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Foundations of a Theory of Prominence in the Decimal System Part III: Perception of Numerical Information, and Relations to Traditional Solution Concepts

by

Wulf Albers

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University of Bielefeld

33501 Bielefeld, Germany

Abstract

The paper introduces the present state of the applications of the theory of prominence of ALBERS-ALBERS (1983) to the perception of numerical information. Basic elements of the theory can be found in the WEBER-FECHNER law (1834,1860) concerning the psychophysical perception of physical stimuli as brightness, loudness, or weight. The rules are (1) evaluation of the intensity of stimuli is logarithmic, (2) stimuli are perceived with a constant relative exactness, and (3) there is a smallest absolute intensity that can be perceived. The same rules can be applied to the perception of stimuli which are presented in a numerical way, as prices, quantities, percentages, or time. The perception of all these different kinds of stimuli is ruled by identical basic laws concerning the perception of numbers, here presented for the decimal system.

Basic elements of the theory is a system of numbers which are most easily perceived, the prominent numbers $\{a * 10^i : a \in \{1,2,5\}, i \,\text{integer}\}$. Comparison of numerical stimuli happens on a scale, on which the prominent number define the full steps. Half steps, quarters, etc. can be defined. The difference of numerical stimuli is given by the difference measured in steps on this scale. Every number is perceived as a sum of prominent numbers, where the coefficients are +1, -1, or 0, i. e. one obtains a number by starting with some high prominent number and refines the number stepwise by adding or subtracting smaller prominent numbers, for instance $17 = 20 - 5 + 2$. The exactness of such a presentation is the smallest prominent number needed in the presentation, the exactness of a number is given by the crudest exactness over all possible presentations of the number, the relative exactness of a number is its exactness divided by the number.

The system of half steps, quarter steps, etc. corresponds to perception with decreasing exactness. As in the WEBER-FECHNER laws (2) and (3) it turns out that – depending only on situation, person, and task – relative exactness and absolute exactness of perception are constants. These rules are insofar different from the WEBER-FECHNER law that 1. rule (3) becomes important since it enables to compare positive and negative payoffs (while the variables of the WEBER-FECHNER approach are allways positive), and 2. the constants of exactness do now also depend on the task. While spontaneous perception usually happens at a level cruder or equal to the halves of the step scale, the absolute exactness of perception, i. e. the smallest perceived step essentially depends on the task. For the evaluation of money amounts the smallest absolute unit is roughly 20% of the largest absolute money value involved in the task.

Accordingly, it is not possible to present the obtained rules of perception of numerical differences by a universal perception function, specificly, it is not possible to give a universal utility function describing the perception of monetary payoffs (as KAHNEMAN-TVERSKY do in their v-function). The nonexistence of such a function creates for instance the possibility that the same prospect can be evaluated differently depending on other prospects with which it is compared. This is the reason for a kind of preference reversal which could be predicted by the theory here, and afterwards verified in the experiment.

The generalized perception function of our model is compared with the evaluation functions of KAHNEMAN-TVERSKY (1992). The new approach could be successfully used to modify traditional fairness concepts for different types of bargaining situations (KALAI-SMORODINSKY's equal concession solution, HARSANYI-

SELTEN's risk dominance, NASH's bargaining solution) in a way that they now seem to be the best predictors for the related experimental behavior.
It may be mentioned that the modificaton of traditional concepts follows certain simple rules, and that in the obtained solutions the variables are only treated on an additive level (no products, no quotients), and all coefficients of involved variables are either +1 or -1. This simplicity might be a general phenomenon of certain boundedly rational models, so that the decision maker has only to decide, whether to apply a given variable or not, and in which direction it works.

Further results that confirm the given approach are presented in Part IV.

Contents

1 The WEBER-FECHNER Law

The WEBER-FECHNER law considers the situation where a subject evaluates the intensity of a physical stimulus as brightness, loudness, etc. It is known that the physical stimulus causes a physiological response which is then mapped to a numerical response. The WEBER-FECHNER law gives three rules of this evaluation. Rules (2) and (3) were observed by WEBER (1834), from these rules FECHNER concluded on the shape of the perception function, rule (1):

(1) **logarithmic perception:** multiple stimuli (measured on the physical scale) cause additive perception of the differences
(by adequate selection of constants this rule can be condensed to the mathematical formula 'stimulus times 2 gives perception plus 1')

(2) **constant smallest perceived relative difference:** the smallest perceived relative difference between two stimuli (measured on the physical scale) is a constant, which only depends on variable and person

The third rule less noticed, since it is of minor importance for the shape of a perception function:

(3) **smallest percieved absolute value:** there is a smallest physical value that can be perceived, it is a constant which only depends on variable and person.

