume that $u = \phi_{i-1}^{\mu(\mathcal{B})}(z^*)$ for some $z^* \in \pi_{i-1}^{-1}(\gamma_{i-1}(u))$ and that $f(u) \Leftrightarrow (f(\gamma_{i-1}(u)) \wedge E^{i-2}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-2}))$ is a tautology.

Case 1; $j \in \underline{n}$: Since $f(u) \Leftrightarrow (f(\gamma_{i-1}(u)) \wedge E^{i-2}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-2}))$ is a tautology and either $f(\pi_i(z)) \Rightarrow k_j(f(\gamma_{i-1}(u))$ is a tautology or $f(\pi_i(z)) \Rightarrow \neg k_j(f(\gamma_{i-1}(u))$ is a tautology we have also that either $(f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})) \Rightarrow k_j f(u)$ or $(f(\pi_i(z)) \wedge E^{i-1}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-1})) \Rightarrow \neg k_j f(u)$ is a tautology, a contradiction.

Case 2, $j = n+1$: The formula $h(\mathcal{P}_{i-1})$ and the world $\pi_{i-1}(z)$ have already determined whether $k_{n+1}(f(\gamma_{i-1}(u))$ or $\neg k_{n+1}(f(\gamma_{i-1}(u))$ is in v or v', and therefore the tautology $f(u) \Leftrightarrow (f(\gamma_{i-1}(u)) \wedge E^{i-2}(h(\mathcal{P}_0)) \wedge \ldots \wedge h(\mathcal{P}_{i-2}))$ settles the claim.

By the claim, Γ is injective. With $\Gamma \circ \phi_\infty^{\mu(\mathcal{B})}$ the identity on $\Omega(X, \underline{n})$, this implies that $\phi_\infty^{\mu(\mathcal{B})}$ and Γ are inverses. That means also that $\phi_\infty^{\mu(\mathcal{B})} \circ \pi_i^{-1}(w)$ is an open set of $\mathbf{Ck}(T(\mathcal{B}))$ for every $w \in \Omega_i(X, \underline{n})$; topological equivalence of $\mu(\mathcal{B})$ and $\mathbf{Ck}(T(\mathcal{B}))$ follows.

Lastly, we must show that $\mu(\mathcal{B})$ and $\mathbf{Ck}(T(\mathcal{B}))$ are isomorphic. Let $\overline{\psi}_n$ be the map from X to subsets of $\Omega(X, \underline{n})$ defining the Kripke structure $\Omega(X, \underline{n})$ and likewise define $\overline{\psi}_{n+1}$. Since $\Omega_0(X, \underline{n}) = \Omega_0(X, \underline{n+1})$ we have that $\Gamma \circ \overline{\psi}_{n+1} = \overline{\psi}_n$. Now we must show for every $j \in \underline{n+1}$ that y and y' in $\mathbf{Ck}(T(\mathcal{B}))$ share the same member of $\mathcal{F}_\infty^j$ of $\Omega(X, \underline{n+1})$ if and only if $\Gamma(y)$ and $\Gamma(y')$ share the same member of the jth agent's partition in the Kripke structure $\mu(\mathcal{B})$.

Case 1; $j \in \underline{n}$: Let us assume that the pair $\Gamma(y), \Gamma(y') \in \Omega(X, \underline{n})$ do not share the same member of $\mathcal{F}_i^j$ of $\Omega(X, \underline{n})$ for some $i \geq 1$. Any difference $\Gamma(y)$ and $\Gamma(y')$ have in the containment of some formula $k_j(f)$ for f in $\mathcal{L}_{i-1}(X, \underline{n})$ must also be present in their images by $\phi_\infty^{\mu(\mathcal{B})}$ in $\Omega(X, \underline{n+1})$, namely $\phi_\infty^{\mu(\mathcal{B})} \circ \Gamma(y) = y$ and $\phi_\infty^{\mu(\mathcal{B})} \circ \Gamma(y') = y'$.

Case 2; $j = n+1$: Assume that $\Gamma(y)$ and $\Gamma(y')$ don't belong to the same member of $\mathcal{P}_i$ for some i. By the definition of the formulas $h(\mathcal{P}_i)$ this implies that $\phi_\infty^{\mu(\mathcal{B})} \circ \Gamma(y) = y$ and $\phi_\infty^{\mu(\mathcal{B})} \circ \Gamma(y') = y'$ differ on the containment of $k_{n+1}(f(w))$ for some i-world $w \in \Omega_i$, and thus y and y' don't belong to the same member of $\mathcal{F}_\infty^{n+1}$ in $\Omega(X, \underline{n+1})$.

The converse of both cases follows by the definition of ϕ. q.e.d.

3 Alienated Extensions: an alternative proof of Corollary 2

In order to show how $S \in \mathcal{CK}$ can be both maximal and un-centered, we need a digression into a technique that delivers Corollary 2 without the use of the Baire Category Theorem. (See [Si].)

Let $0 \le i < \infty$ and let w be any world in Ω_i. Let $\overline{F}_i^j(w)$ be the member of $\overline{\mathcal{F}}_i^j$ containing w. Define $p_{i+1}(w)$ to be that unique $i+1$ world such that $M_i^j(p_{i+1}(w)) = \overline{F}_i^j(w)$ for every $j \in \underline{n}$ [Fa-Ha-Va]. If $w \in \Omega_i$ define $p_i(w) := w$. Define $p_k(w) := p_k(p_{k-1}(w))$ for $k > i$ and $p(w) := \cap_{k=1}^\infty \pi_k^{-1}(p_k(w))$. From the definition of p it follows that if w and w' in Ω_i share the same member of $\overline{\mathcal{F}}_i^j$ for some $j \in \underline{n}$ then $p(w)$ and $p(w')$ share the same member of $\mathcal{F}_\infty^j$. The map $p_i : \Omega_{i-1} \to \Omega_i$ was called the "no-information" extension [Fa-Ha-Va].

For any $w \in \Omega_i$ and $w' \in \Omega_{i+1}$ with $w \in \pi_i \circ \pi_{i+1}^{-1}(w')$ Fagin, Halpern, and Vardi showed that as long as n, the number of agents, is at least two then the adjacency distance between $p_{i+1}(w)$ and w' in Ω_{i+1} is no greater than 2, (see Lemma 4.3 of [Fa-Ha-Va];) therefore the adjacency distance between $p(w)$ and $p(w')$ in Ω is no greater than 2. Together with finite induction this is sufficient to show that all no-information extensions belong to the same dense cell of Ω [Fa-Ha-Va], and that the Kripke structures Ω_i are connected for every $i < \infty$. We need to generalize their results to "alienated extensions."

Let $\mathcal{P}^\infty(\mathbf{N_0})$ be the set of subsets of the whole numbers $\mathbf{N_0} = \{0, 1, 2, \cdots\}$ with infinite cardinality ($S \in \mathcal{P}^\infty(\mathbf{N_0})$ implies $|S| = \infty$.) For any member S of $\mathcal{P}^\infty(\mathbf{N_0})$, $i \in S$, and $w \in \Omega_i$ we will define a special point in $\pi_i^{-1}(w)$ called the alienated extension of w with respect to S, labeled $p^S(w)$. If $i \in S \in \mathcal{P}_\infty(\mathbf{N_0})$ define $n_S(i) := \inf\{k \in \mathbf{N_0} \mid k > i, k \in S\}$. If $i \in S$ and $w \in \Omega_i$ define $p^S_{n_S(i)}(w) := p_{n_S(i)}(\phi^{\Omega_i}_{n_S(i)-1}(w))$ and $p_i^S(w) = w$. For every $k \in S$ and for all $w \in \Omega_i$ with $k \ge i \in S$ and p_k^S already defined, define

$p^S_{n_S(k)}(w)$ to be $p^S_{n_S(k)}(p^S_k(w))$. Lastly, for all $i \in S \in \mathcal{P}^\infty(\mathbf{N_0})$ and $w \in \Omega_i$ define

$$p^S(w) := \bigcap_{l \in S,\ l > i} \pi_l^{-1}(p^S_l(w)).$$

By Lemma 2a ($w \in \Omega_i$ implies $\phi_i^{\Omega_i}(w) = w$) we have that $p^{\mathbf{N_0}} = p$. For any $i \in S \in \mathcal{P}^\infty(\mathbf{N_0})$ and $w \in \Omega_i$ we call $p^S(w)$ the alienated extension of w with respect to S.

Lemma 4: If $S \in \mathcal{P}^\infty(\mathbf{N_0})$ and there are at least two agents, all alienated extensions with respect to S share the same dense cell of Ω.

Proof: If $i \in S \in \mathcal{P}^\infty(\mathbf{N_0})$ and w and w' are members of Ω_i and both are contained in the same member of $\overline{\mathcal{F}}^j_i$, then $p^S(w)$ and $p^S(w')$ are both contained in the same member of $\mathcal{F}^j_\infty$. This follows directly from the definitions of p and ϕ.

Now, given any $i,k \in S$ with $B \in \mathcal{F}_i$ and $D \in \mathcal{F}_k$ consider the pair $p^S_{\max(i,k)}(\pi_i(B))$, $p^S_{\max(i,k)}(\pi_k(D)) \in \Omega_{\max(i,k)}$. The adjacency distance between $p^S(\pi_i(B))$ and $p^S(\pi_k(D))$ in Ω is no more than the adjacency distance between $p^S_{\max(i,k)}(\pi_i(B))$ and $p^S_{\max(i,k)}(\pi_k(D))$ in $\Omega_{\max(i,k)}$. q.e.d.

Define the formula $g_i := f(\phi^{\Omega_i}_{i+1}(\Omega_i))$ of depth $i+1$.

Lemma 5: If $i \geq 1$, $i \in S \in \mathcal{P}^\infty(\mathbf{N_0})$, and $i+1, i+2, \ldots, i+l+1 \notin S$, then $p^S(\Omega_i) \subseteq \alpha^\Omega(E^l(g_i))$.

Proof: Notice that g_i is valid in the Kripke structure Ω_i. $E^l(g_i)$, a formula of depth $i+l+1$, not true at some point of $\phi^{\Omega_i}_{i+l+1}(\Omega_i)$ would be a contradiction with the Stability Lemma and that g_i is common knowledge in $\phi^{\Omega_i}_\infty(\Omega_i)$. q.e.d.

The proof of the following lemma is omitted because it is a direct corollary of the axioms K1, K2, and K3 and Lemma 4.2 of [Fa-Ha-Va].

For fixed n and k, define a function $\xi^n_k : \mathbf{N}_0 \to \mathbf{N}$ in the following way:
$\xi^n_k(0) := 2^{2^k - 1}$ and
$\xi^n_k(i+1) := 2^{-1+(\xi^n_k(i))^{n-1}}$.

Lemma 6: Let k be the cardinality of X. The following is true for every $j \in \underline{n}$ and $0 \leq i < \infty$:
a) if $w \in \Omega_i$, $F \in \mathcal{F}^j_i$ and $w \in \pi_i(F) \in \overline{\mathcal{F}}^j_i$ then the number of members of $\mathcal{F}^j_{i+1}$ contained in F with a non-empty intersection with $\pi_i^{-1}(w)$ is at least

$\xi_k^n(i)$, and
b) if $w \in \Omega_i$ and $\pi_i^{-1}(w) \cap F \neq \emptyset$ for some $F \in \mathcal{F}_{i+1}^j$ then the number of elements of Ω_{i+1} that are in $\pi_{i+1}(\pi_i^{-1}(w)) \cap \pi_{i+1}(F)$ is at least $(\xi_k^n(i))^{n-1}$.

That $\xi_k^n(i)$ is at least two for all $i \geq 0$, $k \geq 1$ and $n \geq 2$, is all that we need until the 7th section.

Lemma 7: If $n \geq 2$ and i, $i+1$ and $i+2$ are in S, then Eg_i is not true at any point of $p^S(\Omega_i)$.

