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Foundations of a Theory of Prominence in the Decimal System

Part I:

Numerical Response as a Process, Exactness, Scales, and Structure of Scales

by

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Abstract

Starting point of the theory of prominence is the observation that the selection of a numerical response is performed by a process of stepwise refinement of a reasonable answer until the available information does not permit a further specification. The procedure starts with a sufficiently high number, and stepwise decides whether to add, subtract or not use the next finer of the set of prominent numbers for the presentation, where the prominent numbers are $\{a * 10^i : a \in \{1, 2, 5\}, i \text{ integer}\}$. The result is the presentation of a number as sum of prominent numbers with coefficients +1, -1, or 0, where every prominent number is 'used' at most once. For instance $17 = 20 - 5 + 2$, or $24 = 20 + 5 - 1$. This presentation is not necessarily unique. Important is the smallest prominent number used by a presentation. It is denoted as the exactness of the presentation. The exactness of a number is the smallest exactness among all presentationce of the number. It informs about the crudest level of exactness on which the number can be perceived, i. e. constructed by a response process. - Central tools for the analysis of numerical responses are two types of scales. $S(r, a)$ -scales are based on the observation that subjects adjust relative exactness r , and absolute exactness a to a given type problem or situation. $M(i, a)$ -scales are constructed by starting with the prominent numbers, and stepwise inserting the respective 'most prominent number' as 'midpoint' between any two neighboured numbers of the preceding scale. Accordingly one obtains scales on the full step, half step, ... level. $M(i, a)$ -scales permit to define a perception function by assuming that the distances of any two neighboured numbers of an $M(i, a)$ -scale are equal, i. e. by applying the usual interpolation principle. Several lemmata concerning the structure of scales are given.

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0 Motivation

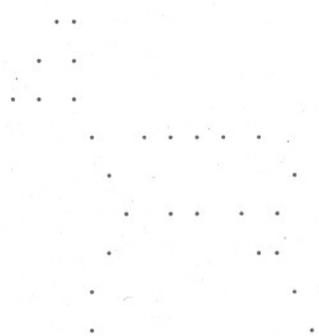
The term 'prominence' has been first introduced by SCHELLING (1960). He noticed that, as part of decision processing, people have the ability to select one (or a few) alternatives from a given set by their 'prominence'. This selection seems to follow certain unwritten rules of minimal entropy, where the entropy is given by the logical and social information of the alternative. The corresponding pattern can be easily demonstrated by coordination problems, and questions for spontaneous numerical responses:

1. Two partisans meet in an area of which both have a map. The map shows a river with a bridge, a forest and several houses. Both persons know the time when to meet, but not the place, and this is common knowledge. Which point will they select? - Most subjects decide to meet at the bridge, which is a 'focal point' of the map.
2. Two subjects are asked to give a number greater than 0, and less than 100. If they select identical numbers, both get a fixed payoff, otherwise they get nothing. - Most subjects answer 50.

- Subjects are asked "how many inhabitants has Cairo". - The response is one number, and this number is typically a multiple of 1 or 1/2 Million, for instance "6 Millions".

The selection pattern seems to be related to 'graneness of judgement' which can be observed in many situations of individual and group decision making. Graneness of judgement can be observed in numerical and nonnumerical decisions. Graneness is not a phenomenon which can be just casually observed, but rather seems to be a general phenomenon caused by the general structure of decision processing. An example of graneness in nonnumerical decisions is

- Show a subject a very imprecise picture of an animal as shown to the right. Ask: "what an animal do you think is this". The answer will frequently be only one animal, usually not more than three alternatives, at most 5 alternatives of animals, where the animals are classified in a sufficiently crude way to permit a reduction of reasonable alternatives to a reasonable number.



Examples for the graneness in numerical decisions are

- Retail prices of clothes between DM 500 and 800 (Karstadt, Germany)

observed prices	549	598	649	698	749
corresponding price levels	550	600	650	700	750

(the corresponding price levels have exactness 50)

- Proposals in an experimental 5-person game (Apex Game)

Observed proposals in the 2-person coalitions are (60,40), (65,35), (70,30), (75,25), (80,20), the answers have exactness 5.

- Spontaneous answers to a questions as "probability of an armed intervention of the US in the Irak within the next year"

response	0	1	2	3	5	7	(8)	10	15	20	(25)	30	50	(60)	70	all
frequency	1	1	2	3	9	1	1	8	7	10	2	7	3	1	4	60

(the answers have relative exactness > .25)

The paper presents a theory that permits to predict numerical selections of persons who are educated in the decimal system. The approach can be modified to nonnumerical responses, and to other numerical systems (for instance the dual system), but this is not done in this paper. We suggest that the described phenomena are natural consequences of some structural feature of the hardware of human brain (related to the restriction of the short term memory to five storage places), and that thereby general principles of

decision creating processes (as the complexity of analysis in urgent decision situations) could be controlled by evolution. Following this idea, the creation of numerical responses is only a special application of a more general procedure which creates the selection of an alternative within extended decision processing.

