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The Pazner-Schmeidler Social Ordering: A Defense

by

Marc Fleurbaey

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University of Bielefeld

33501 Bielefeld, Germany

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Marc Fleurbaey†

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Abstract

It is shown that the Pazner-Schmeidler social ordering appears as a very natural solution to the problem of defining social preferences over distributions of a fixed bundle of divisible goods. The paper follows an approach to preference aggregation which relies only on interpersonally non-comparable preferences, and circumvents Arrow's impossibility by taking account of the shape of indifference curves. Social preferences can then be constructed and justified with fairness principles.

JEL Classification: D63, D71.

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1 Introduction

In a seminal paper, Pazner and Schmeidler [9] proposed a concept of equity, 'egalitarian-equivalence', that was different from the standard no-envy criterion, and relied on the idea that all individuals should be put in a situation that is Pareto-indifferent (i.e., indifferent for all individuals) to a perfectly egalitarian allocation. Although their idea has been applied in many parts

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†CATT, THEMA, IDEP, Université de Pau. Email: marc.fleurbaey@univ-pau.fr

of the theory of fair allocation, it has not been as influential as it could have, for two reasons. First, the concept of egalitarian-equivalence has been, like most of the theory of fair allocation, restricted to the search for equitable first-best allocations and has not been translated into tools for addressing issues of second-best allocation and cost-benefit analysis. Second, it has often attracted the criticism that it is too vague, since in the presence of several goods, there are many ways in which the egalitarian allocation (to which all individuals should be indifferent) can be determined. Pazner and Schmeidler themselves proposed to look for an egalitarian bundle that would be proportional to the total consumption, but this particular choice was not really justified. Since then a suspicion of arbitrariness has always surrounded this concept. It is only very recently that Pazner's and Schmeidler's particular choice of the egalitarian bundle has been axiomatically justified, by Sprumont and Zhou [10]. Nonetheless, their remarkable result only deals with first-best allocation rules, and moreover relies on an axiom of solidarity that is commonly considered demanding, in the field of fair allocation.[1]

In several recent papers (in particular [2], [3]), Fleurbaey and Maniquet have shown that the theory of fair allocation can be extended so as to provide full-fledged rankings of allocations, and not only selections of small subsets of first-best allocations. The main feature of our approach is that, by taking account of enough information about individual preferences, one is able to define social preferences and circumvent Arrow's impossibility theorem, thanks to relaxing his restrictive condition of Independence of Irrelevant Alternatives.[2] Such social preferences aggregate individual (ordinal and noncomparable) preferences and satisfy valuable properties related to principles of efficiency, impartiality and equity. This new approach makes it possible to think about direct applications to second-best allocation problems and cost-benefit analysis.

What do the social preferences obtained in this way look like? In [2] and [3] Fleurbaey and Maniquet provided several results which give support to two different kinds of social preferences. One is related to the concept of Walrasian equilibrium, whereas the other is directly inspired from Pazner's and Schmeidler's concept of "egalitarian-equivalent" allocations, and is called

[1] Their axiom requires that all individuals gain or lose when the population size and/or the preferences change. They show in particular that assuming separately solidarity w.r.t. population changes and solidarity w.r.t. preference changes would not do.

[2] Why this condition can legitimately be relaxed is explained at length in Fleurbaey and Maniquet [3]. See Section 7 below for a definition and a short discussion.

the Pazner-Schmeidler ordering in the sequel. Both are precisely defined later on in this paper. In a nutshell, the Pazner-Schmeidler ordering gives priority to any individual who has the worst position, and such an individual is determined as the one who would accept to exchange his bundle for the lowest share of the total available resources. The Walrasian ordering also gives priority to some individuals, namely those who would willingly exchange their bundle for the lowest share of the total available resources, if the latter was used as an initial endowment for competitive trades at some relevant prices. (These prices are chosen so that this lowest share is the highest possible.)

In this paper, I propose an additional defense of the Pazner-Schmeidler ordering, pursuing with the same method of construction of full-fledged social preferences over allocations. My defense is based on the idea that it is quite appealing to apply egalitarian principles to allocations in which individual bundles are all proportional to each other and to the total available resources. On this basis one is inevitably led to consider the Pazner-Schmeidler ordering as the most justified.