The corresponding stimulus-response function can be subdivided into two steps. Step one is the physiological reaction on the physical stimulus. Step two is the numeric response to the physiological reaction:

physical stimulus $\longrightarrow$ physiological reaction $\longrightarrow$ numerical response

The WEBER-FECHNER law describes the combined effect of the two mappings. The general opinion seems to be that the numeric response describes the physiological reaction adequately (in a linear way), so that (without loss of generality) the second mapping can be interpreted as the identification numerical values. According to this opinion, the rules describe the reaction of the physiological response to the physical stimulus ('law of psychophysics').

The rules describe judgement behavior of subjects, that can be condensed to three basic abilities, namely the qualitative judgements whether a given signal can be noticed (addressed by rule (3)), and which of two given stimuli has greater intensity (addressed by rule (2)), and the quantitative judgement whether the intensity of a stimulus is 'in the middle' of two others (the result of such judgements is addressed by rule (1)).

2 General Comments Concerning the Perception of Numerical Stimuli

Different from the preceding section we now consider the situation that the stimuli are not given in a physical, but in a numerical way. The task is, for instance, to determine the money equivalent of a lottery for instance to get a payoff x with probability 50% and to get nothing otherwise. There are three ways how these stimuli might create reactions: (a) the stimuli induce the reaction by numerical operations, (b) the stimuli induce emotional reactions which guide the decision, (c) the stimuli induce physiological reactions, the physiological reactions induce emotional reactions, and these permit judgements. We do not want to decide which of the alternatives describes the reaction mechanism best. However, our impression is that the decision process is guided by emotions which are induced via the mental perception of the alternatives presented by the task. These emotions serve to select among proposals to solve the task, in the example proposals of money equivalents. They give a spontaneous feeling whether one answer is better than another.

Experimental results indicate that the spontaneous emotional feeling, which of two answers is better, is induced via an internal counting of arguments for the respective alternatives, where the alternative with more arguments wins. In this kind of procedure the weight of an argumental dimension is given by the number of arguments concerning this dimension which are distinguished by the decision maker at a given level of graneness of judgement.

To make the decision more rational, subjects can underly this emotional uncontrolled 'black box' of internal reaction mechanism by rational arguments which support one or the other alternative, or can create alternatives as proposals for the solution. Important decisions seem to be generally made by constructing a (boundedly) rational model, which leeds to the selected decision, and serves to devaluate concurring alternatives. Where the adequacy of the model is checked by compairing its results with the respective emotional decisions. (Part of this interactive process of rational and emotional analysis is to countercheck emotional judgements and results of rational analysis, where certain mechanisms help to solve problems of dissonance.) Complicated schemes of reasoning are usually constructions of subroutines most of which have been checked in different context and have been confirmed as successful tools.

For decision making with numerical in- and outputs the boundedly rational models which can be constructed to support one or the other decision can and do involve (simple) numerical calculations. It can make sense to model the situation in a mathematical way.

We can expect certain purely mathematical mechanisms that support the decision processes. We will see that the rules of certain processes which serve as general tools are very similar to those of the WEBER-FECHNER law. The corresponding type of analysis of purely numeric processing in decision making might be denoted as 'psychomathematics' in contrast to the quotation 'psychophysics' used for the WEBER-FECHNER law.

The identity of the obtained fundamental rule to the WEBER-FECHNER law is striking. It may result from the fact that emotional evaluations of numeric amounts as money, probability, or time create reactions that are similar to those which are induced via the perception of loudness, brightness, or weight. It seems that the observed phenomena are even clearer in psychomathematics. Moreover, the psychomathematical approach permits to identify additional phaenomena which seems to hold in psychophysics as well.

3 The Law of Numerical Perception

The law of numerical perception is similar to the three rules of psychophysics:

(1) **logarithmic perception:** multiple numerical stimuli cause additive perception of the differences (by adequate selection of constants this rule can be condensed to the formula 'stimulus times 2 gives perception plus 1')

(2) **constant smallest perceived relative difference:** the smallest perceived relative difference between two numerical stimuli is a constant, which depends on variable, person, and task

(3) **smallest perceived amount:** there is a smallest numerical amount that can be perceived. This is a constant which depends on variable, person, and task

The main difference between these and the preceding rules is that the constants now depend on the task. For instance, the smallest percieved absolute money amount in decisions about the annual budget of a state may be (at least in early phases of the decision) a billion dollars, while considerations concerning the price of a dinner may have an exactness of between 1 and 5 dollars.

Another basic difference between the two approaches is, that psychomathematics permits the comparison of negative and positive numbers, while in psychophysics only positive stimuli are considered. As we will see, this makes rule (3) important, since it permits to measure distances from the zero-point.

4 Basic Notations of Decimal Perception

In the following we restrict our considerations to the description of decimal perception. Other numerical systems, for instance for the dual system, can be developed in a similar way. In fact the dual system would permit a less complicated model, since the basic element of the approach is a system of numbers, where each number is (roughly) double as great as the preceding one.