Proof: It suffices to show for any $j \in \underline{n}$ and $m \geq 0$ that $k_j(E^m g_i)$ is not true at $p_{i+m+2}(w)$ for any $w \in \Omega_{i+m+1}$. Let $w \in \pi_{i+m+1}(F) \in \overline{\mathcal{F}}_{i+m+1}^j$, for some $F \in \mathcal{F}_{i+m+1}^j$. Because for every $v \in \Omega_i$ the formula g_i prescribes a single member of Ω_{i+1} in $\pi_{i+1}(\pi_i^{-1}(v))$, including $v = \pi_i \circ \pi_{i+m+1}^{-1}(w)$, there is by induction exactly one $u \in \Omega_{i+m+1}$ with $E^m g_i \in u$ and $v = \pi_i \circ \pi_{i+m+1}^{-1}(u)$, (namely $\phi_{i+m+1}^{\Omega_i}(v)$; see the proof of Theorem 4.23 of [Fa-Ha-Va].) From now on let $v = \pi_i \circ \pi_{i+m+1}^{-1}(w)$. By Lemma 6 there is more than one $u \in \Omega_{i+m+1}$ with $u \in \pi_{i+m+1}(F) \cap \pi_{i+m+1}(\pi_i^{-1}(v))$. The $F' \in \mathcal{F}_{i+m+2}^j$ defining $p_{i+m+2}(w)$ must have a non-empty intersection with $\pi_{i+m+1}^{-1}(u)$ for all $u \in \Omega_{i+m+1}$ with $u \in \pi_{i+m+1}(F) \cap \pi_{i+m+1}(\pi_i^{-1}(v))$, and therefore F' is not contained in $\alpha^\Omega(E^m g_i)$. q.e.d.

Corollary 2 (proved again) If there are at least two agents then there is an uncountable number of cells of Ω dense in Ω.

Proof: Define a map $\beta : \mathcal{P}(\mathbf{N_0}) \to \mathcal{P}^\infty(\mathbf{N_0})$ by $\beta(S) := \{0, 1, 2, 4, 8, \cdots\} \cup \{2^i + 1, \cdots, 2^{i+1} - 1 \mid i \in S\}$.

Define an equivalence relation on $\mathcal{P}(\mathbf{N_0})$ by $S \sim T$ if and only if there exists an $m \in \mathbf{N_0}$ such that $S \backslash \{0, 1, 2, \cdots m\} = T \backslash \{0, 1, 2, \cdots, m\}$. The co-sets of this equivalence relation is an uncountable set.

Due to Lemma 4, it suffices to show that if S and T are both subsets of $\mathbf{N_0}$ with $S \not\sim T$ then $p^{\beta(S)}(w)$ does not share the same dense cell as $p^{\beta(T)}(w)$ for some $w \in \Omega_0$. For the sake of contradiction, let us suppose that the adjacency-distance in Ω between $p^{\beta(S)}(w)$ and $p^{\beta(T)}(w)$ equals a finite number $l < \infty$. Because $S \not\sim T$ there exists an $i > log_2((l+2))$ such that $i \in S$ and $i \notin T$, or vice versa. By symmetry, let us assume that $i \in S$ and $i \notin T$. By Lemma 5 applied to $p^{\beta(T)}(w)$ it follows that $p^{\beta(T)}(w) \in \alpha^\Omega(E^{l+1} g_{2^i})$. But because the adjacency-distance between $(p^{\beta(S)}(w)$ and $p^{\beta(T)}(w))$ is l we have that $p^{\beta(S)}(w) \in \alpha^\Omega(E(g_{2^i}))$, a contradiction to Lemma 7. q.e.d.

4 Maximal Common Knowledge

Now we are able to construct an example with three agents for which $S \in \mathcal{CK}$ is maximal and $F(S)$ is uncountable.

Example 1: For $\Omega = \Omega(X, \underline{2})$ we will define a consistent filter of partitions and use Theorem 3. Let $S = \{0\} \cup \{2^k \mid k = 0, 1, 2, \ldots\}$ For all $i \notin S$ define a subset $A_i := \{\phi_i^{\Omega_{2^k}}(v) \mid 2^k < i < 2^{k+1} \text{ and } v \in \Omega_{2^k}\} \subseteq \Omega_i$. We define the $\mathcal{P}_i$ in the following way:
$\mathcal{P}_0 = \{\Omega\}$,
if $i = 2^k$ for some $k \geq 0$ then $\mathcal{P}_i = \mathcal{P}_{i-1}$,
if $i \neq 2^k$ for any k then $\mathcal{P}_i = \mathcal{P}_{i-1} \wedge \{\pi_i^{-1}(A_i), \pi_i^{-1}(\Omega_i \backslash A_i)\}$.
Lemma 6 shows that this is a consistent filter of partitions. For all $2^k \leq i < 2^{k+1} - 1$, $v \in \Omega_{2^k}$ and $w = \phi_i^{\Omega_{2^k}}(v)$ there is an extension of w in A_{i+1} and many in $\Omega_{i+1} \backslash A_{i+1}$, which means that both $\pi_i^{-1}(A_{i+1})$ and $\pi_i^{-1}(\Omega_{i+1} \backslash A_{i+1})$ have a non-empty intersection with all the members $\mathcal{F}_i$ in $P_i(w)$, (where $w \in P_i(w) \in \mathcal{P}_i$.) Otherwise, if $w \neq \phi_i^{\Omega_{2^k}}(v)$ for every $v \in \Omega_{2^k}$ then the member of $\mathcal{P}_i$ containing $\pi_i^{-1}(w)$ will also be a member of $\mathcal{P}_{i+1}$.

First we show that every member of $\mathcal{F}_\infty^1 \vee \mathcal{F}_\infty^2 \vee \mathcal{P}_\infty$ is dense in Ω. By Theorem 3 that would imply that $F(\underline{Ck}(T(\mathcal{B})))$ is not empty and that $\underline{Ck}(T(\mathcal{B}))$ is maximal in $\mathcal{CK}$.

Consider any $z \in \Omega$ and define $A(z) := \{i \in \mathbf{N_0} \backslash S \mid \pi_i(z) \in A_i\}$. Define a new point $z' \in \Omega$ in the following way: Start with any $w_0 = \pi_0(z') \in \Omega_0$. We assume that $\pi_l(z')$ is defined for all $l < i$. If $i \in S$ then let $\pi_i(z')$ be $p_i(\pi_{i-1}(z'))$. If $i \in A(z)$ and $2^k < i < 2^{k+1}$ for some k then let $\pi_i(z')$ be $\phi_i^{\Omega_{2^k}}(\pi_{2^k}(z'))$. If $i \notin A(z)$ and $2^k + 1 < i < 2^{k+1}$ for some k then let $\pi_i(z')$ be $p_i(\pi_{i-1}(z'))$. If $i \notin A(z)$, $2^k + 1 = i$ for some k and F^1 and F^2 are the members of $\overline{\mathcal{F}}_{i-1}^1$ and $\overline{\mathcal{F}}_{i-1}^2$ containing $\pi_{i-1}(z')$, respectively, then let $M_{i-1}^1(\pi_i(z'))$ be F^1, let $M_{i-1}^2(\pi_i(z'))$ be any legitimate proper subset of F^2 containing $\pi_{i-1}(z')$, (by Lemma 6 at least one exists,) and define $\pi_i(z')$ according to $M_{i-1}^1(\pi_i(z'))$ and $M_{i-1}^2(\pi_i(z'))$. z' and z share the same member of $\mathcal{P}_\infty$; by induction on i we have that z' and $p^{\mathbf{N_0} \backslash A(z')}(w_0)$ share the same member of $\mathcal{F}_i^1$ for every $i < \infty$, and thus they also share the same member of $\mathcal{F}_\infty^1$. (z and $p^{\mathbf{N_0} \backslash A(z)}(w_0)$ do not in general share the same member of $\mathcal{P}_\infty$ because $p_i(w) = \phi_i^{\Omega_{i-1}}(w)$ for all $1 \leq i < \infty$ and $w \in \Omega_{i-1}$.) Since $\mathcal{F}_\infty^1 \vee \mathcal{F}_\infty^2 \vee \mathcal{P}_\infty$ is coarser than or equal to $\mathcal{F}_\infty^1 \vee \mathcal{F}_\infty^2$, by Lemma 4 $p^{\mathbf{N_0} \backslash A(z')}(w_0)$, and thus also z, belongs to a member of $\mathcal{F}_\infty^1 \vee \mathcal{F}_\infty^2 \vee \mathcal{P}_\infty$ dense in Ω.

Now we must show that all of Ω cannot belong to the same member of $\mathcal{F}^1_\infty \vee \mathcal{F}^2_\infty \vee \mathcal{P}_\infty$.

Lemma 8: With regard to Example 1, for every $w \in \Omega_{2^i}$ and $0 \leq l \leq 2^i - 2$, $f(\phi^{\Omega_{2^i}}_{2^i+l+1}(w)) \Rightarrow E^l(g_{2^i})$ is valid in the Kripke structure $\mu(\mathcal{B})$.

Proof: We proceed by induction on l. If $l = 0$, then the statement is true from the definition of g_{2^i}. Let us assume that the claim is true for $l - 1 \geq 0$. It suffices to show for all $j \in \underline{3}$ that $f(\phi^{\Omega_{2^i}}_{2^i+l+1}(w)) \Rightarrow k_j E^{l-1}(g_{2^i})$ is valid in the Kripke structure $\mu(\mathcal{B})$. By the Stability Lemma and the fact that $f(\phi^{\Omega_{2^i}}_{2^i+l}(\Omega_{2^i}))$ is common knowledge in $\phi^{\Omega_{2^i}}_\infty(\Omega_{2^i})$ if $j = 1, 2$ then $f(\phi^{\Omega_{2^i}}_{2^i+l+1}(w)) \Rightarrow k_j f(\phi^{\Omega_{2^i}}_{2^i+l}(w)) \in \mathcal{L}(X, \underline{2})$ is a tautology, so that we have the result by the induction hypothesis. For $j = 3$ it follows by the induction hypothesis and from the validity of the formula $h(\mathcal{P}_{2^i+l})$ in $\mu(\mathcal{B})$. q.e.d.

Now we proceed exactly as in the second proof of Corollary 2. We define the same map $\beta : \mathcal{P}(\mathbf{N_0}) \rightarrow \mathcal{P}^\infty(\mathbf{N_0})$ and the same equivalence relation on $\mathcal{P}(\mathbf{N_0})$. We suppose for the sake of contradiction that for some $w \in \Omega_0$ and for some subsets S and T of $\mathbf{N_0}$ with $S \not\sim T$ the adjacency distance between $p^{\beta(S)}(w)$ and $p^{\beta(T)}(w)$ in $\mu(\mathcal{B})$ is $l < \infty$. The only difference is that we use Lemma 8 instead of Lemma 5 to arrive at a contradiction with Lemma 7.

An example of a non-maximal but centered member S of $\mathcal{CK}$ is much easier to find. Any cell is both maximal and centered if and only if it is closed, so by Corollary 1 any infinite cell that is not closed but has at least one isolated point will suffice. A nice example is that of "coordinated attack," presented by Heifetz and Samet [He-Sa]. A more complicated example is presented below in the next section.

5 Increasing Common Knowledge

One could imagine that the correct analogue to Theorem 2 would be that increasing the set of formulas held in common knowledge can never result in a switch from centered to non-centered. But it is possible to have two members S and S' of $\mathcal{CK}$ of actual common knowledge such that $S \subseteq S'$ and S is centered but S' is not centered – such is the case with Example 2.

For any $J' \subseteq \underline{n}$ and $0 \leq k < \infty$ define $E^k_{J'}$ in the following way:

$$E_{J'}(f) := E^1_{J'}(f) := \wedge_{j \in J'} k_j f,$$

$E^0_{J'}(f) := f$, and for $k \geq 1$ $E^k_{J'}(f) := E_{J'}(E^{k-1}_{J'}(f))$.