It is well known, that human decision processing is to a large extent casuistic. The selection of adequate objects (cases, situations) that fit to a given decision situation is crucially important for the fitness of the decision maker. Accordingly, evolution did not only develop a high capacity of (long term) memory, but also the ability of intelligent classification on different levels of generalization, to create and re-identify new objects on different levels of generalization, and to select an adequate level of generalization for every subproblem of a decision process.

Items (i. e. objects or situations) stored in the long term memory are ordered by inclusion (generalization/specification) and nature developed the ability to find for any given actual item a 'most similar' candidate of the memory. The process by which this similar item is found starts with a quite general item that fits to the given item, then stepwise increases the degree of specificity and selects the most adequate item among the more specified ones that are next to the respective selected candidate. The process stops when the available information about the given item does not permit to decide which of the more specified objects to select. This process does not only select a specific item but also a problem-adjusted level of specificity, the 'graneness of judgement' or 'exactness of analysis'.

1 Prominence in the Decimal System

1.1 Principles of the Selection Process for Numerical Responses

To apply the outlined general procedure (of finding an item in the memory that fits a given observed item best) to the problem of numerical response, it was necessary to give numerical responses a similar structure of stepwise increasing precision as they are created in other spaces of alternatives by inclusion or specification.

As an example consider the classification of animals (by persons). The chain 'animal' - 'mammal' - 'ape' - 'anthropoid ape' - 'orang-outang' is a chain of stepwise refinement, where in each step one specific answer is selected from a small set of (usually not more than 5, in exceptional cases up to 7) alternatives. This chain is followed until the available information does not permit a further specification.

A similar structure is generated for decimal responses by the sequence of digits, as 5 - 5.3 - 5.34 ... However, this trivial structure does not accord with the decision process creating numerical responses. Persons do not create decimal numerical responses digit per digit (where the number of alternatives in each step is 10), but by procedures where the fineness of the response is about halved in each step (so that in each step the numerical response is improved by selecting 1 out of 2, and not 1 out of 10 alternatives). Following

such a process until no further specification is possible, permits in the average essentially finer responses than the digit per digit procedure. (Such a process can be compared with the stepwise selection of the respective next digit of a response in the dual system.)

Besides this general idea concerning the fineness of a response system, there are three points that have to be clarified before a model of a process that creates numeric responses for given stimuli can be made, namely

- (1) A person cannot judge whether the distance of a response to a stimulus is high or low.
- (2) But she can judge which of two responses x, y is nearer to the stimulus than the other.
- (3) This judgement permits the four responses 'alternative x ', 'alternative y ', 'there are equally strong arguments for x and y ', and 'cannot say (because my limits of judgement abilities are reached)'. (4) The judgement is knife-edged, i. e. the judgement 'equally strong arguments' nearly never occurs.

Empirical results indicate that we cannot (or do not) decide whether a response hits a given signal 'sufficiently well'. It seems that our brain can only decide, whether a certain response r is better than another response s , or not. Our brain can only compare the quality of responses. Accordingly, the decision, whether a certain response hits the signal 'sufficiently well', has to be created by pairwise comparisons with other responses, or by a procedure that creates a 'winner' in iterated pairwise comparisons. – Another aspect, that has to be clarified, is the question, how precise these pairwise comparisons are. Is there an indifference area which decreases during the procedure generating the response, or are the decisions equally precise, even if very crude responses are compared. (For instance, if the true signal is 51, and a subject is asked whether the signal is nearer to 0 or 100, will she make a very precise judgement when crude alternatives are asked, or will she answer 'near the middle', so that further investigations near the middle are necessary.) Our investigations support the impression that judgements are always as precise as possible and decide between alternatives in a 'knife-edged' way. Smallest differences are noticed, and there is nearly no space for the answer 'cannot say' unless the limits of discrimination are reached. Moreover, in the few cases where the judgement 'there are equally strong arguments for both alternatives' is made, the subject is aware of the quality of the judgement, and does not mix it up with 'cannot say, since the limit of judgement ability is reached'.