In the next section the framework is presented and the Pazner-Schmeidler ordering is defined. Section 3 is devoted to the leading idea of this paper, and contains preliminary results that serve as a basis for four characterizations developed in the four following sections (from 4 to 7). The last two sections (7 and 8) discuss the results and conclude. The proofs are relegated to the appendix.

2 The Pazner-Schmeidler ordering

Following the literature (Pazner and Schmeidler [9], Sprumont and Zhou [10], among many others), this paper deals with the canonical problem of distributing a fixed bundle $\Omega \in \mathbb{R}^{\ell}_{++}$ of ℓ goods ($\ell \geq 2$) to n individuals ($n \geq 2$). Every individual $i = 1, ..., n$ has a preference ordering R_i over $\mathbb{R}^{\ell}_{+}$, which is assumed here to be monotonic ($x_i \geq y_i$ implies $x_i \; R_i \; y_i$ and $x_i \gg y_i$ implies $x_i \; P_i \; y_i$)[3] and continuous. Let $\mathcal{R}$ denote the set of such orderings. (Convexity of preferences could also be required without altering the results.)

An allocation is a list of bundles, one for each agent: $x = (x_1, ..., x_n) \in \mathbb{R}^{n\ell}_{+}$. A *social ordering function* (SOF) is a mapping which, for every economy defined by a number $n \geq 1$, a profile $(R_1, ..., R_n) \in \mathcal{R}^n$ and a bundle $\Omega \in \mathbb{R}^{\ell}_{++}$,

[3] Vector inequalities are denoted $\geq, >, \gg$.

determines a (social) ordering[4] over the set of allocations $\mathbb{R}_+^{n\ell}$.

Pazner and Schmeidler [9] proposed to select allocations which are Pareto-optimal, and such that every individual i is indifferent between his bundle and a particular bundle proportional to Ω, that is, such that for some real number λ, one has $x_i \ I_i \ \lambda\Omega$ for all i. As they themselves mention in their paper, notably in the proof of the existence of 'egalitarian-equivalent' allocations, this solution to the distribution problem can also be described by referring to the following SOF, denoted R_{PS} :

$$x \ R_{PS} \ y \Leftrightarrow \min_i v_i(x_i) \geq \min_i v_i(y_i),$$

where v_i is a representation of i's preferences defined by:

$$v_i(x_i) = \min\{v \mid v\Omega \ R_i \ x_i\}.$$

This SOF can be refined by replacing its maximin form by the lexicographic refinement of maximin, also called leximin. This yields a SOF which satisfies stronger versions of the Pareto principle in particular. But we will not focus on such refinements here, because for all practical purposes, the maximin criterion is enough. That is, in most conceivable applications, the maximin version of the Pazner-Schmeidler SOF presented here determines a unique, or essentially unique, best allocation. All refinements of the Pazner-Schmeidler SOF are then practically equivalent to it, and do not provide any better selection. For later purposes, let say that an ordering R' refines an ordering R whenever for any pair of allocations x, y,

$$xR'y \Rightarrow xRy.$$

3 The basic idea

Because the problem is to distribute a bundle Ω, it is quite natural to focus on allocations which are proportional distributions of this bundle. Let us say that allocation x is proportional to Ω whenever for all i, $x_i = \lambda_i\Omega$ for some non-negative real number λ_i. For such an allocation, let $\min_i x_i$ denote $(\min_i \lambda_i)\,\Omega$.

If equality of shares is retained as a guiding principle, then it is tempting to apply it to allocations proportional to Ω, in the sense of the maximin criterion. Namely, this would amount to the following property:

[4] An ordering is a reflexive, transitive and complete binary relation.

Maximin for Proportional Allocations: If x and y are two allocations proportional to Ω, then

$$\min_i x_i > \min_i y_i \Rightarrow x\ P\ y.$$

Another condition one would like to see satisfied by a social ordering is the Pareto criterion, which will be retained in its weak version here:

Weak Pareto: If x and y are such that for all i, $x_i\ P_i\ y_i$, then $x\ P\ y$.

Now, the combination of these two axioms lead us directly to the Pazner-Schmeidler SOF, as stated in the following lemma.

Lemma 1 *Any SOF which satisfies Weak Pareto and Maximin for Proportional Allocations is the Pazner-Schmeidler SOF or a refinement.*

This basic result paves the way for a series of characterizations, below, that exploit the fact that Maximin for Proportional Allocations, arguably quite demanding and restrictive, is actually not very hard to justify with more elementary axioms. This is the topic of the next sections.