Example 2: Let $X = \{x, y\}$ with $x \neq y$ and let $n = 2$. Let Ω stand for $\Omega(\{x,y\}, \underline{2})$. Define the following consistent filter $\mathcal{B}$ of partitions:
$\mathcal{P}_0 = \{\alpha^\Omega(\neg x)\} \cup \{\{\pi_0^{-1}(w)\} \mid x \in w \in \Omega_0\}$,
and for $i > 0$
$\mathcal{P}_i = \{\{\alpha^\Omega(\neg x)\} \cup \{\{\pi_i^{-1}(w)\} \mid x \in w \in \Omega_i\}$.
The limit partition is therefore $\mathcal{P}_\infty = \{\alpha^\Omega(\neg x)\} \cup \{\{z\} \mid x \in z \in \Omega\}$.

First, we claim that the set $\mathbf{Ck}(T(\mathcal{B})) \cap (\cup_{i=0}^\infty \alpha^{\Omega(X,\underline{3})}(\neg E^i_{\{1,2\}} x))$ is a dense cell of $\mathbf{Ck}(T(\mathcal{B}))$. If $z \in \Omega$ and $\neg E^i_{\{1,2\}} x \in z$ then there is some $j = 1$ or $j = 2$ and some $z' \in \alpha^\Omega(\neg E^{i-1}_{\{1,2\}} x)$ that shares the same member of $\mathcal{F}^j_\infty$, the partition of Ω, with z. By induction we have that z shares the same member of $\mathcal{F}^1_\infty \vee \mathcal{F}^2_\infty$ with some member of $\alpha^\Omega(\neg x)$. But all members of $\alpha(\neg x)$ share the same member of $\mathcal{P}_\infty$, so by Theorem 3 all members of $\mathbf{Ck}(T(\mathcal{B})) \cap (\cup_{i=0}^\infty \alpha^{\Omega(X,\underline{3})}(\neg E^i_{\{1,2\}} x))$ belong to the same cell of $\mathbf{Ck}(T(\mathcal{B}))$. Since the set $\cup_{i=0}^\infty \alpha^\Omega(\neg E^i_{\{1,2\}} x)$ is dense in Ω, we need to show that no point of $\phi_\infty^{\mu(\mathcal{B})}(\mathbf{Ck}(\{x\}))$ can belong to this same cell of $\mathbf{Ck}(T(\mathcal{B}))$. By the discretion of the partition $\mathcal{P}_\infty$ in $\alpha^{\mu(\mathcal{B})}(x)$, x is also held in common knowledge in $\phi_\infty^{\mu(\mathcal{B})}(\mathbf{Ck}(\{x\}))$, which completes the claim by Lemma 0.

Second, we claim that $\underline{Ck}(T(\mathcal{B}) \cup \{x\}) \in \mathcal{CK}$ is not centered and of actual common knowledge. For $\underline{Ck}(\{x\}) \subseteq \mathcal{L}(X, \underline{2})$, $F(\underline{Ck}(\{x\}))$ is not empty since every subset of Ω_0, including $\alpha^{\Omega_0}(x)$, is Kripke closed [Fa-Ha-Va]. $\underline{Ck}(\{x\}) \subseteq \mathcal{L}(X, \underline{2})$ is not centered by Theorem 2. The rest follows by Theorem 3.

6 Determinate

J. Halpern, (in private communication,) suggested the following definition of a property he called "determinate." Let $F \in \mathcal{F}^j_\infty$ be contained in a cell C. F is defined to be "determinate" if there exists some $i < \infty$ and an i-world $u \in \Omega_i$ such that $F \supseteq \pi_i^{-1}(u) \cap C$. A cell is called determinate if it contains a some determinate $F \in \mathcal{F}^j_\infty$ for some agent j. By Corollary 1, a determinate cell is centered and a cell C is determinate if and only if it contains a point z such that $R^1_\infty(z)$ is not meagre in C. But are all centered cells also determinate? The answer is no; and to show this, we use Theorem 3 again.

Example 3: Let n equal 2, with $\Omega = \Omega(X, \underline{2})$. We define another consistent filter of partitions in the following way:

For every $0 < i < \infty$ define $A_i = \{p_i(w) \mid w \in \Omega_{i-1}\}$. Define $\mathcal{P}_0 = \{\Omega\}$ and $\mathcal{P}_i = \mathcal{P}_{i-1} \wedge \{\pi_i^{-1}(A_i)\ ,\ \Omega \backslash \pi_i^{-1}(A_i)\}$.

For either $j = 1$ or $j = 2$ consider the member F^1 of $\mathcal{F}^1_\infty$, the partition of $\Omega(X, \underline{2})$, containing $p(w_0)$ for any $w_0 \in \Omega_0$. For any $i < \infty$ consider $F' \in \mathcal{F}_i^1$ satisfying $F^1 \subset F'$ and a $u \in \Omega_i$ with $u \in \pi_i(F') \in \overline{\mathcal{F}}_i^1$. Let F'' and F^* be the members of $\mathcal{F}^1_{i+1}$ and $\mathcal{F}^2_{i+1}$, respectively, defining $p_{i+1}(u)$. By Lemma 6 there will be a $v \in \Omega_{i+1}$ with $v \in \pi_{i+1}(\pi_i^{-1}(u))$, $v \in \pi_{i+1}(F'')$, but $v \notin \pi_{i+1}(F^*)$. By finite induction this means that for every subset $S \subseteq \mathbf{N}$ there is an nested sequence $C_i \in \mathcal{F}_i$, $i = 1, 2, \ldots$, such that $C_i \subseteq \pi_i^{-1}(\pi_i(F^1))$ but for every i $\pi_i(C_i) = p_i(\pi_{i-1}(C_i))$ if and only if $i \in S$. By the compactness of F^1 this implies that $z = \cap_{i=1}^\infty C_i \in F^1$ and $z \in \pi_i^{-1}(A_i)$ if and only if $i \in S$. This means that F^1 intersections every member of $\mathcal{P}_\infty$ and the adjacency distance with respect to the Kripke structure $\mu(\mathcal{B})$ between $p(w_0)$ and every element of $\Omega(X, \underline{2})$ is no more than 2. On the other hand, Lemma 6 implies that every member of $\mathcal{F}^j_\infty$ for $j = 1, 2$ is a meagre set of Ω. Likewise, no member of $\mathcal{P}_\infty$ can contain an open set of Ω. Theorem 3 implies that $\phi_\infty^{\mu(\mathcal{B})}(\Omega)$ is a centered cell that is not determinate.

Lemma 9: For any $S \in \mathcal{CK}$ with $F(S) \neq \emptyset$ if a cell of $F(S)$ is not determinate and for every $i < \infty$ and every world $w \in \Omega_i$ with $\pi_i^{-1}(w) \cap \mathbf{Ck}(S) \neq \emptyset$ there is at least one cell $C \subseteq \mathbf{Ck}(S)$ of countable (finite or infinite) cardinality with $\pi_i^{-1}(w) \cap C \neq \emptyset$, (necessarily the case if there is at least one countable cell in $F(S)$,) then S is not centered.

Proof: Let us assume, for the sake of contradiction, that S is centered, so that by Theorem 1 there is a cell C with $F(S) = \{C\}$, an $i < \infty$ and a world $w \in \Omega_i$ such that $C \supseteq \pi_i^{-1}(w) \cap \mathbf{Ck}(S) \neq \emptyset$. By the hypothesis of the lemma, we must assume that C has countably many members. Therefore C is a countable union of closed meagre sets of $\mathbf{Ck}(S)$, namely the sets $\{F \in \mathcal{F}^j_\infty \mid j \in \underline{n}, F \subseteq C\}$. By the Baire Category Theorem, C could not have contained $\pi_i^{-1}(w) \cap \mathbf{Ck}(S)$. q.e.d.

Question 1: For which of the questions Examples 1, 2, and 3 were designed to answer can one find alternative examples with only two agents?

7 Unique extendibility

Theorem 2 presents two different ways to have centeredness, first from the maximality of the set of formulas held in common knowledge and second from the cardinality of the cell. Sections 4 and 5 focused on the first property, let us focus now on the second.

From Lemma 9 one must conclude that Example 3 was of a centered cell with uncountable cardinality. In this section an example will be shown of a cell that is countable and dense in Ω (thus not centered by Corollary 2;) and in the next section an example will be shown of an uncentered $S \in \mathcal{CK}$ for which all members of $F(S)$ are countable. Remarkable about both these examples is that these cells have the additional property that all of their information sets are finite.

For a finite set X of propositional variables, a finite set $\underline{n}$ of agents and every ordinal number α there is an associated canonical Kripke structure $B_\alpha(X, \underline{n})$ for which $\Omega(X, \underline{n})$ is the canonical Kripke structure $B_\omega(X, \underline{n})$ associated with the first infinite ordinal ω. We write B_α if there is no ambiguity. B_α is defined inductively so that an element of B_α is defined by the knowledge of the agents concerning subsets of B_β for all β less than α. Additionally, for $\beta < \alpha$, every element of B_α is an extension of some element of B_β and every element of B_β is extendible to an element of B_α. Furthermore, for every ordinal α and every Kripke structure defined with the same sets X and $\underline{n}$ there is a canonical map of the Kripke structure to $B_\alpha(X, \underline{n})$. (For the above, see [Fa] and [He-Sa].)

The "fan-out" of an element $z \in \Omega$ is the set $R^1_\infty(z) = \cup_{j \in \underline{n},\ z \in F \in \mathcal{F}^j_\infty} F$. An important property is whether for all $\alpha > \omega$ there is a unique extension from a $z \in B_\omega = \Omega$ to an element of B_α, which is equivalent to the property that the fan-out of every member of the cell containing z is finite [Fa-Ge-Ha-Va]. We can add that in order for a connected Kripke structure μ to be determined by the set of formulas held in common knowledge in the image of ϕ^μ_∞ (modulo a non-essential duplication of elements) it must be isomorphic to a cell that is both centered and has the unique extension property. We say that a cell has finite fan-out if every member of the cell has finite fan-out.

Since all cells from alienated extensions are both dense in Ω and have some uncountably infinite information sets, an example of a cell dense in Ω with finite fan-out shows that the property of having a unique extension to higher transfinite levels is a property of cells not determined always by the set of formulas held in common knowledge.

For every $0 \le i < \infty$ define two elements w and w' in Ω_i to be adjacent if they have adjacency distance of one in Ω_i, or in other worlds if w and w' share the same member of $\overline{\mathcal{F}}_i^j$ for some $j \in \underline{n}$.

Example 4: Let $S \in \mathcal{P}^\infty(\mathbf{N_0})$ with $\inf S \ge 1$. For every $i \in S$ let $n_S := S \to S$ be defined as in Section 3. Define $n_S^1 := n_S$ and for $k \ge 2$ define $n_S^k := n_S \circ n_S^{k-1}$. For every $i \ge \inf S$ we define inductively two subsets A_i and B_i of Ω_i. We start with any element w_0 of $\Omega_{\inf S}$ and let $B_{\inf S} = \{w_0\}$ and $A_{\inf S} = \emptyset$. We assume that A_k and B_k have been defined for all $\inf S \le k < i$, and show how to define A_i and B_i. We define a extension function $\gamma_i : A_{i-1} \cup B_{i-1} \to A_i \cup B_i$ for all $i > \inf S$. It suffices to determine for every $1 \le j \le n$, the set $M_{i-1}^j(\gamma_i(w))$ for all $w \in A_{i-1} \cup B_{i-1}$. Let $F^j(w)$ be the member of $\overline{\mathcal{F}}_{i-1}^j$ containing $w \in A_{i-1} \cup B_{i-1}$. If $w \in A_{i-1}$ then $M_{i-1}^j(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^j(w)$. If $w \in B_{i-1}$ shares $F^j(w)$ with some member of A_{i-1}, then $M_{i-1}^j(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^j(w)$; otherwise if $A_{i-1} \cap F^j(w) = \emptyset$, then $M_{i-1}^j(\gamma_i(w)) := F^j(w)$. We see that $\gamma_{\inf S+1}(w_0) = p_{\inf S+1}(w_0)$. If $i \in S$ we define B_i to be the set $\{p_i(w) \mid w \in \Omega_{i-1} \backslash (A_{i-1} \cup B_{i-1}),\ w \in F^j(b)$ for some $b \in B_{i-1}$, $F^j(b) \cap A_{i-1} = \emptyset\}$. If $i \notin S$ we define B_i to be the set $\gamma_i(B_{i-1})$. If $i \in S$ we define A_i to be the set $\gamma_i(A_{i-1} \cup B_{i-1})$. If $i \notin S$ we define A_i to be the set $\gamma_i(A_{i-1})$. For any $i > \inf S$, $l \ge 0$, and $w \in A_{i-1} \cup B_{i-1}$ we define $\gamma_{i+l}(w) = \gamma_{i+l} \circ \ldots \circ \gamma_i(w)$ and we define $\gamma(w) := \cap_{k=i}^{\infty} \pi_k^{-1} \gamma_k(w)$. We define C to be $\{\gamma(w) \mid i \ge \inf S, w \in A_i\}$.