Applying these decision elements repeatedly the following model to identify the position of a signal can be modeled (it is the well known procedure used for instance to find the zero-point of a function):

Signal Identification Process

- start: select a sufficiently large number y , set $x = 0$
- step: decide whether the signal is nearer to x or y
 - if nearer to x then replace y by $mid(x, y)$, repeat step
 - if nearer to y then replace x by $mid(x, y)$, repeat step
- end: if x, y are equally near then respond $mid(x, y)$
otherwise respond x , or y

Notice that the process stops, when the limit of judgement ability of a subject is reached, i. e. when the subject is not any more able to decide whether the signal is nearer to x or y . Moreover, it may be remarked that for diffuse numerical signals the notation 'the signal is nearer to x than to y ' must be interpreted as 'the best number to characterize the signal is nearer to x than to y '.

Given the information that the signal is between 0 and y , and that all values in this interval have the same probability, then the given procedure is the shortest way to obtain the result, in the sense that there is no procedure with less steps leading on the average to a more precise result in a given number of steps. (Notice that the shortness of the procedure is closely related to the structure of iterated midpoints.)

It is not unreasonable that nature selected such a kind of process for numerical responses. Of course we cannot expect that the procedure as a tool of boundedly rational behavior uses the computation of midpoints for arbitrary numbers, but is only able to respond midpoints for certain numbers that are easily accessed in the memory. – For simplicity assume that the exactness of analysis is restricted such that a distance between signal and response below 1 cannot be perceived. Then all calculations are 'simple', if the initial number y is selected as an integer power of 2. In this case the process can be reformulated such that in every step the decision maker decides whether to add, subtract, or not to use the respective next lower term 2^i . (The process is similar to an interpolation process.) – The respective state of this modified process can be described two parameters x, p : where x is the respective present answer, $|p|$ is the present level of exactness (power of 2), $\text{signum}(p)$ gives the direction, where the signal is (expected) compared to the respective present answer. Accordingly we obtain a process model for cultures, where the dual numbers (= integer powers of 2) are most easily accessed:

Numerical Response Process (for the dual system)
 start: select a sufficiently large power of 2 ($p = 2^i, i$ integer),
 set $x = 0, y = p$,
 step: decide whether x or y is nearer to the signal
 if x is nearer then $p = +p/2, x = x, y = x + p$, repeat step
 if y is nearer then $p = -p/2, x = y, y = x + p$, repeat step
 end: if x, y are equally near then respond $x + p/2$
 otherwise respond x , or y

The process decides for every term of the decreasing sequence 2^i (i integer), with which of the signs +1, -1, or 0 it shall be added to the respective present result. The process ends, when a further specification of the response is not possible.

1.2 Prominent Numbers and the Response Process for the Decimal System

Our culture decided for the decimal instead of the dual system, probably related to the fact that we have ten fingers that we can use for simple calculations. In this system the powers

of ten are the most prominent alternatives which have highest priority to be selected as responses, or terms by which given responses should be modified. Iterated application of halving is not compatible with this structure, since iterated halving creates sequences as ..., 100, 50, 25, 12.5, - Accordingly, the sequence of powers of 2 of the preceding model has to be replaced by a sequence of numbers with the following properties: 1. the sequence contains the powers of ten, 2. the relation between two subceding numbers should be about 1/2, the numbers should be as 'simple' as possible, they should in their decimal presentation not have more than two digits different from zero. The solution with minimal deviation from the quotients 1/2 is the sequence of type ..., 100, 50, 20, 10, 5, 2, 1, This is the set of prominent numbers which is the basic tool of numeric perception. Accordingly we obtain the following operator to transform the preceding rule to the application in the decimal system:

- replace the sequence of powers of 2 by the sequence of prominent numbers
- replace the operation ' $p/2$ ' (of the process) by 'select next lower of the sequence of prominent numbers'

where

Definition: The prominent numbers are $\{a * 10^i : a \in \{1, 2, 5\}, i \text{ integer}\}$.

This gives the

Numerical Response Process (NRP) (for the decimal system)^a

start: select a sufficiently large prominent number p , set $x = 0$
 step: decide whether the signal is nearer to x or to $x + p$
 if nearer to x then $p = +p/2$, repeat step
 if nearer to $x + p$ then $x = x + p$, $p = -p/2$, repeat step
 end: if $x, x + p$ equally strong then respond $x + p/2$
 otherwise respond x , or $x + p$

^a $p/2$ denotes *signum*(p) times 'greatest prominent number below $|p|$ '