> **Notations:** Basic numbers of decimal perception are the 'prominent numbers' $\{a * 10^i : a \in \{1, 2, 5\}, i \, \text{integer}\}$, i. e. ...,1,2,5,10,20,50,100,.... Every (real) number can be presented as a sum of prominent numbers where every prominent number is used as most once, and all coefficients are +1, -1, or 0. The exactness of a presentation is the smallest prominent number used in the presentation. The exactness of a number is the maximal exactness among all presentations of the number. The relative exactness of a number $x \neq 0$ is its exactness divided by $|x|$. The exactness of 0 is ∞, its relative exactness is 1. (Example: The number 17 can be presented as $10 + 5 + 2$, $20 - 5 + 2$, or $20 - 2 - 1$. The exactness of the presentation $10 + 5 + 2$ is 2, that of $20 - 5 + 2$ is 2, that of $20 - 2 - 1$ is 1. The exactness of 17 is 2. The relative exactness of 17 is 2/17.)

The numbers of the decimal system that are most easily accessed are the powers of ten. If one assumes that the distance of any two neighbours of the sequence of powers of ten is perceived as equal, one obtaines the evaluation function $\log(x)$, which follows rule (2). Related to the fact that persons most easily compare in steps of doubles or halves, there are two additional numbers introduced between any two powers of ten, namely $2 * 10^i$ and $5 * 10^i$. Under the condition that a system of numbers contains the powers of ten, consists of round numbers, and has a relation of about two between any two numbers, there are only two candidates of pairs of numbers between 1 and 10, namely the pair 2,4 and the pair 2,5. The second pair has the advantage that it fits to the idea of presentation of numbers better, since $10-5=5$. Accordingly, the system of prominent numbers has been selected.

The general idea, namely that the steps of the scale are obtained by douplications fits to logarithmic structure. The selected numbers are the best integer replacements for two logarithmically equal steps between 1 and 10, which are 2.15 and 4.62. The conflict between the two aims to obtain a scale by iterated doubling, and to obtain in every third step integer powers of ten is hidden as long as the relative exactness of perception does not permit to distinguish 2.5 and 2, i. e. on a level of relative exactness cruder .25. This is a structural reason which supports to keep the exactness of perception cruder than .25, i. e. at the level if spontaneous numbers.

5 Scales

The surprising fact of decimal perception is that exactness and relative exactness as defined above, via the smallest number used in the presentation, fits to rules (2) and (3), so that by a smallest relative exactness r (rule (2)), and a smallest absolute exactness a (rule (3)) the following type of scales can be defined:

> **Definition:** A **scale** $S(r,a)$ is the number 0, and set of all numbers with a relative exactness $\geq r$, and with an absolute exactness $\geq a$. **Two elements** x, y of a scale are **perceived as equal**, if $|y-x|/max(|x|,|y|) < r$.

The second condition is related to the fact that under decimal perception it can happen that on a given level of relative exactness two different numbers can be responded, although these numbers cannot be distinguished on this level. Such numbers are defined to belong to the same 'step' of the scale. – This insufficiency is related to the fact that 5 is more than the double of 2, so that $10+2=12$ and $10+5-2=13$ give different values, both of which can be used as notations for the step between 10 and 15 with exactness 2. Which of these two numbers is selected by a subject seems to depend on which of them is mentally more easy accessed. This selection seems to be guided by the

> **Shortness Rule:** Among two numbers in the same step of a scale that number is preferred as response, which has a shorter presentation (less prominent numbers in its presentation).

For instance, by many subjects the response $18 = 20-2$, is preferred to $17 = 20-5+2 = 10+5+2$, and $12 = 10+2$ is preferred to $13 = 10+5-2 = 20-5-2$. – The shortness rule does not decide whether to select 7 or 8 on exactness level 2, since $7 = 5+2$, $8 = 10-2$. In fact both numbers can be found as responses on exactness level 2. Under ordinary conditions the 7 seems to be preferred (its relative exactness is finer), but for percentages we observed that the response 8% is more frequent than 7%, which suggests that 10% (which serves to reach 8%) is more easily accessed than 5% (which serves to reach 7%).

(In this context see also the experiments of VOGT-ALBERS 1993.)

Examples of scales are

```
S(100%,5)= ..., -100,          -50,        -20,       -10, -5,  0,  5, 10,      20,      50, 100, ...
S( 26%,5)= ..., -100,   -70, -50,    -30, -20, -15, -10, -5,  0,  5, 10, 15, 20, 30, 50, 100, ...
S( 10%,0)= ...,     10, 12=13,  15, 17=18,  20,  25,  30, 35, 40, 45, 50, 60, 70, 80, 90, 100, ...
```

The numbers with relative exactness cruder than 25% are called **spontaneous numbers**. They are selected by very spontaneous responses (as an example see the probability estimations just before negotiations in section 9). Resale prices in Germany (after rounding amounts as 1.98 to 2.00) reach a relative exactness of about 10% to 5%, and a finer level only in exceptional cases.