An additional point worth being mentioned is that under a continuity requirement the above lemma can be strengthened. Continuity of the social ordering is a convenient property in applications where optimization over a compact subset of allocations is performed.

Continuity: For all economies, the social ordering is continuous.

Lemma 2 *The Pazner-Schmeidler SOF is the only one satisfying Weak Pareto, Maximin for Proportional Allocations and Continuity.*

4 First characterization

A first idea is to rely on the theorems that characterize the maximin or the leximin in the theory of social choice. In particular, Hammond's [6] concept of equity can be adapted here quite naturally. His initial formulation had to do with comparable welfare levels, and said that it is socially acceptable to reduce the inequality of utilities between two individuals, by increasing the worse-off's utility and decreasing the better-off's utility. Here we deal with bundles and preferences, not with utilities, but if one is interested in the egalitarian sharing of resources, the same idea can be applied to bundles of commodities whenever they are unambiguously comparable. This is indeed the case for allocations proportional to Ω, since all bundles are then comparable by vector inequalities.

Hammond Equity for Proportional Allocations: If x and y are two allocations proportional to Ω, such that for two agents i and j,

$$y_i > x_i > x_j > y_j,$$

whereas for all other agents k, $x_k = y_k$, then $x \; R \; y$.

Notice that, from Fleurbaey and Trannoy [4], it can be deduced that such an axiom must be written for allocations proportional to one vector Ω only, and would be incompatible with Weak Pareto if it were required to hold for more than one vector (more precisely, for more than one direction in the space of goods). If one has to choose only one direction in the space of goods to formulate such a condition, the direction defined by Ω is certainly the most appealing.

Tungodden [13] has recently shown that, if one is interested in the maximin criterion and not in the leximin refinement, then Hammond Equity can be weakened substantially, and be replaced with something like the following axiom. This axiom says that it is enough to focus on the worst-off and the best-off agents in the whole population. If the inequality between the subgroups of worst-off and best-off agents is reduced, while these agents remain the worst-off and the best-off, then the situation is not considered worsened. Let $W(x)$ and $B(x)$ denote the subgroups of worst-off agents and best-off agents in an allocation x proportional to Ω :

$$\begin{aligned} W(x) &= \{i \mid x_i = \min_j x_j\} \\ B(x) &= \{i \mid x_i = \max_j x_j\} \end{aligned}$$

Minimal Equity for Proportional Allocations I: If x and y are two allocations proportional to Ω, with $W(x) = W(y)$ and $B(x) = B(y)$, and such that

$$\begin{aligned} \forall i &\in W(x),\; y_i < x_i \\ \forall i &\in B(x),\; y_i > x_i, \end{aligned}$$

whereas for all agents k outside $W(x) \cup B(x)$, $x_k = y_k$, then $x \; R \; y$.

Another possible weakening of Hammond Equity, inspired by some other axioms proposed by Tungodden, is exemplified by the next axiom. In words, the next axiom says that if there is one worst-off agent and one best-off agent, then it is socially acceptable to reduce the inequality between the two, provided the best-off is not pulled down below the mean, which is denoted, for any allocation x, $\mu(x) = \frac{1}{n}\sum_i x_i$.

Minimal Equity for Proportional Allocations II: If x and y are two allocations proportional to Ω, with $B(y) = \{i\}$ and $W(y) = \{j\}$, and such that

$$y_i > x_i > \max\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y)\} > x_j > y_j,$$

whereas for all other agents k, $x_k = y_k$, then $x \, R \, y$.

It is easy to check that these two axioms are logically implied by Hammond Equity. Combining Tungodden's results and the above lemma, one gets the following result, whose proof is given for the sake of completeness.

Theorem 1 *Any SOF which satisfies Weak Pareto and Minimal Equity for Proportional Allocations (I or II) is the Pazner-Schmeidler SOF or a refinement. Under the additional requirement of Continuity it is the Pazner-Schmeidler SOF.*

5 Second characterization

A variant of the above idea is to use the Hammond Equity condition at a more general level, relying on an axiom proposed by Fleurbaey and Maniquet [3]. This condition applies to individuals with identical preferences, when they agree that one of them has a strictly better bundle than the other, and this inequality is reduced, in particular by giving a better bundle to the one who has the worse bundle. The appeal of this axiom is rather obvious, and can be related to the idea of a Rawlsian priority given to the agents who have the worst bundles, when there is agreement among concerned individuals about how bundles should be ranked. It can also be justified on grounds of reducing the intensity of envy, since the agent with the worse bundle envies the other one, while the other does not envy him.