The violation of the principle of iterated halving by the system of prominent numbers has it's price: this process cannot find numbers which are in an open interval $(X + 2 * 10^i, X + 5 * 10^i - 2 * 10^i)$, i integer, where the presentation of X only contains prominent numbers $q > 5 * 10^i$. For example, 22 is not found since the process selects the numbers 0, ..., 0, 20, 20, ..., and continues searching below 20. This problem is related to the fact that 20 is less than half of 50. To obtain every real number, the process has to be modified. One way is, to insert an additional switch when p has the shape $20 * 10^i$:

insert as first lines of step:
 if $p = 2 * 10^i$ (i integer) then decide if signal is above or below x
 if (above and $p < 0$) or (below and $p > 0$) then $p = -p/2$

The modification assumes - in addition to the discriminatory abilities required above - that subjects can decide whether the signal is above or below a given signal (but the decision can as well be obtained by asking whether the signal is nearer to $x + p$ or to $x - p$).

Again the loop of the obtained process is iterated until the exactness reaches the boundary of the judgement ability of the subject. It can also be used to determine a unique decimal presentation for a given number by performing the procedure uninfinitely often, or until x hits the signal (in the latter case the response has to be x , not x or y).

That the response is not specified, but permits x and y as answers is related to the observation that - in doubt which of two responses to select - some subjects prefer to select the cruder number (here x), others prefer to give an answer that shows the level of exactness they reached during the analysis (here y).

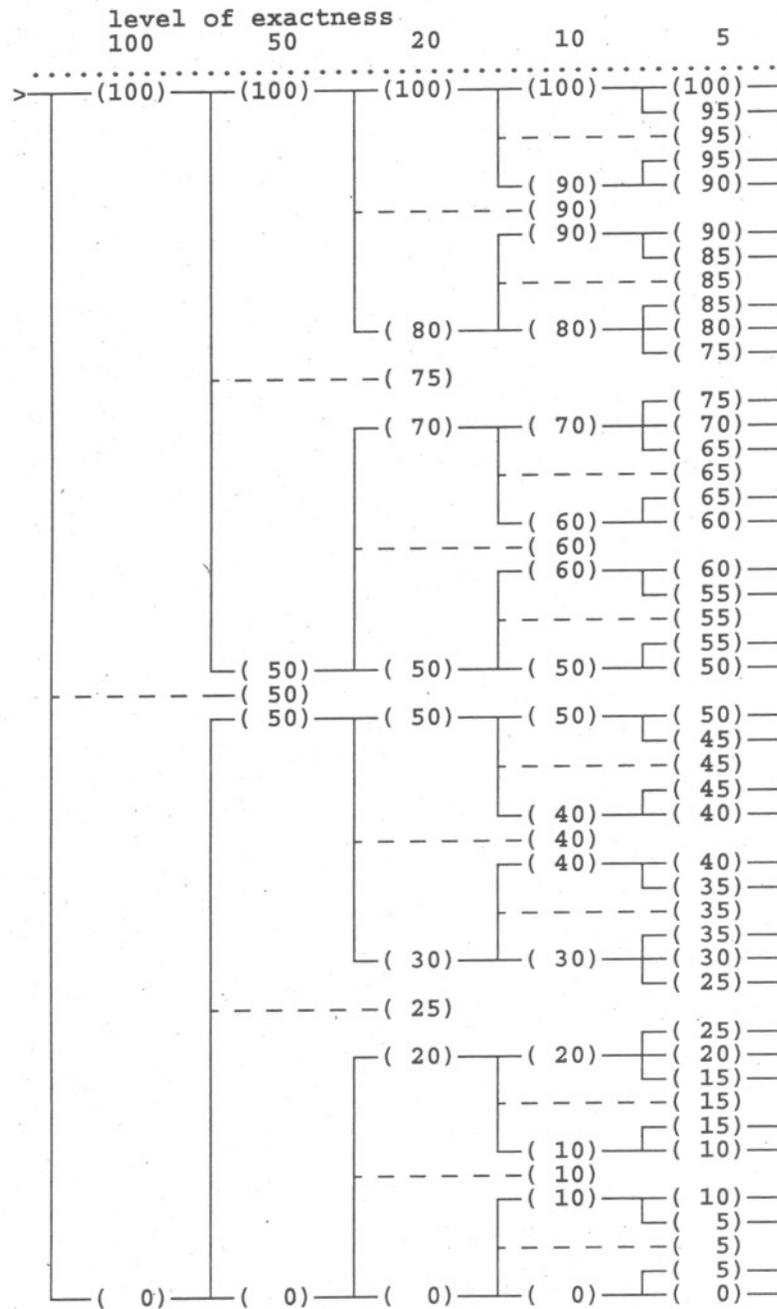
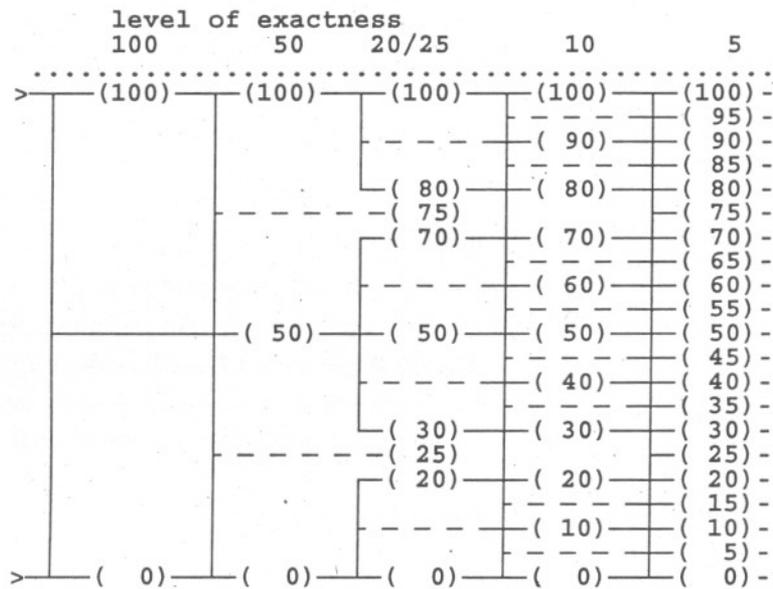