A comment concerning '25' as a prominent number: The identification of steps of a scale happens for numbers on exactness level $2 * 10^i$, as for instance $20 = 30$ when the relative exactness is 40%, or $70 = 80$, when the relative exactness is 20%. In these cases the responses '20' and '30' can be understood as verbal expressions, which both denote 'the number at half step between 10 and 50', or '70' and '80' denote 'the number at half step between 50 and 100'. In these situations, some subjects denote the class $20 = 30$ by the expression '25', or the class $70 = 80$ by '75'. In this context the '25' and '75' can be evaluated as responses with absolute exactness 20. ALBERS-ALBERS (1983) therefore defined the prominent numbers as $\{a * 10^i : a \in \{1, 2, 2.5, 5\}, i\,\text{integer}\}$. However, the 25 can substitute 20 as a prominent number only, when 25 is the smallest term of the presentation.

6 The Perception Function

Another presentation of scales is obtained if one assumes, that in the decision process leeding to a response, subjects stepwise refine the level of relative exactness by inserting 'midpoints' until they reach the boundary of their discrimination ability. This instrument of stepwise refining scales permits to measure the distance of numerical stimuli in full steps (given by the prominent numbers), half steps (given by the spontaneous numbers), quarter steps, etc. The corresponding scales are[1]

```
full steps  ...  10        20        50         100  ...  (prominent numbers)
half steps  ...       15        30        70         ...  (spontaneous numbers)
quarters    ...    12    18    25    40    60    80    ...  (quarters)
etc.        ...  ......................................(etc.)
```

Notation: This construction permits to identify a response function which maps numerical stimuli to responses, we denote it as *per*: stimuli $\longrightarrow$ responses.

[1]Some subjects replace the notations '70' for the half step, and '60','80' for the quarters by '80' for the half step, and '70', '90' for the quarters. This is induced by the two different notations, '70' and '80', for the half step between '80' and '100'.

Table 1 gives the step structure of integer responses for the numbers between 10 and 100. It seems reasonable to assume that the function describing numerical responses and the function describing the perception of numerical stimuli are closely related. It is assumed that a numerical stimulus activates a numerical response according to the response function, where relative and absolute exactness are given by the respective task.

An open question is, what happens, when the numerical information is finer than the level of exactness of analysis. Do subjects round, or truncate that part of the information which is too fine? It seems reasonable to presume that they cut off that part of the response which is finer than the given level of exactness, which is quite similar to rounding. (But since we did not yet invertigate this problem, we presently select all numbers presented to the subjects on a level that is not finer than half steps. And we suggest to do so also to others, if they want to avoid noise in their data.) Notice that the shape of the obtained function near the zero-point essentielly depends on the smallest perceived full step. – It may be remarked, that in this presentation prominent numbers can have the property to be half steps, if they are between 0 and the smallest perceived full step. An example:

```
full steps ... -20           -10         0         +10           +20 ...
half steps ...       -15           -5        +5           +15        ...
quarters   ...    -18    -12    -7   -2  +2   +7    +12    +18      ...
etc.       ..........................................................
```

Loosely speaking, the shape of the perception function is 'logarithmic' for absolute values greater than the smallest perceived full step, and linear in the range between the smallest perceived full steps.

Comments: 1. Presently we perform all of our calculations under the assumption that the level of relative exactness (full steps, half steps, or finer level) on which numbers are perceived in a given task does not depend on the size of the numbers, what (roughly) accords with rule (1). – 2. The classification of numbers into full steps, half steps, etc. suggests that halves are mentally more easily reached than quarters, quarters more easily than eighths, etc. However, this principle does not fit to the assumption that the easiness of perception is monotonous with relative exactness. For instance, 18 is on a quarter step, 35 is on an eighth step, but the relative exactness of 35 is 1/7, while that of 18 is 1/9. We did not yet clarify which of the concepts describes simplicity of perception better. Both responses, 18 and 35, are anyway outside the normal exactness of spontaneous answers. Related to this problem is the question whether the aproach by $S(r,a)$ scales (section 5) or the procedure of iterated midpoints with refinement to halves, quarters, etc. is a better frame to describe subjects' choices. By and large the two scales create the same kind of perception. They coincide for spontaneous numbers. – 3. Although the steps obtained near the zero-point are generally equal under both scale concepts, the motivations why a lower boundary is introduced are slightly different: In the $S(r,a)$ approach the cut off point is determined by a smallest perceived amount, a, that can result as a payoff. In the repeated midpoint approach the cutoff point is given by a smallest perceived full step, which is the border between linear and logarithmic perception (all values between the smallest full steps may be interpreted to be obtained by a linear interpolation procedure). But these differences are only on the level of interpretation. The obtained steps in the range between zero-point and smallest full step are identical, this may be illustrated by an example: by a smallest full step of 10, and relative exactness of quarters one obtains the steps 0,2,5,7,10,12,15,18,20,25,30,40,50; the same values (and 35) are obtained by $S(11\%,2)$. – 4. If one wants that the operation of inserting 'midpoints' fulfills the reasonable condition that the absolute exactness of numbers monotonically increases for the obtained scale, then certain insertions (denoted by '–' in the table below) have to be postponed to the respective 'next rounds', where every line of the table refers to one round. The condition, that the relative exactness of the numbers of the scale increases with every step leeds to unreasonably many structural breaks. These problems can be avoided by keeping the relative exactness cruder than 1/7.