Claim : γ_i is well defined for every $i \ge \inf S$ and if $b \in B_i$ is adjacent in Ω_i to $a \in A_i$, sharing the same member of $\overline{\mathcal{F}}_i^j$, and k is the largest member of S less than or equal to i, then $a = \gamma_i(b')$ for some $b' \in B_{k-1}$ with $F^j(b') \cap A_{k-1} = \emptyset$.

Proof of Claim: $\gamma_{\inf S+1}(w_0) = p_{\inf S+1}(w_0)$ is well defined. We assume that γ_k is well defined for all $k < i$. If $i-1 \notin S$ and $i > \inf S$ then the well definition of γ_{i-1} shows that γ_i is also well defined.

For the following cases, let $i-1 \in S$, $w \in A_{i-1} \cup B_{i-1}$, and for any given $j \in \underline{n}$ let us assume that $v \in \Omega_{i-2}$ satisfies $\pi_{i-2}^{-1}(v) \cap \pi_{i-1}^{-1}(F^j(w)) \ne \emptyset$. We need to show that $\pi_{i-1}(\pi_{i-2}^{-1}(v)) \cap M_{i-1}^j(\gamma_i(w)) \ne \emptyset$.

Case 1; $w \in A_{i-1}$ and $v \in A_{i-2} \cup B_{i-2}$: $\gamma_{i-1}(v)$ is in $F^j(w)$, because v and $\pi_{i-2} \circ \pi_{i-1}^{-1}(w)$ share the same member of $\mathcal{F}_{i-2}^j$.

Case 2; $w \in A_{i-1}$ and $v \notin A_{i-2} \cup B_{i-2}$: This is possible only if $\pi_{i-2} \circ \pi_{i-1}^{-1}(w) \in B_{i-2}$ and $F^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(w)) \cap A_{i-2} = \emptyset$; since $v \in F^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(w))$

we have $p_{i-1}(v) =\in B_{i-1} \cap F^j(w)$.

Case 3; $w \in B_{i-1}$ **and** $F^j(w) \cap A_{i-1} \neq \emptyset$**:** Let $a \in F^j(w) \cap A_{i-1}$. By the second part of the induction hypothesis $\pi_{i-2} \circ \pi_{i-1}^{-1}(a) \in B_{i-2}$ with $F^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(a)) \cap A_{i-2} = \emptyset$; it follows that $v \in F^j(\pi_{i-2} \circ \pi_{i-1}^{-1}(a))$ and $p_{i-1}(v) \in B_{i-1} \cap F^j(w) = B_{i-1} \cap F^j(a)$.

Case 4; $w \in B_{i-1}$ **and** $F^j(w) \cap A_{i-1} = \emptyset$**:** Since $w \in p_{i-1}(\Omega_{i-2})$ we have that $p_{i-1}(v) \in F^j(w)$.

For the rest of the claim, suppose for the sake of contradiction that $b' := \pi_{k-1} \circ \pi_i^{-1}(a) \in A_{k-1}$. b' shares the same member of $\mathcal{F}_{k-1}^j$ with $c := \pi_{k-1} \circ \pi_i^{-1}(b) \in \Omega_{k-1} \backslash (A_{k-1} \cup B_{k-1})$. For every $j \in \underline{n}$ and $B \in \mathcal{F}_{k-2}$ if the member of $\mathcal{F}_k^j$ defining $\gamma_k(b') = \pi_k \circ \pi_i^{-1}(a)$ intersects B then it intersects B at only one member of $\mathcal{F}_{k-1}$; therefore by Lemma 6 it is different from the member of $\mathcal{F}_k^j$ defining $p_k(c) = \pi_k \circ \pi_i^{-1}(b)$, a contradiction. Furthermore, if $F^j(b') \cap A_{k-1} \neq \emptyset$ then likewise $\pi_k \circ \pi_i^{-1}(a)$ and $\pi_k \circ \pi_i^{-1}(b)$ would not share the same member of $\overline{\mathcal{F}}_k^j$. q.e.d.

The second part of the claim shows that if $i \in S$, $b \in B_{i-1}$ and $\gamma_i(b) = a \in A_i$ then for every $k \geq 1$ the only members of $A_{n_S^k(i)} \cup B_{n_S^k(i)}$ adjacent to $\gamma_{n_S^k(i)}(a)$ are already in the set $\gamma_{n_S^k(i)}(A_i \cup B_i)$. Therefore $C = \{\gamma(w) \mid i \geq \inf S, w \in A_i\}$ is a cell with finite fan-out.

Lemma 10: Given that the number of agents, n, is at least 2,

(a) the adjacency diameter of Ω_i is $2i + 1$,

(b) if w and w' in Ω_i are not adjacent to any member of $p_i(\Omega_{i-1})$ then there is an adjacency path $w = w_0, w_1, \ldots, w_l = w'$ between w and w' of length $l \leq$ adjacency diameter $(\Omega_{i-1})+2$ such that $w_m \notin p_i(\Omega_{i-1})$ for all $1 \leq m \leq l-1$.

Proof:

(a) First we show that the diameter of Ω_i is at least $2i+1$. The proof of (b) can be repeated without the conditions on adjacency with and containment in $p_i(\Omega_{i-1})$; this establishes that the diameter is exactly $2i + 1$.

Consider the element w in Ω_i containing the formula $E^i(x)$ for some $x \in X$ and the element w' containing $E^i(\neg x)$. Any element adjacent to w will contain $k_j E^{i-1} x$ for some $j \in \underline{n}$, and therefore also $E^{i-1} x$. By induction, if v is i adjacency steps from w we have that $k_j x \in v$ for some $j \in \underline{n}$ and therefore $x \in v$. The same is true for w', but with the containment of $\neg x$. Therefore no element of Ω_i can be within an adjacency distance of i of both

w and w'.

(b) We proceed by induction on i. If $i = 0$, then every element of Ω_0 is adjacent to every other. If $i > 0$, consider an adjacency path $v_1, v_2, \dots, v_l$ in Ω_{i-1} with $l - 1 \leq$ adjacency diameter (Ω_{i-1}) and $v_1 = \pi_{i-1} \circ \pi_i^{-1}(w)$ and $v_l = \pi_{i-1} \circ \pi_i^{-1}(w')$. We assume that v_k and v_{k+1} share the same member $F_k^{j_k}$ of $\mathcal{F}_{i-1}^{j_k}$ for all $1 \leq k \leq l-1$. Define an extension $\beta(v_k) \in \Omega_i$ for every k in the following way. If possible, let $M_{i-1}^{j_k}(\beta(v_k)) = M_{i-1}^{j_k}(\beta(v_{k+1}))$ be any proper subset of $F_k^{j_k}$ containing v_k and v_{k+1} and with a non-empty intersection with $\pi_{i-1}(D)$ for every $D \in \mathcal{F}_{i-2}$ with $\pi_{i-1}^{-1}(F_k^{j_k}) \cap D \neq \emptyset$. If this is not possible, then let $M_{i-1}^{j_k}(\beta(v_k)) = M_{i-1}^{j_k}(\beta(v_{k+1})) = F_k^{j_k}$. By Lemma 6 this is not possible only if $n = 2$, $|X| = 1$, and $\pi_{i-1}^{-1}(F_k^{j_k})$ has a non-empty intersection with only one member of $\mathcal{F}_{i-2}$ and contains in this intersection only two member of $\mathcal{F}_{i-1}$ – a special situation we will deal with later. Let $M_{i-1}^{j}(\beta(v_1)) = M_{i-1}^{j}(w)$ for at least one $j \neq j_1$ and $M_{i-1}^{j}(\beta(v_l)) = M_{i-1}^{j}(w')$ for at least one $j \neq j_{l-1}$. If $n \geq 3$ then for any $M_{i-1}^{j}(v_k)$ not yet defined let $M_{i-1}^{j}(v_k)$ by any set satisfying the appropriate conditions. $w, \beta(v_1), \dots, \beta(v_l), w'$ is a path connecting w and w', allowing possibly for the identity of w and $\beta(v_1)$ or of w' and $\beta(v_l)$.

Now we must consider the possibility that $\beta(w_k) = p_i(w_k)$. By the assumption that w and w' were not adjacent to any element in $p_i(\Omega_{i-1})$, we have that $k \neq 1$ and $k \neq l$. If $F \in \mathcal{F}_{i-1}^{j}$ and F has a non-empty intersection with only one member of $\mathcal{F}_{i-2}$ and contains in this intersection only two members of $\mathcal{F}_{i-1}$, then we must conclude the same (for the next lower level) for all members of $\mathcal{F}_{i-2}^{j'}$ intersecting F for all $j' \neq j$, since otherwise by Lemma 6 and Lemma 4.2 of [Fa-Ha-Va] one member of $\mathcal{F}_{i-2}^{j'}$ intersecting F not satisfying this special condition would have generated more than two members of $\mathcal{F}_{i-1}$ in F. By induction this is only possible if $\pi_{i-1}(F)$ contains one of the two members of Ω_{i-1} defined by the containment of either $E^{i-1}(x)$ or $E^{i-1}(\neg x)$ for the single element x in X. Since $k \neq 1$ and $k \neq l$ and the adjacency diameter of $\Omega_0(\{x\}, \underline{2})$ is one, we must assume that $i - 1 \geq 1$, and therefore these two special elements of Ω_{i-1} are at least an adjacency distance of 3 from each other. Therefore $\beta(w_k) = p_i(w_k)$ implies that w_k is one of these two special elements, say the one defined by the containment of $E^{i-1}(x)$. Assuming that w_{k-1}, w_k and w_{k+1} are mutually distinct we know exactly what are w_{k-1} and w_{k+1}. One is defined by the containment of $k_1 E^{i-2} x$ and $\neg k_2 E^{i-2} x$ and the other by the containment of $k_2 E^{i-2} x$ and $\neg k_1 E^{i-2} x$. We can create a new adjacency path that bypasses the element containing $E^{i-1} x$, replacing it with the element v defined by the containment of $E^{i-2}x$, $\neg k_1 E^{i-2}x$ and $\neg k_2 E^{i-2}x$; (there are only four elements of $\Omega_{i-1}(\{x\},\underline{2})$ containing $E^{i-2}x$.) Because the members of $\overline{\mathcal{F}}^1_{i-1}$ and $\overline{\mathcal{F}}^2_{i-1}$ containing this fourth element v have a non-empty intersection with $\alpha^{\Omega_{i-2}}(\neg E^{i-2}x)$, we can choose $\beta(v)$ as before so that it won't be in $p_i(\Omega_{i-1})$. q.e.d.