Figure 1.2.1: Possible Paths of the Numerical Response Process

It is clear that the process (only) makes sense, when

- the signal is 'single peaked', and
- the signal is precise enough that (in all cases with response unequal zero) the decision whether the signal is nearer to $p/2$ or p can be made for every prominent number p .

A simple way to present presentations is obtained by omitting terms with coefficients 0, as $17 = 20 - 5 + 2$, $28 = 50 - 20 - 2$, $13 = 20 - 10 + 5 - 2$, etc.



Rules:

- after a point one of the following next points is selected:
1. straight, 2. one up, and 3. one down
- stop when "cannot decide", whether to go upward, downward or straight
- decide for compromise (dashed line), when there are equally strong arguments to go upward and straight (/ to go downward and straight)
- stop after having reached a point via a dashed line

Figure 1.2.2: A Path-Model Permitting Errors in the Search Process

1.3 Presentations

The term presentation can be generalized as follows:

Definition: A decimal presentation is a mapping $a : P \rightarrow \{+1, -1, 0\}$ which assigns to every prominent number one of the coefficients $+1$, -1 , or 0 .

Presentations need not be unique, for instance $17 = 20 - 5 + 2 = 10 + 5 + 2 = 20 - 2 - 1$. And it is in no way important for the theory to have a unique presentation. Nevertheless, we give a condition that selects a unique presentation:

Definition: Let a, b presentations of the same number x . For every prominent number p let $d(a, x, p) := \text{sum}(a(q) * q : q \leq p)$. Select p^* as the maximal prominent number such that $d(a, x, p^*) \neq d(b, x, p^*)$. a is a **lexicographically better** presentation of x , if $d(a, x, p^*) < d(b, x, p^*)$.

Lemma: For all presentations $a, b (a \neq b)$, all real numbers x , either a is a lexicographically better presentation of x than b , or b is a lexicographically better presentation of x than a .

Corollary: For every real number x there is a unique lexicographically best presentation.

Lemma: For every real number x , the modified NRP described above selects the lexicographically best presentation.

We remark that the Signal Identification Process and the Dual Numerical Response Process are theoretical constructs used to introduce the decimal numerical response process in an understandable, structured way. The Signal Identification Process is well known as a quick simple procedure to identify a signal, and, in our opinion, it is very reasonable that persons who are thinking in the dual system will create their responses as the Dual Numerical Response process describes.

A question is, under which conditions the unmodified NRP creates 'correct' answers. The criterion is

Lemma: If the relative exactness of a number is cruder equal 10% then it is presented in the same way by the modified and the unmodified NRP.

(Proof: Without loss of generality we consider the interval (20,30). 25 is found by the process. All other numbers have a relative exactness (= smallest prominent number of the presentation of x with nonzero coefficient divided by x) below 10%.)

This result suggests that subjects have a somewhat different behavior to numbers with relative exactness below 10%. In fact, empirical data indicate that the relative exactness of 10% is a natural upper bound for precise spontaneous responses.