Table 6.1: Iterated Insertion of 'Midpoints' in the Ranges 10-50, and 50-90

10		20					50
	15			30			
12 ..	.. 18		25			40	
-- .. 13	17 .. --	22 ..	.. 28		35		45
11 14	16 19	-- .. 23	27 .. --	32 ..	.. 38	42 ..	.. 48
		21 24	26 29	-- .. 33	37 .. --	-- .. 43	47 .. --
				31 34	36 39	41 44	46 49

50							
				70		..	
		--		..		80	
		60					90
	55		65		75		85
52 ..	.. 58	62 ..	.. 68	72 ..	.. 78	82 ..	.. 88
-- .. 53	57 .. --	-- .. 63	67 .. --	-- .. 73	77 .. --	-- .. 83	87 .. --
51 54	56 59	61 64	66 69	71 74	76 79	81 84	86 89

-- refers to insertions that are postponed to the next round of insertion
.. indicates that the the corresponding notations may as well introduced in the other order, for instance 13 before 12, 17 before 18, or 80 before 70

7 Relation to KAHNEMAN-TVERSKY's Evaluation Functions

Different from classical utility theory, and different from the approach of KAHNEMAN-TVERSKY the approach here gives a perception of 'utility' which depends on the task. As already mentioned, state budget decisions may be performed and perceived with an absolute exactness of a billion dollars, private consumption decision for certain goods may be made with an absolute exactness of a dollar. People adjust their absolute exactness. The different absolute exactnesses create different utility functions.

Common part of two perception functions with different levels of absolute exactness are the logarithmic pieces for sufficienty large (or sufficiently low) numbers. The shape of the perception function in between, i.e. around the zero point (and the vertical distance of the two logarithmic parts) depends on the respective absolute exactness (see Figure 7.1).

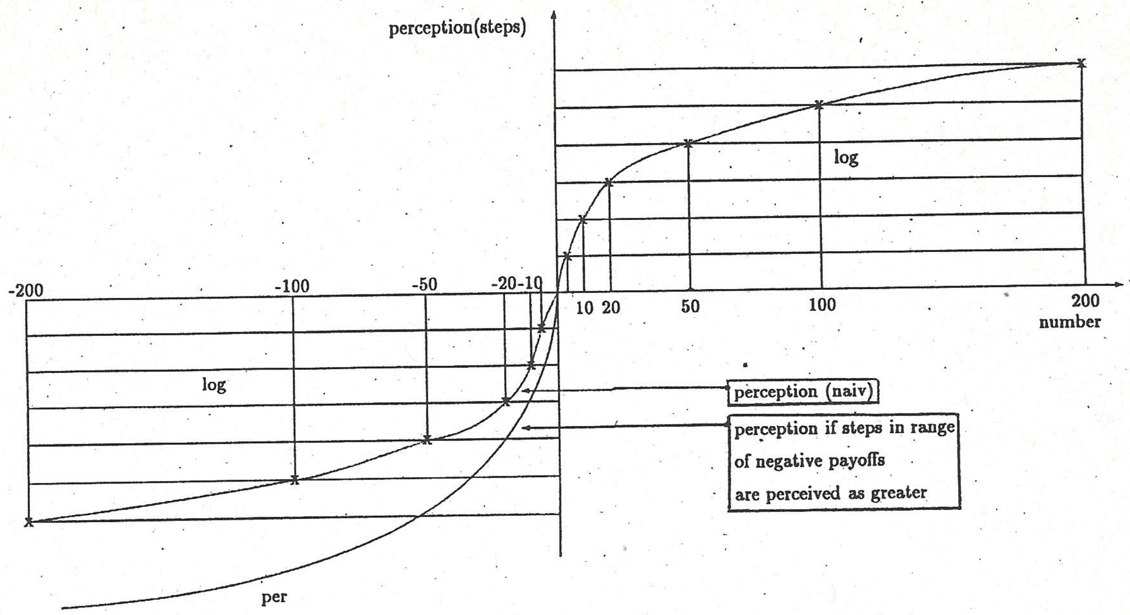

Figure 7.1: The Perception Function for Money

Another difference of the approach here to that of KAHNEMAN-TVERSKY is that their value function has a kink in the zero-point and the negative part is shifted downward by a factor. This modification is necessary and can be explained by the rule