Maximin Treatment of Equals: If x and y are two allocations, and i and j are two agents with identical preferences denoted R_0, such that

$$y_i \; P_0 \; x_i \; P_0 \; x_j \; P_0 \; y_j,$$

whereas for all other agents k, $x_k = y_k$, then $x \; R \; y$.

We will actually make use here of a much weaker version of this axiom, in particular allowing for any degree of inequality aversion, in the fashion of the Pigou-Dalton transfer principle extended to multidimensional bundles as in Fleurbaey and Trannoy [4]. The standard Pigou-Dalton principle says that in a uni-dimensional setting (individuals are described by their income, for instance), it is socially acceptable to transfer a given amount of resource from one agent to another who is poorer, if the transfer does not reverse the ranking. This can be extended to multi-dimensional bundles by considering a transfer of a part of a bundle which has more of all goods than another bundle.

Again, Fleurbaey and Trannoy [4] show that the multi-dimensional Pigou-Dalton principle should not be applied to individuals with different preferences, because of a resulting incompatibility with Weak Pareto. Restricting its application to individuals with identical preferences avoids this problem. The next axiom, then, says that it is socially acceptable to transfer part of a bundle from one agent to another when, both ex ante and ex post, the agent who gives has more of all goods.

Transfer Principle for Equals: If x and y are two allocations, and i and j are two agents with identical preferences, such that for some $\delta \gg 0$,

$$y_i - \delta = x_i \gg x_j = y_j + \delta,$$

whereas for all other agents k, $x_k = y_k$, then $x \; R \; y$.

The motivation that led to the formulation of the axioms of the previous sections can now be formulated also in a more general way, as the idea that proportional allocations can be assessed independently of individual preferences. This condition can be justified on grounds of simplicity, or by the argument that our ethical intuition is firm enough in the case of proportional allocations, so that we do not have to look at individual preferences. But, contrary to the axioms of the previous section, this one does not say anything about the *content* of social preferences over proportional allocations. They could be anti-egalitarian.

Independence of Preferences for Proportional Allocations: If x and y are two allocations proportional to Ω, and R, R' are the social orderings for any profiles $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ respectively, then

$$x \, R \, y \Leftrightarrow x \, R' \, y.$$

One then obtains the following result:

Theorem 2 *Any SOF which satisfies Weak Pareto, Transfer Principle for Equals and Independence of Preferences for Proportional Allocations is the Pazner-Schmeidler SOF or a refinement. Under the additional requirement of Continuity it is the Pazner-Schmeidler SOF.*

It is worth emphasizing that none of the axioms involved in this result implies a positive inequality aversion by itself. It is only when combining them that a priority for the worst-off is derived.

6 Third characterization

A somewhat different route can be proposed, which also introduces the infinite inequality aversion typical of the maximin criterion and related orderings, in a quite indirect way.

Let us indeed consider a minimal requirement saying essentially that inequality aversion must be non-negative, but arbitrarily low, regarding allocations proportional to Ω. That is, equalizing a proportional allocation by giving the average bundle to all agents is socially acceptable. This axiom is very intuitive, and can be related to a traditional notion of fairness, according to which no individual should ever be given less than the average consumption (Steinhaus [11]). Sprumont and Zhou [10], as well as Fleurbaey and Maniquet [3], also make use of similar conditions.

Minimal Egalitarianism for Proportional Allocations: If x and y are two allocations proportional to Ω, such that for all i,

$$x_i = \frac{1}{n} \sum_j y_j,$$

then $x \, R \, y$.

Another feature of some interesting SOFs is that, when assessing a particular allocation, they focus on the positions of the various indifference curves observed in the economy, without taking account of the size of the subgroups of the population located at the various positions. This property is satisfied by the Pazner-Schmeidler SOF but also by other interesting SOFs (see next section). This is not just a feature that relates to informational parsimony, it also has ethical implications that philosophers (e.g. Parfit [8]) describe as the precept that in some relevant circumstances, "numbers do not count". Saving one individual from a severe predicament is sometimes a priority that should override a small gain given to affluent people, no matter how numerous they are. This idea that numbers do not count can be formulated as follows.