Finally may be remarked that all judgements used in the NRP are implicitly using a linear scale. For example, the question 'is the signal nearer to 100 or zero' discriminates between the range above and below 50, and accordingly the next question either addresses 0 and 50, or 50 and 100. This is surprising, since the general obtained structure with steps at 1, 2, 5, 10, 20, 50, 100, ... has logarithmic properties.

1.4 Prominence of Presentations

Consider a subject in a numeric decision situation. For example, assume you asked the subject for the number of inhabitants of Cairo. (In our model) she starts with some sufficiently high prominent number as an upper bound, say 100 millions, and then performs a sequence of pairwise comparisons (all numbers in millions inhabitants). Let the corresponding judgements be '0 better than 100', '0 better than 50', '0 better than 20', '10 better than 0', '5 better than 10', '7 better than 5', 'cannot say whether 6 better than 5'. The obtained presentation is $7 = 10 - 5 + 2$. The process stops when no further judgement is possible, since the limit of the judgement ability of the subject is reached. This means that the subject can decide whether the 2 should be added to the $10 - 5$ or not, but it cannot decide, whether 1 has to be subtracted from $10 - 5 + 2$. Accordingly, the smallest number of the presentation informs about the exactness with which the decision is made:

Definition: The **exactness** of a presentation is the smallest prominent number of the presentation with a coefficient unequal zero. The exactness of 0 is defined as ∞ .

Notice that the exactness of a presentation informs about the exactness with which the corresponding judgement has been made only insofar that the exactness of the judgement is finer than or equal to the exactness of the obtained response. Accordingly, we define

Definition: A response has **level of exactness or prominence** p (p a prominent number) if its exactness is cruder or equal p .

During the process a subject creates essentially more information than she reveals by giving her response. Instead of a number as 60, she knows on which side of the response the signal is, and she knows the distance within which the signal might be. The following table illustrates the possible results of NRP's and the corresponding obtained information (every connection line refers to a decision which of two alternatives is 'better', the tree is continued below the respective better alternative, ties are not shown, the process is stopped when a further decision is not possible.). Accordingly, the complete information reached in a state of the process is a response (as '50'), usually the direction, in which the response is found (except for cases, where the insert of the NRP applies), and the exactness of the response (as '50 with level of exactness 10') (compare Figure 1.3.1).

(It can also be seen that the response (25) is only reached if the responder in the decision between (0+) and (50-) has equally many arguments for both alternatives.)

More important than the (absolute) exactness of a response is its relative exactness:

Definition: The **relative exactness** of a response $x \neq 0$ is its exactness divided by $|x|$. The relative exactness of 0 is 1. (The level of relative exactness is defined accordingly.)

Table 1.4.2 gives exactness and relative exactness for the integers 1-20.

Table 1.4.2: Selected presentations of the integers 1 to 20

no=presentation	exact.rel.exact.	no=presentation	exact.rel.exact.
1=1	1 100%	11=20-10+1 = 10+1	1 9%
2=2	2 100%	12=20-10+2 = 10+2	2 17%
3=5-2 = 2+1 ^a	2 67%	13=20-10+5-2 = 10+5-2	2 15%
4=5-1	1 20%	14=20-10+5-1 = 10+5-1	1 14%
5=10-5 = 5	5 100%	15=20-10+5 = 10+5	5 33%
6=10-5+1 = 5+1	1 17%	16=20-5+1 = 10+5+1	1 6%
7=10-5+2 = 5+2	2 28%	17=20-5+2 = 10+5+2	2 13%
8=10-2 = 5+2+1	2 25%	18=20-2 = 20-5+2+1	2 11%
9=10-1	1 11%	19=20-1	1 5%
10=10	10 100%	20=20	20 100%

^athe presentations according to the corollary of 1.3 are in first place

This definition is closely related to the perceptual abilities of subjects. It seems that for similar tasks subjects develop identical levels of relative exactness. For instance, very spontaneous answers are given at a level cruder or equal 25%, spontaneous answers on a level cruder or equal 10%. Price sensitivity of consumers is usually not finer than 10%, retail prices in the food sector in Germany are usually not finer than 5% (after rounding the respective last digit, as replacing 3.98 by 4.00, or 3.49 by 3.50).

For sets of empirical data, the (relative) exactness of the data set is defined as the crudest (relative) exactness which is fulfilled by 75% of the data. (SELTEN 1987 defined a different measure, which is related to his 'measure of success of a theory'. The problem has similarity with the problem of factor analysis, to decide which factors are relevant.)