> **Rule of Perception of Negative steps:** Steps within the range of negative numbers are evaluated as greater than steps in the range of positive numbers

In the analysis below we come to the conclusion that steps in the negative range are evaluated as double. KAHNEMAN-TVERSKY's factor is rather 1.3 than 2. In this point our experiments do not confirm their result. The given mathematical theory permits to recognize deviations as psychological effects. There are neither questions about the effect, nor about its multiplicative character, only about the factor.

The π-function evaluates probabilities. In the example, the shape for probabilities below 50% is similar to that induced by an absolute exactness 5% on the full-step level, which gives the half steps 0%, 2%, 5%, 8%, 10%, 15%, 20%, 30%, 50%. Evaluating counter-probabilities in the same way gives the steps 50%, 70%, 80%, 85%, 90%, 92%, 95%, 98%, 100%. The two parts are stitched in 50% by identifying the step levels.

The π-function of KAHNEMAN-TVERSKY shows a different shape. Its value at 50% is lower than half the distance between the values of 0% and 100%. We see three possible explanations for this deviation from the naiv theoretical construct:

(a) Subjects have a general aversion against lotteries. This psychological effect causes a devaluation which is maximal for 50%-50% lotteries.

(b) Since subjects are more sensitive to missing probabilities than to probabilities, they perceive counterprobabilities with a finer absolute exactness than probabilities. Assuming that the absolute exactness of the perception of counterprobabilities is by one step more exact than that of prabilities, one obtains two more half steps on the side of counterprobabilities and thereby reduces the position of the 50% point by 2/18, which is about 10% of the total distance. This fits to the shape of KAHNEMAN-TVERSKI's π-function.

(c) KAHNEMAN-TVERSKY describe the π-function as the result of evaluating lotteries of type [1000(p%),0(100%-p%)]. It is not clear whether they computed the answers in money equivalents or in the perception space (i. e. in steps). If the values are money equivalents, then the inverse of the money evaluation function has to be applied to obtain the correct evaluation function for probabilities. Applying this function would increase the values in the middle range of the π-function.

Our observations do not indicate an aversion to 50%-50% lotteries, as suggested in (a). Concerning (c) we will ask KAHNEMAN-TVERSKY, but we suggest that they the remapping to the perception space is done in their analysis. We can follow argument of (b). However we do not follow the construction of the π-function as it is done.

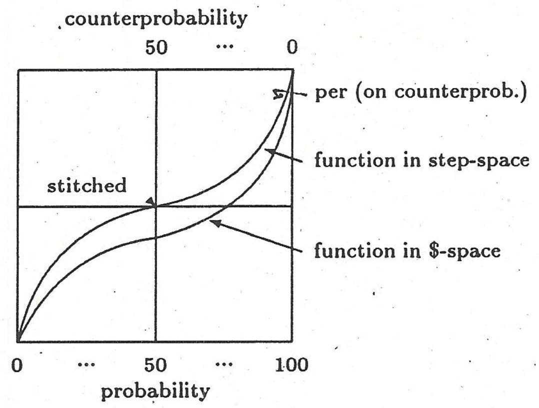

Figure 7.2: The π-Function

From our point of view, the π-function cannot be presented as a unique function serving for the evaluation of arbitrary prospects. In our opinion, the absolute exactness depends on the task. The high fineness at the 100% tale does only effect the perception, if probabilities with high percentage (as 98%) are perceived. Such an evaluation will not happen in the same prospect, where 50% has to be evaluated. Accordingly, we argue that the evaluation of the probabilities of a 50%-50%-prospect happens on a scale with equal smallest absolute probabilities at both anchor points,so that 50% is evaluated to be in the middle between 0% and 100%. (For instance, 1/5 of 50% gives an absolute exactness of 10%, and thereby the full steps 0%, 10%, 20%, 50%, 80%, 90%, 100%, where 50% is in the middle.)

The remarkable result is that the assumption of a task-independent π-function creates severe contradictions which do not occur in the approach here.

(That the same problem of variable relative exactness did not that clearly arise for the evaluation function of money, is related to the fact that the money amounts of the lotteries in the investigation of KAHNEMAN-TVERSKY that were different from zero were in a limited range (may be between 300 and 3000 dollars).