With respect to Example 4, Lemma 10 implies that the removal of B_i does not disconnect $\Omega_i \backslash B_i$. This follows from $B_i \subseteq p_i(\Omega_{i-1})$, that every element of $p_i(\Omega_{i-1})$ is adjacent to an element outside of $p_i(\Omega_{i-1})$, and that every element adjacent to $p_i(\Omega_{i-1})$ but not in $p_i(\Omega_{i-1})$ is also adjacent to an element not adjacent to $p_i(\Omega_{i-1})$.

Lemma 11: With respect to Example 4, if $i \in S$ and the shortest adjacency paths within $\Omega_i \backslash B_i$ between $w \in \Omega_i \backslash B_i$ and $\pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)}) \subseteq \Omega_i$ are of length $k \geq 1$, then there is an $1 \leq l \leq k$ with $p_{n_S^{l+1}(i)}(w) \in B_{n_S^{l+1}(i)}$.

Proof: We proceed by induction on k. If $k = 1$, let $c \in \pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)})$ be adjacent to w and let j be the member of $\underline{n}$ such that w and c share the same member of $\overline{\mathcal{F}}^j_i$. c could not have shared the same member of $\overline{\mathcal{F}}^j_i$ with a member of B_i, since otherwise w also sharing this same member of $\overline{\mathcal{F}}^j_i$ would imply that $p_{n_S(i)}(w) \in B_{n_S(i)}$, a contradiction to $k = 1$.

Assume the claim is true for $k - 1 \geq 1$. Let $v \in \Omega_i$ be the next element after c in one of the shortest adjacency paths within $\Omega_i \backslash B_i$ from $c \in \pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)})$ to w. Let v share with c a member of $\overline{\mathcal{F}}^j_i$. By the same argument as above we have that $p_{n_S^2(i)}(v) \in B_{n_S^2(i)}$ and also $p_{n_S(i)}(v) \notin B_{n_S(i)}$ and $p_{n_S(i)}(u) \notin B_{n_S(i)}$ for all $u \in \Omega_i$ in the adjacency path from v to w, including v and w, (otherwise this path would not have been one of the shortest.) Therefore we have an adjacency path of length $k-1$ within $\Omega_{n_S(i)} \backslash B_{n_S(i)}$ between $p_{n_S(i)}(v) \in \pi_{n_S(i)} \circ \pi^{-1}_{n_S^2(i)} B_{n_S^2(i)}$ and $p_{n_S(i)}(w)$. Whether or not it is one of the shortest adjacency paths of this kind we have our conclusion by the induction hypothesis. q.e.d.

Proposition 4: The cell $C = \{\gamma(a) \mid i \in S, a \in A_i\}$ of Example 4 is dense in Ω.

Proof: By Lemma 11 and the fact that B_i does not disconnect $\Omega_i \backslash B_i$, we need only show that $\pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)})$ is not empty for every $i \in S$; but also by Lemma 11, the non-emptiness of $\pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)})$ and the existence of some $w \in \Omega_i \backslash B_i$ that is of positive but finite adjacency distance from

$\pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)})$ implies the non-emptiness of $\pi_{n_S(i)} \circ \pi^{-1}_{n^2_S(i)}(B_{n^2_S(i)}) \subseteq \Omega_{n_S(i)}$. All elements of $B_i \cup (\pi_i \circ \pi^{-1}_{n_S(i)}(B_{n_S(i)}))$ are within an adjacency distance of $|S \cap \{1, 2, \ldots, i\}| \le i$ from $\gamma_i(w_0)$, yet the diameter of Ω_i is $2i+1$ by Lemma 10a. q.e.d.

For fixed n and k, define the functions $\chi^n_k : \mathbf{N}_0 \to \mathbf{N}$ and $\zeta^n_k : \mathbf{N}_0 \to \mathbf{N}$ in the following way:
$\chi^n_k(0) := 2^{2^k-1}$, $\zeta^n_k(0) := 2^k$,
$\zeta^n_k(i+1) := (\chi^n_k(i))^{n-1}\zeta^n_k(i)$,
$\chi^n_k(i+1) := (2^{\chi^n_k(i)^{n-1}-1})(2^{\chi^n_k(i)^{n-1}} - 1)^{\zeta^n_k(i)-1}$.

The proofs of parts a) through c) of the following lemma are omitted because they are direct corollaries of the axioms K1, K2, and K3 and Lemma 4.2 of [Fa-Ha-Va].

Lemma 12: Let k be the cardinality of X. The following is true for every $j \in \underline{n}$ and $0 \le i < \infty$:
a) if $w \in \Omega_i$, $F \in \mathcal{F}^j_i$ and $w \in \pi_i(F)$ then the number of members of $\mathcal{F}^j_{i+1}$ contained in F with a non-empty intersection with $\pi_i^{-1}(w)$ is no more than $\chi^n_k(i)$,
b) if $F \in \mathcal{F}^j_i$, then the cardinality of $\pi_i(F)$ is no more than $\zeta^n_k(i)$,
c) if $w \in \Omega_i$ and $\pi_i^{-1}(w) \cap F \neq \emptyset$ for some $F \in \mathcal{F}^j_{i+1}$ then the number of elements of Ω_{i+1} that are in $\pi_{i+1}(\pi_i^{-1}(w) \cap F)$ is no more than $(\chi^n_k(i))^{n-1}$,
d) for all $n \ge 2$ and $k \ge 1$ *except* for $n = 2$ and $k = 1$
$\zeta^n_k(i) < \xi^n_k(i)$ for all $i \ge 2$,
and if additionally $n = 3$ and $k = 1$ do not hold then also for all $i \ge 1$.

Proof of d): Replace ζ^n_k and χ^n_k by functions larger than both, respectively, namely $\tilde{\zeta}^n_k(0) := \zeta^n_k(0)$, $\tilde{\chi}^n_k(0) := \chi^n_k(0)$, $\tilde{\zeta}^n_k(i+1) := (\tilde{\chi}^n_k(i))^{n-1}\tilde{\zeta}^n_k(i)$, and $\tilde{\chi}^n_k(i+1) := 2^{\tilde{\chi}^n_k(i)^{n-1}\tilde{\zeta}^n_k(i)} = 2^{\tilde{\zeta}^n_k(i+1)}$.

We make a stronger claim: $\xi^n_k(i+1) \ge 4\tilde{\zeta}^n_k(i)(\tilde{\chi}^n_k(i))^{n-1}$ for all $i \ge 0$ (for all $i \ge 1$ if $n = 3$ and $k = 1$.)

We check for $n = 2$, $k \ge 2$ and $i = 0$. We have $\xi^2_k(1) = 2^{2^{2^k-1}-1}$ and $4\tilde{\zeta}^2_k(0)\tilde{\chi}^2_k(0) = 2^{2^k+k+1}$. It suffices that $2^{2^k-1} - 1 \ge 2^k + k + 1$. This is an equality for $k = 2$ and an inequality for $k \ge 3$; (but it is not true for $k = 1$.)

We check for $i = 0$ and $n \ge 3$ and $k \ge 1$, excluding the case of $n = 3$ and $k = 1$. We have $\xi^n_k(1) = 2^{(2^{2^k-1})^{n-1}-1}$ and $4\tilde{\zeta}^n_k(0)(\tilde{\chi}^n_k(0))^{n-1} = 2^{(n-1)(2^k-1)+k+2}$. It suffices that $2^{(n-1)(2^k-1)} \ge (n-1)2^k - n + k + 4$, (which is not true for $n = 3$ and $k = 1$.)

We check for $i = 1$, $n = 3$ and $k = 1$: $\xi_1^3(1) = \zeta_1^3(1) = \tilde{\zeta}_1^3(1) = 8$ and $\tilde{\chi}_1^3(1) = 2^8$. We have $\xi_1^3(2) = 2^{63}$ and $4\tilde{\zeta}_1^3(1)(\tilde{\chi}_1^3(1))^2 = 2^{21}$.

Notice that $(\tilde{\chi}_k^2(i))^2 \geq (\tilde{\zeta}_k^2(i))^3$ for all $i \geq 0$ and $k \geq 2$ with equality only if $k = 2$ and $i = 0$. Notice that both functions are big enough so that $(4\tilde{\zeta}_k^n(l)(\tilde{\chi}_k^n(l))^{n-1})^{n-1} > (7/3\tilde{\zeta}_k^n(l)(\tilde{\chi}_k^n(l))^{n-1})^{n-1} + 1$ for all $l \geq 0$. The smallest value for $\tilde{\chi}_k^2(i)$ with $k \geq 2$ is $\tilde{\chi}_2^2(0) = 8$ and $\tilde{\chi}_k^n(i)$ is always greater or equal to $\tilde{\zeta}_k^n(i)$, with equality only for $k = 1$ and $i = 0$. We use also that $a^b \geq ab$ for all $a, b \geq 2$.

We proceed by induction:

$$\xi_k^n(i+1) = 2^{(\xi_k^n(i))^{n-1}-1} \geq 2^{\left(4\tilde{\zeta}_k^n(i-1)(\tilde{\chi}_k^n(i-1))^{n-1}\right)^{n-1}-1} >$$

$$2^{\left((7/3)(\tilde{\zeta}_k^n(i-1)(\tilde{\chi}_k^n(i-1))^{n-1}\right)^{n-1}} \geq 2^{\left(\tilde{\zeta}_k^n(i-1)(\tilde{\chi}_k^n(i-1))^{n-1}\right)\left((n-1)(7/3^{n-1})\right)}$$

$$= \left(\tilde{\chi}_k^n(i)\right)^{\left((n-1)(7/3^{n-1})\right)}.$$

For $n = 2$ and $k \geq 2$ the above is at least $\geq 8^{2/3}(\tilde{\chi}_k^2(i))^{5/3} \geq 4\tilde{\chi}_k^2(i)\tilde{\zeta}_k^2(i)$. For $n \geq 3$ the above is at least $\geq (\tilde{\chi}_k^n(i)^{n-1})^{49/9} > 4(\tilde{\chi}_k^n(i)^{n-1})^{3/2} \geq 4\tilde{\chi}_k^n(i)^{n-1}\tilde{\zeta}_k^n(i)$. q.e.d.

Theorem 4: If $n \geq 2$ and $n = 2$ and $|X| = 1$ do not hold then there are uncountably many cells dense in $\Omega(X, \underline{n})$ with finite fan-out.

Proof: From Lemma 6, $(\xi_k^n(i))^{n-1}$ is a lower bound on the size of a member of $\mathcal{F}_{i+1}^j$. If $C = \{\gamma(a) \mid i \in S, a \in A_i\}$ is a cell generated from Example 4 with $\inf S \geq 2$, by Lemma 12 one can conclude from the size of the information sets in C which subset $S \in \mathcal{P}^\infty(\mathbf{N_0})$ was used to create C. Since there are uncountably many such members of $\mathcal{P}^\infty(\mathbf{N_0})$ there must be uncountably many such distinct cells C. q.e.d.

We suspect that there are also uncountably many cells dense in $\Omega(X, \underline{2})$ with finite fan-out even if $|X| = 1$.

8 Small Kripke structures

We define a Kripke structure $\mu := (S; \underline{n}; \mathcal{P}^1, \ldots, \mathcal{P}^n; X; \psi)$ to be "small" if the map $\phi_\infty^\mu : S \to \Omega(X, \underline{n})$ is injective. The following lemmas show that we can consider small Kripke structures as subsets of Ω with partitions finer than or equal to $\mathcal{F}^j_\infty, j \in \underline{n}$.

Lemma 13: Let $\mu = (S; \underline{n}; \mathcal{P}^1, \ldots, \mathcal{P}^n; X; \psi)$ be a Kripke structure. Let $j \in \underline{n}$ and $P \in \mathcal{P}^j$. $\phi^\mu_\infty(P)$ is a dense subset of F for some $F \in \mathcal{F}^j_\infty$.