1.5 Structural Prominence

Besides the prominence induced by the numerical presentation system, other kinds of structural prominence in the sense of SCHELLING (1962) occur. They are not addressed by the theory of prominence in the decimal system, but may be mentioned. We distinguish three types of structural prominence:

1.5.1 Structural Prominence in Division Tasks

Structural prominence can occur in division tasks, where a given amount X has to be divided among a set of n persons, as for instance in n -person games with sidepayments. Solving such a task, the following operations seem to be used (and may be applied repeatedly for different sets $S \subseteq N$):

- divide a set S of persons into a partition $S = S_1 + \dots + S_r$
- assign the payoff $X = p(N)$ to N
- select a payoff $p(S)$ for a set S
- compute the remainder of payoffs for the last subset of a partition $S = S_1 + \dots + S_r$, i. e. $p(S_r) = p(S) - p(S_1) - \dots - p(S_{r-1})$

- divide a payoff equally within a partition $S = S_1 + \dots + S_r$, i. e. $p(S_i) = p(S)/r$ for all S_i

Only the third type of operation involves free selection of a number. In each decision of this type the value $p(S)$ has a certain exactness. The minimum of these exactnesses over all subdecisions, i. e. $\min\{\text{exactness}(p(S)): p(S) \text{ selected in a subdecision}\}$ is the exactness of the result of the division task. (In case that the result can be explained by more than one process, that process has to be selected that gives the crudest exactness).

1.5.2 Structural Prominence in Coordination Tasks

Coordination tasks are such that two or more subjects have to agree upon a joint answer. For self-organized processes that generate such a coordination see Part II, Section 10. Part of such processes can be to agree upon the arithmetic mean of individual responses. In the mentioned experiments with free communication this happened only in the very last step, when the positions of the subjects were on two points which were neighboured with respect to the exactness and relative exactness generated by the decision problem.

1.5.3 Structural Prominence by Arguments

In negotiations and decision problems statistical data (that are introduced into the decision process), evaluations by experts, and results of reasonable computations can become focal points, which are noticed at any level of exactness. This type of structural prominence is not considered here.

A similar type of prominence that we repeatedly observed in price setting (especially for used cars) is to use 'lucky numbers' as 7777 or 3333. May be that there are persons who think that changing the price means losing the quality of a 'lucky car'?

2 Scales

2.1 Construction of Scales by Iterated Selection of Midpoints: Scales of Type $M(i, a)$

Another principle that can be detected in numerical response behavior is the (iterated) construction of midpoints. Thereby the basic scale of prominent numbers, where the distance of any two numbers is perceived as one step, is refined to half steps, quarter steps, etc. For these steps, the following notations are used (by the subjects):

full steps	10		20		50		100
half steps		15		30		70	
quarter steps		12	18	25	40	60	80
.....

This step structure starts with the prominent numbers, and the respective next midpoints are obtained by the following rule:

Midpoint Selection Rule: The number x denoting the midpoint of two neighbored points a, b fulfills: (1) There is no number between a and b with a cruder relative exactness than x . (2) Among all numbers that fulfill (1) x has the shortest presentation. (Where the length of a presentation is its number of nonzero coefficients.)

It may be remarked that Condition (2) would not have been necessary for the dual presentation. It has to manage the insufficiency of the decimal system. It selects the number with the shorter presentation, for example $12 = 10 + 2$ against $13 = 10 + 5 - 2$ as 'midpoint' between 10 and 15, or $18 = 20 - 2$ against $17 = 20 - 5 + 2$ as 'midpoint' between 15 and 20.

Lemma: The Midpoint Selection Rule selects a unique midpoint except for the midpoint of $5 * 10^i$ and $10 * 10^i$ (i integer). For this midpoint the values $7 * 10^i$ and $8 * 10^i$ are permitted.

Empirical data accord with this rule. They indicate that in fact some subjects select $7 * 10^i$, some $8 * 10^i$ as midpoint of $5 * 10^i$ and $10 * 10^i$. (For percentages the selection of $8\% = 10\% - 2\%$ seems to be essentially more frequent, what may be related to the fact that 10% is very easily perceived.) Nevertheless, we suggest to add the condition

(3) If conditions (1) and (2) characterize more than one point, then that with the smaller absolute value is selected.

to the Midpoint Selection Rule, if the reader wants to have a unique prescription for all cases. According to this extended condition we introduce the following notation for the obtained scales:

Definition: Let i a positive integer, a a prominent number. Then $M(1, a)$ is the set of all numbers with relative exactness 1, and absolute exactness $\geq a$. $M(i, a)$ is the set obtained from $M(1, a)$ by applying the Midpoint Selection Rule (1) - (3) i times.

Remark: The definition of iterated midpoint scales follows the general idea that differences of numbers are perceived in steps (and parts of steps). Under this assumption the limit of the perception scales, $\lim(M(i, a) : i \rightarrow \infty)$, can be interpreted as a perception function.