An interesting point of the construction of the π-function is that certain operations on scales seem to be possible, namely the use of anchor-points (here 0% and 100%), and the stitching of scales (here in the point 50%). In this context it may be remarked, that the empirical data of a probability evaluation task show that about 20% of the subjects introduce 50% as an additional anchor point, so that 3 stitching points are obtained. Their absolute exactness at the 50% anchor point was clearly cruder than at 0% and 100%. This confirms that the absolute exactness can depend on the anchor point (as suggested in explanation (b)).

8 Rules Concerning Relative and Absolute Prominence of Bargaining

To complete the theory needs predictions concerning the selection of absolute and relative exactness. The best rule concerning the absolute smallest money amount which can be deduced from subjects' behavior (in different selection situations) is:

> **Absolute Prominence Selection Rule:** When two numbers are involved in a judgement then the median smallest absolute amount (in full steps) considered by subjects is 20% of the amount of the larger absolute number.

The concrete values can essentially differ between subjects. The value of 20% is obtained for students for substantial monetary amounts (for German students these are over DM 100). Moreover, under certain conditions students pay special attention to losses, and replace the above 20%-value by 20% of the absolute value of the largest negative amount, if this amount is greater than DM 100, and less than the 20%-value above (we denote this appendix as rule (*))[2].

To define a rule for the relative exactness, we recall which situations have to be modelled. Imagine, there is a problem for which a subject has to make a numerical decision. The idea is that (controlled and uncontrolled) activities of the brain generate proposals for answers. Certain proposals are (clearly) rejected by the subject. An interesting observation is that in such situations the most natural question seems to be which is the highest, and which is the lowest reasonable answer (best case, and worst case). Accordingly, it seems to make sense to assume that for every such situation subjects can give a 'range of reasonable alternatives'. The following rule is based on this range:

> **Relative Prominence Selection Rule:** The relative exactness of a numerical response is selected such that in the range of reasonable alternatives there are between 3 and 5 numbers with this or a higher level of relative exactness.

[2] I personally have a cruder level than 20%, I am at the crudest level which permits to perceive the respective decision in a nonlinear way, and do never apply rule (*). I suggest that the data of business men are near to my personal value than to that of the subjects, as long as the amounts are not too extreme.

This rule has been supported by several empirical investigations. (For example, see ALBERS-ALBERS 1983, VOGT-ALBERS 1993.)

9 The Golden Rule of Boundedly Rational Choice

In the preceding sections we presented an instrumentarium for the evaluation of numerical stimuli. This new – and from our point of view comparatively precise – approach permits to reconsider experimental tasks as equilibrium selection, bargaining, joint numerical responses of groups, evaluation of 50%-50%-prospects in a more detailed way than previous studies. The adjustment of the perception function to the principles of boundedly rational decision making opened the chance to detect general rules of boundedly rational decision processing. The result is

> **Golden Rule of Boundedly Rational Aggregation of Information:**
> Given a decision problem with numerical stimuli $x_1, ..., x_n$ of the same dimension. Boundedly rational decision processing
> - attributes signs (-1, +1, or 0) to the stimuli,
> - evaluates the stimuli according to the perception function, and
> - adds up

For monetary payoffs the obtained function has the shape $\Sigma_i(\epsilon_i * per(x_i))$, where all $\epsilon_i \in \{-1, +1, 0\}$. Of course $\epsilon_i = 0$ means that x_i is unimportant, and should not be added to the list of attended variables.

The Golden Rule permits to adjust traditional fairness criteria to boundedly rational decision processing. Assume the criterion is given by a formula using variables $x_1, ..., x_n$, and the operators/quantifiers '*', '/', '+', '-', 'and', 'or', '>', '<', '=', 'max', 'min', '=max!', '=min!', and using brackets according the usual mathematical rules.[3] Then the corresponding mapping is

> **Mapping Induced by the Golden Rule:**
> - write the given criterion or formula in one line
> - replace x_i by their percieved values for all variables
> - replace the operators '*' by '+', '/' by '-'
> - leave all other operators as they are

An example is the tranformation $(a_i - c_i)/(b_i - m_i) \longrightarrow (per(a_i) - per(c_i)) - (per(b_i) + per(m_i))$. – The surprising result is that subjects behave as if they cannot distinguish between logarithmic and linear evaluation, and just add up the variables independent from their character.

In this context we mention a study about the prices of flats in Jerusalem[4] with the result that the best predictor for the preference is just the number of criteria that support the

[3] it may be necessary first to apply logarithmic or exponential transformation to obtain a shape of this kind.

[4] unfortunately we do not remember the author(s)

respective choice, where positive arguments are evaluated by '+1', negative arguments by '-1'. It seems that - at least in certain decision situations - subjects use a constant 'graneness of analysis' over all possible criteria, and continue separating arguments until all of them have the same weight. (Notice that the available numerical values as size, distance from the center of the town, etc. did not enter the function. This seems reasonable since it may be hard to compare quantities of different dimensions on a boundedly rational level of analysis. – Similar results have been obtained in a study concerning the ranking of the number of inhabitants of towns in Germany.