Proof: By the definition of ϕ, $\phi^\mu_\infty(P)$ must be contained in a single member F of $\mathcal{F}^j_\infty$. Let us suppose for the sake of contradiction that $\phi^\mu_\infty(P)$ is not dense in F. Then there is an i and a world $w \in \Omega_i$ with $\pi_i^{-1}(w) \cap F \neq \emptyset$ and $\pi_i^{-1}(w) \cap \phi^\mu_\infty(P) = \emptyset$. The former implies that $\neg k_j \neg f(w) \in z$ for every $z \in F$ and the latter implies that $k_j \neg f(w) \in v$ for every $v \in \phi^\mu_{i+1}(P)$ and therefore $k_j \neg f(w) \in z$ for every $z \in \phi^\mu_\infty(P)$, a contradiction. q.e.d.

If $\mu := (S; \underline{n}; \mathcal{P}^1, \ldots, \mathcal{P}^n; X; \psi)$ is small define the partitions $\mathcal{P}^1_*, \ldots, \mathcal{P}^n_*$ of $\phi^\mu_\infty(S)$ by $\mathcal{P}^j_* = \{\phi^\mu_\infty(A) \mid A \in \mathcal{P}^j\}$; and for every $x \in X$ define $\psi_*(x) := \overline{\psi}(x) \cap \phi^\mu_\infty(S)$.

Lemma 14: If μ is small then $\mu_* := (\phi^\mu_\infty(S); \underline{n}; \mathcal{P}^1_*, \ldots, \mathcal{P}^n_*; X; \psi_*)$ is a Kripke structure isomorphic with fixed ground set and agents to μ, using the map ϕ^μ_∞. Conversely, any subset $S \subseteq \Omega(X, \underline{n})$ and partitions $\mathcal{P}^1, \ldots, \mathcal{P}^n$ of S with the property that for every j and every $A \in \mathcal{P}^j$ A is a dense subset of some member of $\mathcal{F}^j_\infty$ define a small Kripke structure $\mu = (S, \underline{n}; \mathcal{P}^1, \ldots \mathcal{P}^n; X; \overline{\psi}|_S)$ with $\overline{\psi}|_S(x) := \overline{\psi}(x) \cap S$ such that $\phi^\mu_\infty(s) = s$ for every $s \in S$.

Proof: By the definition of "small," the map $\phi^\mu_\infty : S \to \phi^\mu_\infty(S)$ is bijective, which implies the isomorphism of μ and μ_* from the definition of μ_*.

For the converse statement, it suffices to show for every $z \in S$ that $\phi^\mu_\infty(z) = \phi^\Omega_\infty(z)$, or equivalently that $\alpha^\mu(f) = \alpha^\Omega(f) \cap S$ for every $f \in \mathcal{L}$. We proceed by induction on the structure of formulas. From the definition of $\overline{\psi}|_S$ the claim is true for the formulas $x \in \mathcal{L}$ with $x \in X$. If the claim is true for $f \in \mathcal{L}$ and $g \in \mathcal{L}$ then the definition of $\alpha^\mu(f \wedge g)$ and $\alpha^\mu(\neg f)$ show that is it true for $f \wedge g$ and $\neg f$. We assume the claim is true for $f \in \mathcal{L}$ and consider $k_j f$. If $z \in \alpha^\Omega(k_j f) \cap S$ and F is the member of $\mathcal{F}^j_\infty$ containing z then F is contained in $\alpha^\Omega(f)$. Since $z \in A \in \mathcal{P}^j$ implies that $A \subseteq F$, we have by induction that $A \subseteq \alpha^\mu(f)$ and $z \in \alpha^\mu(k_j f)$. On the other hand, assume that $z \in \alpha^\mu(k_j f)$ and $z \in A \in \mathcal{P}^j$. Let F be the member of $\mathcal{F}^j_\infty$ containing A, and let us suppose, for the sake of contradiction, that F is not contained in $\alpha^\Omega(f)$, or that $F \cap (\Omega - \alpha^\Omega(f)) \neq \emptyset$. Since $\Omega - \alpha^\Omega(f)$ is an open set and we assume that A is dense in F we must conclude that A is not contained in

$\alpha^{\Omega}(f)$, which means by the induction hypothesis that A is not contained in $S \cap \alpha^{\Omega}(f) = \alpha^{\mu}(f)$, a contradiction. q.e.d.

Corollary 3: If D is a subset of a cell C of Ω and there are partitions $\mathcal{P}^1, \dots, \mathcal{P}^n$ of D such that for every $j \in \underline{n}$ $P \in \mathcal{P}^j$ implies that P is a dense subset of some member of $\mathcal{F}^j_\infty$, then D is dense in C.

This follows directly from Lemma 2b and Lemma 14.

A. Heifetz has proven that, as long as the number of agents is at least two, there are 2^c distinct small Kripke structures for which the map into Ω is also surjective, where c is the cardinality of the continuum. [He]

For any subset A of Ω, define a Kripke structure $\mathcal{V}(A) := (A; \underline{n}; \mathcal{F}^1_\infty|_A, \dots, \mathcal{F}^n_\infty|_A; X; \overline{\psi}|_A)$ by $\mathcal{F}^j_\infty|_A = \{F \cap A \mid F \cap A \neq \emptyset,\ F \in \mathcal{F}^j_\infty\}$ and $\overline{\psi}|_A(x) = \overline{\psi}(x) \cap A$ for all $x \in X$.

From Lemma 14, if there exists a small Kripke structure $\mu := (S; \underline{n}; \mathcal{P}^1, \dots, \mathcal{P}^n; X; \overline{\psi}|_S)$ with $S \subseteq \Omega(X, \underline{n})$ and $\phi^{\mu}_\infty(s) = s$ for all $s \in S$ then $\mathcal{V}(S)$ is also a small Kripke structure with $\phi^{\mathcal{V}(S)}_\infty(s) = s$ for every $s \in S$.

Lemma 15:
(a) For every cell C there is a countable subset A of C such that $\phi^{\mathcal{V}(A)}(z) = z$ for every $z \in A$,
(b) if C is a countable cell and $\mu := (S; \underline{n}; \mathcal{P}^1, \dots, \mathcal{P}^n; X; \overline{\psi}|_S)$ is a small Kripke structure with S contained in C and $\phi^{\mu}_\infty(s) = s$ for all $s \in S$, then $\mu = \mathcal{V}(S)$,
(c) additionally, if C from (b) is centered then $\mathcal{V}(S)$ is connected.

Proof:

(a) If C is countable, equivalent to every point of C having a countable (finite or infinite) fan-out, there is nothing to prove. Otherwise, we must assume that there is some $j \in \underline{n}$ and $F \in \mathcal{F}^j_\infty$ with F an uncountable subset of C. We choose any countable dense subset $F' \subseteq F$ and let $A_1 := F'$. We define inductively a sequence of countable sets $A_1, A_2, \dots$ in the following way. For every $k > 1$, every $j \in \underline{n}$, every $z \in A_{k-1}$, and F^* the member of $\mathcal{F}^j_\infty$ containing z, let $A^j_k(z)$ be a countable subset of F^* such that $A^j_k(z) \cup (F^* \cap A_{k-1})$ is a dense subset of F^*. We define A_k to be $A_{k-1} \cup_{j \in \underline{n}, z \in A_{k-1}} A^j_k(z)$. The countable set $A = \cup_{k=1}^{\infty} A_k$ satisfies the necessary conditions.

(b) By the Baire Category Theorem every compact and countable set $F \in \mathcal{F}^j_\infty$ must contain an isolated point and therefore there cannot be two disjoint dense subsets of F. Therefore for every $j \in \underline{n}$ and every $P \in \mathcal{P}^j$ the set P is the intersection of S with the member of $\mathcal{F}^j_\infty$ containing P.

(c) By Lemma 9 there must be an $F \in \mathcal{F}^j_\infty$ contained in C with $W \cap C \subseteq F$ for some open set W of Ω. Any connected component of $\mathcal{V}(S)$ must be dense in C by Corollary 3, and therefore it has a non-empty intersection with F – by the proof of (b) there can be only one connected component. q.e.d.

The countability of C is necessary for the conclusion of Lemma 15c.

Proposition 5: With regard to Example 3, define $A := \{p^S(w) \mid S \in \mathcal{P}^\infty(\mathbf{N_0}), i \in S, w \in \Omega_i\} \subseteq \Omega(X, \underline{2})$ and define $B := \phi^{\mu(\mathcal{B})}_\infty(A) \subseteq \mathbf{Ck}(T(\mathcal{B})) \subseteq \Omega(X, \underline{3})$, (where $\mathcal{B}$ is defined as in Example 3.) $\phi^{\mathcal{V}(B)}_\infty(s) = s$ for every $s \in B$ and $\mathcal{V}(B)$ has uncountably many connected components, (although B is a subset of the centered cell.)

Proof: Let $z = p^S(w) \in \Omega(X, \underline{2})$ for some $i \in S \in \mathcal{P}^\infty(\mathbf{N_0})$ and $w \in \Omega_i$. Let $j \in \underline{2}$, $z \in F \in \mathcal{F}^j_\infty$, and $F \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k(X, \underline{n})$ with $k \in S$ and $k \geq i$. Since v shares the same member of $\mathcal{F}^j_k$ with $\pi_k(z)$ we have that $p^S(v) \in F$. Otherwise let $z \in P \in \mathcal{P}_\infty$ and let $P \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k(X, \underline{n})$ with $k \in S$ and $k \geq i$. Likewise $p^S(v) \in P$, since v shares the same member of $\mathcal{P}_k$ with $\pi_k(z)$ and for every $l > k$ $\pi_l \circ p^S(v)$ and $\pi_l(z)$ are no-information extensions of an $l-1$ world if and only if $l \in S$ or $l-1 \in S$. $\phi^{\mathcal{V}(B)}_\infty(s) = s$ for every $s \in B$ follows by Theorem 3 and Lemma 14.

Fix $w_0 \in \Omega_0(X, \underline{2})$. Next we assume that the adjacency distance between $p^S(w_0)$ and $p^T(w_0)$ *within the set* A as a subset of the Kripke structure $\mu(\mathcal{B})$ is $l < \infty$ for some pair $S, T \in \mathcal{P}^\infty(\mathbf{N_0})$ both containing $\{0\}$. Let $p^S(w_0) = z_0, z_1, \ldots, z_l = p^T(w_0)$ be a sequence of members of A such that for every $0 \leq k \leq l-1$ z_k and z_{k+1} share the same member of $\mathcal{F}^1_\infty$, $\mathcal{F}^2_\infty$, or $\mathcal{P}_\infty$, and for every $0 \leq k \leq l$ $z_k = p^{S_k}(v_k)$ for some $S_k \in \mathcal{P}^\infty(\mathbf{N_0})$, $v_k \in \Omega_{n_k}(X, \underline{2})$ and $n_k \in S_k$ (with $S_0 = S$, $S_l = T$, and $v_0 = v_l = w_0$.) Without loss of generality we can assume for all $0 \leq k \leq l$ that there is no $U \subseteq \mathbf{N_0}$ such that S_k is a proper subset of U and $p^U(v_k) = p^{S_k}(v_k)$. For every k let $n'_k := n_{S_k}(n_k)$ be the next member of S_k greater than n_k. Let $N = \max_{0 \leq k \leq l}(n'_k)$. If z_k and z_{k+1} share the same member of $\mathcal{P}_\infty$ then by the definition of $\mathcal{P}_i$ for Example 3 we have that $S_k \backslash \{0, 1, \ldots, N\} = S_{k+1} \backslash \{0, 1, \ldots, N\}$.