Independence of Cloning: Let R denote the social ordering in a particular economy with n agents, and R' the social ordering in another economy derived from the first one by introducing an additional agent $n+1$ identical to agent i. Then, for any pair of allocations x and y,

$$x \, R \, y \Rightarrow (x_1, ..., x_n, x_i) R' (y_1, ..., y_n, y_i).$$

One can, however, criticize this neglect of numbers as being too demanding. In particular, it is easy to show that it is incompatible with stronger versions of the Pareto principle, for impartial social orderings. For instance, reasoning on real numbers (or proportional allocations) for simplicity, the distribution (1,2) is, for any impartial social ordering, socially equivalent to (2,1). But cloning the first agent leads to the distributions (1,1,2) and (2,2,1), respectively, which should not be considered equivalent. Notice that, nonetheless, the maximin criterion itself is indifferent between them. This problem with the maximin criterion is usually solved by refining it in a lexicographic way, leading to the leximin criterion. The latter, however, is not continuous, and, for practical purposes (choosing the best allocation in a compact subset), is in no way superior to the maximin. The Pazner-Schmeidler SOF itself is of the maximin sort and totally neglects the demographic sizes of social strata.

We will nonetheless introduce the idea of neglecting numbers, here, in a less abrupt way, and the condition we will use is fully compatible with lexicographic refinements, and strong versions of the Pareto principle. We achieve this by combining cloning of agents with increments to initial bundles.

Incremental Cloning: Let R denote the social ordering in a particular economy with n agents, and R' the social ordering in another economy derived from the first one by introducing an additional agent $n+1$ identical to agent i. Then, for any pair of allocations x and y, and any bundle $b \in \mathbb{R}^{\ell}_{++}$,

$$x \, R \, y \Rightarrow (x_1 + b, ..., x_n + b, x_i + b) R' (y_1, ..., y_n, y_i).$$

One then obtains the following result.

Theorem 3 *Any SOF which satisfies Weak Pareto, Minimal Egalitarianism for Proportional Allocations and Incremental Cloning is the Pazner-Schmeidler SOF or a refinement. Under the additional requirement of Continuity it is the Pazner-Schmeidler SOF.*

7 Discussion and fourth characterization

Some light may be shed on the above results, by looking for other SOFs satisfying different combinations of the various axioms. It is immediate to check that the sets of axioms used in the characterizations are logically independent. We will rather focus here on the role played by the various axioms, as illustrated by other SOFs.

First, the Weak Pareto condition is the only one to imply strict preference over some allocations. The SOF that is totally indifferent between all allocations does satisfy all the other axioms. If one is looking for a non-degenerate SOF, one may consider the SOF $R_{\min\max}$ defined by:

$$x \, R_{\min\max} \, y \Leftrightarrow \max_i v_i(x_i) \leq \max_i v_i(y_i),$$

which also satisfies all axioms except Weak Pareto. (A table at the end of this discussion recalls all the axioms.)

Second, let us consider an example of SOF for which proportional allocations do not play the same important role. The Walrasian SOF R_W introduced by Fleurbaey and Maniquet [2] can be defined by:

$$x \, R_W \, y \Leftrightarrow \max_p \min_i u_i(x_i, p) \geq \max_p \min_i u_i(y_i, p),$$

where u_i is a money-metric utility function computed as the fraction of the value of Ω that the agent needs in order to reach the current satisfaction:

$$u_i(x_i, p) = \frac{1}{p\Omega} \min \left\{ pq \mid q \in \mathbb{R}^{\ell}_{+},\ q\ R_i\ x_i \right\}.$$

An equivalent, more graphical, definition, goes by saying that this SOF relies on the minimal bundle proportional to Ω and contained in the convex hull of the union of the individual closed upper contour sets. Now, looking at properties, the Walrasian SOF does not satisfy any of the axioms focussing on allocations proportional to Ω, but it does satisfy the others: Weak Pareto, Maximin Treatment of Equals and Independence of Cloning (and the weaker counterparts of the last two).