It also seems worth mentioning that the presentation of numbers as sum of prominent numbers with coefficients +1, -1, or 0 follows the Golden Rule. The different evaluation of negative payoffs indicates, that positive and negative payoffs are not genuinely perceived as the same dimensions in the sense of the Golden Rule. However, (in this case) the two dimensions can be easily adjusted by evaluating steps in the range of negative payoffs double. – Concerning the perception of prices and quantities (as in consumption analysis) the Golden Rule basically predicts logarithmic perception on every dimension. Accordingly, price-demand functions should be linear as functions of log(price) versus log(demand).[5]

The term 'variables of same dimension' used in the Golden Rule needs an interpretation. As we learnt from the examples above (and those presented below), same dimension means that all variables should have the same dimension, as money (measured in DM), probability (measured in percent), weight (measured in kilogram), etc. – Positive and negative parts of the same dimensions are not genuinely interlinked, they originally belong to different scales that can be stitched in the zero-point, but may have different evaluations for the size of steps in the different parts, as the money scale. – The perception of the probability scale as two parts (probabilities [0,50], and counterprobabilities [50,100]) also needs stitching of two parts to a combined perception function where steps in both parts are perceived as equal. – Complex problems, as the evaluation of prospects, can involve more than one dimension in one problem, as money and probability. We do not (yet) have a general rule for the information processing in these situations.

10 Modifications of Traditional Concepts

The following examples show, how the transformation works. They also show that the obtained boundedly rational rules are simpler than the corresponding traditional formulas. The predictions fit to experimental results.

10.1 The Bargaining Problem

Task: Given a closed convex set $X \subset R^n$, and a payoff $c = (c_1, \ldots, c_n)$ which is enacted if the players 1,2 do not all agree to a joint solution (where

[5]This principle could be recently supported by an empirical study using scanner data of the food sector of a big supermarket (see for instance Fegel (1997).

$c \subset X - R^n_+$. Select a fair compromise $x = (x_1, \ldots, x_n)$ from X.

Denote $b = (b_1, \ldots, b_n)$, with $b_i := max\{x_i : x \in X\}$ for all i, as the bliss-point of the problem. – The approach of KALAI-SMORODINSKY selects the PARETO-optimal proportional solution, which can be supported by the criterion that the concession terms $(b_i - x_i)/(x_i - c_i)$ are identical for all players. By the The concession terms transform to

$$(b_i - x_i)/(x_i - c_i) \longrightarrow (per(b_i) - per(x_i)) - (per(x_i) - per(c_i))$$

The obtained criterion computes for every player the difference of the amount of her concessions, $per(b_i) - per(x_i)$, and the amount she received, $per(x_i) - per(c_i)$. For two-person situations this model has been checked, and turned out to be the concept with the best predictions (see VOGT-ALBERS 1997).

10.2 Modified Risk-Dominance

Task: Given a 2x2 bimatrix game with two equilibrium points in pure strategies, where Player 1 prefers (a_1, b_2), Player 2 prefers (b_1, a_2). If both play their favorite strategy, they get the conflict payoff, (c_1, c_2), if both deviate, they get the outcome of miscoordination, (m_1, m_2). (c_1, c_2) and (m_1, m_2) are in both components worse than the equilibrium payoffs.

b_1, a_2	c_1, c_2
m_1, m_2	a_1, b_2

The HARSANYI-SELTEN criterion of 'risk dominance' is to select that equilibrium point for which the concession term $(a_i - c_i)/(b_i - m_i)$ is higher. This transforms to

$$(a_i - c_i)/(b_i - m_i) \longrightarrow (per(a_i) - per(c_i)) - (per(b_i) + per(m_i))$$

The obtained concession term simply evalutates all quantitative arguments which can be posed by the payoffs of a player with the same weight, and only cares if the direction (sign) of the arguments. Again, the obtained modified concept turned out to be the best predictor in a series of experiments where the payoffs in the conflict point were different negative amounts, the miscoordination point was (0,0) (and not attended), and the subjects made their decisions by the strategy method after preplay negotiations. The fact that the criterion really hits the argumentation of the subjects in their preplay negotiations could be strongly supported by an analysis of the arguments of the players. (See VOGT 1994 or 1997.)

10.3 Cooperative Solution of Equilibrium Selection

Task: as in risk dominance, but conflict point and miscoordination point are not attended.

In this case the modified NASH-criterion which is transformed to

$$x_1 * \ldots * x_n = max \longrightarrow per(x_1) + \ldots + per(x_n) = max$$

gives the best predictor for the cooperative bargaining problem (see the experiments of VOGT 1994 or 1997).

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