Now assume that z_k and z_{k+1} share the same member of $\mathcal{F}^1_\infty$, (respectively $\mathcal{F}^2_\infty$.) Likewise, we will conclude that $S_k \backslash \{0, 1, \ldots, N\} = S_{k+1} \backslash \{0, 1, \ldots, N\}$. This shows that $S \backslash \{0, 1, \ldots, N\} = T \backslash \{0, 1, \ldots, N\}$. The second proof of Corollary 2 shows that there is an uncountable subset $\mathcal{U}$ of $\mathcal{P}^\infty(\mathbf{N_0})$ such that any two member of $\mathcal{U}$ are not similar and for every $U \in \mathcal{U}$ there is no $U' \in \mathcal{P}^\infty(\mathbf{N_0})$ with $U \subseteq U'$ and $p^U(w_0) = p^{U'}(w_0)$ other than $U' = U$. With Theorem 3 this is sufficient to show that $\mathcal{V}(B)$ has uncountably many connected components.

By the maximality assumption on S_k and the fact that $\phi^{\Omega_i}_{i+1}(w) = p_{i+1}(w)$ for every $i \geq 0$ and $w \in \Omega_i$ we can assume for all $i \geq 0$ that $i \in S_k$ and $i+2 \in S_k$ imply that $i+1 \in S_k$ (and likewise for S_{k+1}.) By symmetry we can assume that $n'_k \geq n'_{k+1}$.

First, if $i \geq \max(n_k, n_{k+1})$ it is not possible for i to be in both $\mathbf{N_0}\backslash S_k$ and $\mathbf{N_0}\backslash S_{k+1}$ without the largest member of S_k smaller than i, call it i', being equal to the largest member of S_{k+1} smaller than i, call it i^*. By symmetry we suppose for the sake of contradiction that $i' < i^*$. Since $p_{i^*+1}(w) = \phi^{\Omega_{i^*}}_{i^*+1}(w)$ for every $w \in \Omega_{i^*}$, $\neg k_1 \neg f(w)$ would be in both $p^{S_{k+1}}(v_{k+1})$ and $p^{S_k}(v_k)$ for every $w \in \Omega_{i^*}$ sharing the same member of $\mathcal{F}^1_{i^*}$ with $\pi_{i^*} \circ p^{S_{k+1}}(v_{k+1})$ and $\pi_{i^*} \circ p^{S_k}(v_k)$. By Lemma 6 this is impossible, since $\neg k_1 \neg f(w) \in p^{S_k}(v_k)$ for only one $w \in \Omega_{i^*}$ that is an extension of $\pi_{i'} \circ p^{S_k}(v_k)$.

Second, if $i \geq \max(n_k, n_{k+1})$ it is not possible for i and $i+1$ to be in $\mathbf{N_0}\backslash S_k$ (respectively in $\mathbf{N_0}\backslash S_{k+1}$) without $i+1$ being in $\mathbf{N_0}\backslash S_{k+1}$ (respectively in $\mathbf{N_0}\backslash S_k$.) This follows directly from the proof of Lemma 7.

If $n'_k \notin S_{k+1}$ then by the maximality assumption on S_{k+1} we have that either $n'_k - 1 \notin S_{k+1}$ or $n'_k + 1 \notin S_{k+1}$. In either case the last two paragraphs and the maximality assumption on both S_k and S_{k+1} generate contradictions. Once $n'_k \in S_k \cap S_{k+1}$ is established, the maximality condition on S_k and S_{k+1} and the last two paragraphs imply by induction that $S_k \backslash \{0, 1, \ldots, n'_k - 1\} = S_{k+1} \backslash \{0, 1, \ldots, n'_k - 1\}$. q.e.d.

Theorem 5: If a cell C is centered and contains an infinite information set, then there exists a proper subset $A \subset C$ with $\phi^{\mathcal{V}(A)}_\infty(z) = z$ for all $z \in A$.

Proof: By Lemma 15a and Lemma 9 it suffices to prove the theorem for the case that C is countable and determinate.

Let $F_0 \in \mathcal{F}^j$ be any infinite information set contained in C. We choose any cluster point z_0 of F_0, and define the set $E_0 := \{z_0\}$. Next, for every $k > 0$ and $j \in \underline{n}$ we define the set E^j_k to be the union of all $F \in \mathcal{F}^j_\infty$ such that $F \subseteq C$ and there is an isolated point of F contained in E_{k-1}. Define $E_k := E_{k-1} \cup_{j \in \underline{n}} E^j_k$ and $A := C \backslash \cup_{k=0}^\infty E_k$. ($F_0$ could be contained in $\cup_{k=2}^\infty E_k$.) If $F \in \mathcal{F}^j_\infty$ and $F \cap A \neq \emptyset$, then for every $k \geq 0$ no point of $F \cap E_k$ is an isolated point of F. By the Baire Category Theorem we must conclude that $F \cap A$ is dense in F, and therefore as long as A is non-empty it satisfies the necessary properties for $\phi_\infty^{\mathcal{V}(A)}(z) = z$ for all $z \in A$.

Claim: If $j \neq j'$, $F \in \mathcal{F}_\infty^j$ and $F' \in \mathcal{F}_\infty^{j'}$ are two information sets of Ω with a non-empty intersection, S is the set of formulas held in common knowledge in the cell containing F and F', F is determinate, and $z \in F \cap F'$ is an isolated point of F, then F' is also determinate and there is some formula $h \in \mathcal{L}$ with $\{z\} = \alpha^\Omega(h) \cap \mathbf{Ck}(S)$.

The proof of the claim is as follows: Let $f \in \mathcal{L}$ be a formula such that $\mathbf{Ck}(S) \cap \alpha^\Omega(f) \subseteq F$, and let $g \in \mathcal{L}$ be a formula such that $\{z\} = \alpha^\Omega(g) \cap F$. Consider the formula $h := g \wedge (\neg k_j \neg f)$. Since $F = \alpha^\Omega(\neg k_j \neg f) \cap \mathbf{Ck}(S)$, z is the only point of $\mathbf{Ck}(S)$ where h is true. The containment of z in F' implies that F' is also determinate, which settles the claim.

To conclude the proof of the theorem, by Lemma 9 we need show only that no determinate information set of Ω is contained in $\cup_{k=1}^\infty E_k$. By the claim and induction this would imply that there exists a formula f_0 such that z_0 is the only point of $\mathbf{Ck}(S)$ where f_0 is true, a contradiction to z_0 being a cluster point of F_0. q.e.d.

Question 2: Does there exist a countable non-centered cell C and an $A \subsetneq C$ with $\phi_\infty^{\mathcal{V}(A)}(a) = a$ for every $a \in A$ such that $\mathcal{V}(A)$ has infinitely many connected components?

Question 3: Does Theorem 5 hold if one drops the assumption that C is centered?

Question 4: Is it possible for $\phi_\infty^{\mathcal{V}(A)}(A)$ to be contained and dense in A, C, or the closure of C for some subset A of a cell C without $\phi_\infty^{\mathcal{V}(A)}$ being the identity map on A?

Question 5: For every centered but not determinate cell C does there exist a small Kripke structure μ such that μ has uncountably many connected components and the image of ϕ_∞^μ is in C?

9 Bounded fan-out

Can one have an uncentered $S \in \mathcal{CK}$ such that each member of $F(S)$ has finite fan-out? The answer is yes, and it is also possible to have a bound on the fan-out for all members of $F(S)$.

For all $i < \infty$ define the subset of "k-bounded" i-worlds $\overline{A}_i^k \subseteq \Omega_i$ induc-

tively in the following way:
$\overline{A}_0^k = \Omega_0$,
for every $i > 0$ $w \in \overline{A}_i^k$ if and only if for every $j \in \underline{n}$ and $\pi_i^{-1} w \in F \in \mathcal{F}_i^j$ it follows that $\pi_{i-1} \circ \pi_i^{-1}(F)$ is a subset of $\overline{A}_{i-1}^k$ and $|\pi_{i-1} \circ \pi_i^{-1}(F)| \leq k$. Likewise define a member F of $\mathcal{F}_i^j$ to be k-bounded if and only if $\pi_{i-1} \circ \pi_i^{-1}(F)$ is a subset of $\overline{A}_{i-1}^k$ and $|\pi_{i-1} \circ \pi_i^{-1}(F)| \leq k$. Define $T_k \subseteq \mathcal{L}$ to be the set of formulas $\{f(\overline{A}_i^k) \mid i < \infty\}$.

Lemma 16: $\mathbf{Ck}(T_k)$ is the union of all cells C with $|F| \leq k$ for all information sets F contained in C.

Proof: Since $\wedge_{i=0}^{\infty} \mathcal{F}_i$ is the discrete partition of Ω, if $z \in F \in \mathcal{F}_\infty^j$ and $|F| > k$ then $|\pi_{i-1}(F)| > k$ for some $i < \infty$. Let $F^* \in \mathcal{F}_i^j$ contain z. Since $\pi_{i-1} \circ \pi_i^{-1}(F^*) = \pi_{i-1}(F)$, $f(\overline{A}_i^k)$ won't be true at z, and therefore T_k cannot be common knowledge in the cell containing z.

On the other hand, if T_k is not common knowledge in a cell C, then by Lemma 0 there is some $z \in C$ and some $i < \infty$ such that $f(\overline{A}_i^k)$ is not true at z. By the Stability Lemma this implies that the i-world $w \in \Omega_i$ satisfying $w = \pi_i(z)$ is not a k-bounded world. By induction there is an $l \leq i$, a $v \in \Omega_l$ and an $F \in \mathcal{F}_l^j$ such that $v \in \pi_l(F) \in \overline{\mathcal{F}}_l^j$, $\pi_l^{-1}(v) \cap C \neq \emptyset$ and $|\pi_{l-1}(F)| > k$. By the Consistency Property, for any $z' \in \pi_l^{-1}(v)$ if $z' \in F^* \in \mathcal{F}_\infty^j$ then $|F^*| > k$. q.e.d.

We consider the special case of only two agents. We will show that there are uncountably many cells that are dense in $\mathbf{Ck}(T_2) \subseteq \Omega(X, \underline{2})$.

Define a "chain" of level i to be a sequence $\gamma = (w_k \in \Omega_i \mid m_1 < k < m_2)$ of two-bounded i-worlds in $\overline{A}_i^2$ such that m_1 is an integer or $-\infty$, m_2 is an integer or $+\infty$, and
1) for every $m_1 < k < m_2 - 1$ with k even the worlds w_k and w_{k+1} share the same member F of $\overline{\mathcal{F}}_i^1$ and if $|\pi_{i-1} \circ \pi_i^{-1}(F)| = 2$ then $\{\pi_{i-1} \circ \pi_i^{-1}(w_k), \{\pi_{i-1} \circ \pi_i^{-1}(w_{k+1})\} = \pi_{i-1} \circ \pi_i^{-1}(F)$,
2) for every $m_1 < k < m_2 - 1$ with k odd the worlds w_k and w_{k+1} share the same member F of $\overline{\mathcal{F}}_i^2$ and if $|\pi_{i-1} \circ \pi_i^{-1}(F)| = 2$ then $\{\pi_{i-1} \circ \pi_i^{-1}(w_k), \{\pi_{i-1} \circ \pi_i^{-1}(w_{k+1})\} = \pi_{i-1} \circ \pi_i^{-1}(F)$,
3) If m_1 is finite and $m_1 + 1$ is even (respectively odd) and F is the member of $\overline{\mathcal{F}}_i^2$ (respectively $\overline{\mathcal{F}}_i^1$) containing w_{m_1+1} then $|\pi_{i-1} \circ \pi_i^{-1}(F)| = 1$,
4) the symmetrical statement as 3) but with the condition that m_2 is finite.

Notice that for k even (respectively k odd) it is allowed for w_k to equal w_{k+1} as long as the F in $\mathcal{F}_i^1$ (respectively $\mathcal{F}_i^2$) containing w_k satisfies $\pi_{i-1} \circ \pi_i^{-1}(F) = \{\pi_{i-1} \circ \pi_i^{-1}(w_k)\}$.