2.2 Characterization of Scales by Relative and Absolute Exactness: Scales of Type $S(r, a)$

Another approach to scaling uses exactness and relative exactness of the numbers.

Definition: Let $r \geq a$ prominent numbers. The scale $S(r, a)$ is the set of all numbers with relative exactness $\geq r$, and absolute exactness $\geq a$.

Definition: Two numbers x, y (with $0 < x < y$ or $y < x < 0$) of a scale $S(r, a)$ belong to the same step, if $(y - x)/x < r$.

Some examples of such scales are

$$S(100, 10) : \dots, -100, -50, -20, -10, 0, 10, 20, 50, 100, 200, \dots$$

$$S(26, 10) : \dots, -100, -70, -50, -30, -20, -10, 0, 10, 20, 30, 50, 70, 100, 150, \dots$$

$$S(20, 10) : \dots, -100, -80 = -70, -50, -30, -20, -10, 0, 10, 20, 30, 50, 70 = 80, 100, \dots$$

$$S(10, 10) : \dots, 100, 120 = 130, 150, 170 = 180, 200, 250, 300, 350, 400, 450, 500, 600, 700, 800, 900, 1000, \dots$$

$$S(5, 10) : \dots, 100, 110, 120, \dots, 180, 190, 200, 220 = 230, 250, 270 = 280, 300, 320 = 330, 350, \\ 370 = 380, 400, 450, 500, 550, 600, 650, \dots, 850, 900, 950, 1000, \dots$$

There are several firms in Germany which seem to use scales of type $S(r, a)$ as their main guideline for the selection of retail prices (prices before reduction by ϵ amounts as 398 instead of 400, or 249 instead of 250, steps with two numbers as 270=280 are typically priced by 279).

As can be seen from the examples, there are at most two numbers that belong to the same step of a scale, and these numbers have a special shape:

Lemma: There are at most two numbers a, b on the same step of a scale. These numbers can be presented as $X + 20 * 10^i$, $X + 30 * 10^i$, where the exactness of X is cruder than $20 * 10^i$.

Remark: Subjects sometimes denote a step as 120=130 by '125', or 170=180 by 175. This use of the term $25 * 10^i$ caused us to denote the terms $25 * 10^i$ (i integer) also as prominent numbers (see ALBERS-ALBERS, 1983). However, the 25 is only used as a notation, it can be used within a presentation only as finest prominent number, i. e. as last term of a presentation. Therefore we introduce this modification only as a notational, and not as a structural component.

Different from the iterated midpoint approach, the $S(r, a)$ -approach permits the subdivision of ranges between prominent numbers in various ways (see Table 2.2.1), while the iterated midpoint approach only permits to obtain integer powers of 2.

The question arises, whether the iterated midpoint approach selects the respective most prominent numbers, i. e. if the scales $M(i, a)$ can be presented as scales $S(r, a)$. The answer is

Lemma: $M(2, a) = S(.26, a)$ for all a . - No other scale $M(i, a)$ ($i > 1$) can be presented as $S(r, a)$.

Denote a scale as **regular**, if it is obtained by successive addition of a step with the respective crudest relative exactness.

Lemma: $M(1, a)$ and $M(2, a)$ are regular for all a . All other scales $M(i, a)$ ($i > 2$) are not regular. – The scales $S(r, a)$ are regular for $r > 20\%$.

We presented different ways to construct scales. While in the $S(r, a)$ construction new elements are selected by the relative exactness, the iterated midpoints principle selects new elements according to their 'role' as halves, quarters, etc. These principles coincide for scales with relative exactness cruder than 25%, i. e. the scale of steps and half steps. Empirical data indicate that these numbers are responded and perceived on a crude spontaneous level. This justifies the name

Definition and Lemma: The numbers $\{a \cdot 10^i : a \in \{1, 1.5, 2, 3, 5, 7\}, i \in \mathbb{Z}\}$ are denoted as **spontaneous numbers**. (1) They are obtained as the positive numbers of the sets $M(2, 0) = S(26\%, 0)$. (2) They are the set all sums of pairs of subsequent prominent numbers.

Remark: Presently we cannot definitely say how numbers with finer relative exactness than the spontaneous numbers are perceived. May be that they are rounded to the respective next spontaneous number, may be they are truncated, or that finer levels of exactness are reached. Empirical data indicate that the fineness of perception depends on the task. According to these open questions (which have to be clarified) we presently only use spontaneous numbers in the tasks of our experiments. We suggest others to do the same to avoid noise in their data.

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