For a chain γ define the Kripke structure $\mathcal{S}(\gamma)$ to be $(\{k \in \mathbf{Z} \mid m_1 < k < m_2\}; \underline{2}; \mathcal{P}^1, \mathcal{P}^2; X, \psi^*)$ where $\mathcal{P}^1 = \{\{k, k+1\} \mid m_1 < k < k+1 < m_2,\ k$ is even $\} \cup \{\{m_1+1\} \mid m_1+1$ is odd and finite $\} \cup \{\{m_2-1\} \mid m_2-1$ is even and finite$\}$, $\mathcal{P}^2$ is defined symmetrically, and $\psi^*(x) := \{k \mid x \in w_k\}$ for all $x \in X$.

Lemma 17: For every chain γ of level i $\quad \phi_i^{\mathcal{S}(\gamma)}(k) = w_k$ for all $m_1 < k < m_2$.

The proof of Lemma 17 is almost identical to that of Lemma 2a.

Define a partial chain of level i to be a sequence $(w_k \in \Omega_i \mid m_1 < k < m_2)$ of two-bounded i-worlds such that m_1 and m_2 are finite and conditions 1 and 2 are satisfied but we don't assume that conditions 3 and 4 hold. The length of a partial chain is defined to be $m_2 - m_1 - 2$.

Lemma 18: Every partial chain can be extended to a chain.

If Conditions 3 and 4 are satisfied, then the partial chain is also a chain. If not, by symmetry we can assume Condition 3 is not satisfied for an m_1+1 that is odd. Let F be the member of $\overline{\mathcal{F}}_i^1$ containing w_{m_1+1} with $|\pi_{i-1} \circ \pi_i^{-1}(F)| = 2$. Let $v = \pi_{i-1} \circ \pi_i^{-1}(w_{m_{1}+1}) \in \pi_{i-1} \circ \pi_i^{-1}(F)$ and let $u(\neq v)$ be the other member of $\pi_{i-1} \circ \pi_i^{-1}(F)$. Choose for w_{m_1} any member of Ω_i such that $w_{m_1} \in F$ and $\pi_{i-1} \circ \pi_i^{-1}(w_{m_1}) = u$. Continue this extension process as needed.

Lemma 19:

(a) If $i \geq 1$, $F \in \mathcal{F}_i^j$ is two-bounded and $v \in \pi_{i-1}(F)$ then there are $2^{|X|}$ two-bounded members of Ω_i in $\pi_i(\pi_{i-1}^{-1}(v)) \cap \pi_i(F)$.

(b) For every agent j and every two-bounded $w \in \overline{F} \in \overline{\mathcal{F}}_i^j$ there are $2^{|X|}$ two-bounded members of $\mathcal{F}_{i+1}^j$ contained in $\pi_i^{-1}(\overline{F})$ and with non-empty intersection with $\pi_i^{-1}(w)$.

(c) For every two-bounded world $w \in \Omega_i$ there are $2^{|X|+1}$ two-bounded worlds in Ω_{i+1} that are contained in $\pi_{i+1}(\pi_i^{-1}(w))$, one for each pair of possible two-bounded members of $\mathcal{F}_{i+1}^1$ and $\mathcal{F}_{i+1}^2$ from (b).

Proof: We proceed to prove (a) and (b) together by induction on i. For $i = 0$ we need to prove only (b) and (c). For every $w \in \Omega_0$ there are exactly $2^{|X|} - 1$ two-subsets of Ω_0 containing w and one one-subset of Ω_0 containing

w, namely $\{w\}$. The number of possibilities for both players is $2^{|X|+1}$.

Now assume that all three claims are true for $i-1 \geq 0$.

(a) Let F' be the two-bounded member of $\mathcal{F}_{i-1}^{j'}$ containing $\pi_{i-1}^{-1}(v)$ for $j' \neq j$. By the induction hypothesis and (b) there are $2^{|X|}$ different two-bounded members of $\mathcal{F}_i^{j'}$ contained in F' and intersecting $\pi_i^{-1}(v)$. By Lemma 4.2 of (Fa-Ha-Va] each one combined with F defines a two-bounded world of Ω_i contained in $\pi_i(F)$ and extending v.

(b) Case 1; $\pi_{i-1} \circ \pi_i^{-1}(\overline{F}) = 2$: Let $F = \pi_i^{-1}(\overline{F})$. Let v be the member of $\pi_{i-1}(F)$ such that $\pi_{i-1} \circ \pi_i^{-1}(w) \neq v$. By (a) there are $2^{|X|}$ different two-bounded i-worlds contained in $\pi_i(\pi_{i-1}^{-1}(v))$ that are also members of $\overline{F}$. This means that there are $2^{|X|}$ different two-subsets of $\overline{F}$ such that one member is in $\pi_i(\pi_{i-1}^{-1}(v))$ and the other is w.

(b) Case 2; $\pi_{i-1} \circ \pi_i^{-1}(\overline{F}) = \{u\} \in \Omega_{i-1}$: By (a) there are $2^{|X|}$ two-bounded members of Ω_i in $\overline{F}$. There are $2^{|X|}-1$ two-subsets of $\overline{F}$ containing $w \in \Omega_i$ and one one-subset containing w, namely $\{w\}$.

(c) The result follows directly from (b). q.e.d.

Lemma 20: For every pair $w, w' \in \overline{A}_i^2$ of two-bounded worlds and either agent $j \in \underline{2}$ there is a partial chain of level i of length $2i+1 = m_2 - m_1 - 2$ from w and w' such that the information sets $\{w = w_{m_1+1}, w_{m_1+2}\}$ and $\{w_{m_2-2}, w' = w_{m_2-1}\}$ belong to agent j.

Proof: We proceed by induction on i. If w and w' both belong to Ω_0 then (w, w') is a chain.

Without loss of generality, assume that $j = 2$. Define $i-1$ worlds v and v' in the following way. Let F and F' be the two-bounded members of $\mathcal{F}_i^1$ that contain $\pi_i^{-1}(w)$ and $\pi_i^{-1}(w')$, respectively. If $|\pi_{i-1}(F)| = 2$ then let v be the member of $\pi_{i-1}(F)$ other than $\pi_{i-1} \circ \pi_i^{-1}(w)$, and otherwise let v equal $\pi_{i-1}\pi_i^{-1}(w)$. Define v' symmetrically. By the induction hypothesis there is a partial chain $(v = v_k, v_{k+1}, \ldots, v_{k+2i-1} = v')$ of level $i-1$ with k even. Define $v_{k-1} := \pi_{i-1} \circ \pi_i^{-1}(w)$ and $v_{k+2i} := \pi_{i-1} \circ \pi_i^{-1}(w')$. If $k \leq m \leq k+2i-1$ and m is even then define $w_m \in \Omega_i$ by $M_{i-1}^1(w_m) = \{v_m, v_{m+1}\}$ and $M_{i-1}^2(w_m) = \{v_{m-1}, v_m\}$. If $k \leq m \leq k+2i-1$ and m is odd then define $w_m \in \Omega_i$ by $M_{i-1}^1(w_m) = \{v_{m-1}, v_m\}$ and $M_{i-1}^2(w_m) = \{v_m, v_{m+1}\}$. The sequence $(w, w_k, \ldots, w_{k+2i}, w')$ is a partial chain connecting w with w'. q.e.d.

Theorem 6: $F(T_2)$ is an uncountable set.

Proof: If we show that $\mathbf{Ck}(T_2 \cup \{g\})$ is meagre in $\mathbf{Ck}(T_2)$ for any $g \notin \underline{Ck}(T_2)$ it follows by the Baire Category Theorem that $F(\underline{Ck}(T_2))$ is not empty. Lemma 19 shows that no information set contained in $\mathbf{Ck}(T_2)$ is determinate, and therefore from the non-emptiness of $F(\underline{Ck}(T_2))$ and Lemma 9 it would follow that $\underline{Ck}(T_2)$ is not centered.

We suppose for the sake of contradiction that $\mathbf{Ck}((T_2) \cup \{g\})$ is not meagre in $\mathbf{Ck}(T_2)$ for some $g \notin \underline{Ck}(T_2)$ and let d be the depth of g. Therefore there would be a two-bounded world $w \in \Omega_l$ such that $\mathbf{Ck}(T_2) \cap \pi_l^{-1}(w) \neq \emptyset$ and for all $z \in \mathbf{Ck}(T_2) \cap \pi_l^{-1}(w)$ the formula g is common knowledge at z. By $g \notin \underline{Ck}(T_2)$ there would be a point $z \in \mathbf{Ck}(T_2)$ with $g \notin z$, and therefore by the Stability Lemma there would be a two-bounded world $w' \in \Omega_d$ where g is not true at any point of $\pi_d^{-1}(w')$. Without loss of generality we can assume that $w, w' \in \Omega_i$ for $i = \max(l, d)$. By Lemma 20 we have a partial chain of level i starting at w and ending at w', and by Lemma 18 the partial chain can be extended to a chain γ of level i. Since chains are connected Kripke structures, consider the cell containing the image of $\phi_\infty^{S(\gamma)}$, which is also contained in $\mathbf{Ck}(T_2)$ by Lemma 16. By Lemma 17 this cell has a non-empty intersection with both $\pi_i^{-1}(w)$ and $\pi_i^{-1}(w')$, a contradiction to Lemma 0. q.e.d.

A special kind of chain is a chain of two-bounded i-worlds with $m_1 = \infty$, $m_2 = \infty$, and a fixed natural number m such that $w_k = w_{k+m}$ for all integers k. We will call such a chain a "loop," and we will represent such a loop as $[w_k, \ldots, w_{k+m-1}]$.

Question 6: Does there exist for every finite set X and every level $i < \infty$ a loop $[w_0, \ldots, w_{m-1}]$ of level i such that $m = |\Omega_i|$ and $\Omega_i = \{w_0, \ldots, w_{m-1}\}$?

10 Acknowledgements

The inquiry into the number of cells that are dense in Ω was inspired by communication with Robert Aumann and Dov Samet, and later with Aviad Heifetz. Joseph Halpern informed me of related literature and helped me to explore some of the problems that inspired Part II.

11 References

[Au1] Aumann, R. (1976), "Agreeing to Disagree," Annals of Statistics 4, pp. 1236-1239.

[Au2] Aumann, R. (1989), "Notes on Interactive Epistemology," Cowles Foundation for Research in Economics, working paper.

[Fa-Va] Fagin, R., Vardi, M.Y. (1985), Proceedings of the 17th Annual A. C. M. Symposium on Theory of Computing, Providence, R. I., May 6-8, 1985, pp. 305-315.

[Fa-Ha-Va] Fagin, R., Halpern, Y.J. and Vardi, M.Y. (1991), "A Model-Theoretic Analysis of Knowledge," Journal of the A.C.M. 91 (2), pp. 382-428.

[He-Sa] Heifetz, A., Samet, D. (1993), "Universal Partition Spaces," IIBR working paper 26, Tel Aviv University.

[Fa-Ge-Ha-Va] Fagin, R., Halpern, Y.J., and Vardi, M.Y. (1996), "The Expressive Power of The Hierarchical Approach to Modeling Knowledge and Common Knowledge," in preparation.

[He] Heifetz, A. (1995), "How Canonical is the Canonical Model? A Comment on Aumann's Interactive Epistemology," CORE Discussion Paper 9528, Universite Catholique de Louvain.

[Hu-Cr] Hughes, G.E., Cresswell, M.J. (1968), An Introduction to Modal Logic, Routledge.

[Le] Lewis, D. (1969), Convention: A Philosophical Study, Harvard University Press.

[Li-Mo] Lismont, L. and Mongin, P. (1993), "Belief Closure: A Semantics of Common Knowledge for Modal Propositional Logic," Mathematical Social Science, to appear.

[Si] Simon, R. (1995), "Alienated Extensions and Common Knowledge Worlds," Working Paper 242, (April 1995), Institute of Mathematical Economics, University of Bielefeld, Germany.

[Si2] Simon, R. (1996), "The Difference Between Common Knowledge of Formulas and Sets: Part I," Working Paper 252, (March 1996), Institute of Mathematical Economics, University of Bielefeld, Germany.