Another direction is to abandon the inequality aversion embodied in the Pazner-Schmeidler SOF. Consider a utilitarian-like SOF R_{U-PS} defined as follows:

$$x\ R_{U-PS}\ y \Leftrightarrow \sum_i v_i(x_i) \leq \sum_i v_i(y_i).$$

This SOF does satisfy Independence of Preferences for Proportional Allocations, as well as Minimal Egalitarianism for Proportional Allocations. It also satisfies Weak Pareto, but none of the other axioms. For the Minimal Equity axioms (I and II), Maximin Treatment of Equals, as well as for Incremental Cloning, this is not surprising because they involve either a strong inequality aversion, or a neglect of numbers, that are incompatible with an additive criterion. It must be noted, however, that no such incompatibility plagues the axiom of Transfer Principle for Equals. This axiom is indeed satisfied by a more standard utilitarian SOF R_U which, for every possible individual preference relation R, defines a concave utility function U_R, and aggregates over the utilities thus obtained:

$$x\ R_U\ y \Leftrightarrow \sum_i U_{R_i}(x_i) \leq \sum_i U_{R_i}(y_i).$$

This SOF satisfies Transfer Principle for Equals but not Maximin Treatment of Equals. It is, therefore, only by combination with an axiom focussing on proportional allocations that Transfer Principle for Equals eventually leads to an infinite inequality aversion. The SOF R_U also satisfies Weak Pareto, but none of the other axioms. This discussion is summarized in the following table.

	R_{PS}	$R_{\min\max}$	R_W	R_{U-PS}	R_U
Minimal Equity I for P. A.	√	√			
Minimal Equity II for P. A.	√	√			
Ind. of Pref. for P. A.	√	√		√	
Minimal Egalitarianism for P. A.	√	√		√	
Weak Pareto	√		√	√	√
Maximin Treatment of Equals	√	√	√		
Transfer Principle for Equals	√	√	√		√
Independence of Cloning	√	√	√		
Incremental Cloning	√	√	√		

A natural question, now, is whether combining Transfer Principle for Equals with Minimal Egalitarianism for Proportional Allocations, instead of Independence of Preferences for Proportional Allocations, would also push us in the direction of the Pazner-Schmeidler SOF. Notice that the combination of Independence of Preferences for Proportional Allocations and Transfer Principle for Equals, featured in Theorem 2 above, does imply Minimal Egalitarianism for Proportional Allocations, so that the question raised has to do with a strengthening of Theorem 2.

The answer to the question is prima facie negative, since the SOF that coincides with R_{PS} when there are individuals with identical preferences, and with R_{U-PS} otherwise, does satisfy Weak Pareto, Transfer Principle for Equals, and Minimal Egalitarianism for Proportional Allocations. But this peculiar SOF is not appealing because it is too sensitive to slight changes in preferences. In particular, it violates the following condition, due to Hansson [7]. It is a weakening of Arrow's Independence of Irrelevant Alternatives (Arrow [1]). While Arrow's condition requires social preferences on a pair of allocations to depend only on individual preferences over this pair, Hansson's condition requires social preferences on a pair of allocations to depend only on individual closed upper contour sets at these allocations. In other words, only indifference curves at the bundles under consideration should matter, and the rest of the preference relations can be disregarded. This condition is very appealing because it guarantees that social preferences will not be sensitive to far-fetched details of the preferences. At the same time, it makes it possible to take account of relevant features, such as marginal rates of substitution, which are excluded by Arrow's extremely restrictive condition.

Hansson's condition is satisfied by all examples of SOFs in the above table, except R_U.

Hansson Independence: Let x and y be two allocations, and R, R' be the social orderings for two profiles $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ respectively. If for all i, all $q \in \mathbb{R}^\ell_+$,

$$\begin{array}{rcl} x_i \; I_i \; q & \Leftrightarrow & x_i \; I'_i \; q \\ y_i \; I_i \; q & \Leftrightarrow & y_i \; I'_i \; q, \end{array}$$

then

$$x \; R \; y \Leftrightarrow x \; R' \; y.$$

Now, if one adds this condition to the list, then the Pazner-Schmeidler SOF is remarkably singled out once more.

Theorem 4 *Any SOF which satisfies Weak Pareto, Transfer Principle for Equals, Minimal Egalitarianism for Proportional Allocations and Hansson Independence is the Pazner-Schmeidler SOF or a refinement. Under the additional requirement of Continuity it is the Pazner-Schmeidler SOF.*

This characterization is the most striking of the four provided here, because two important features of the Pazner-Schmeidler SOF are introduced in a very indirect way: the infinite inequality aversion, and the focus on proportional allocations. Infinite inequality aversion is not present in either Transfer Principle for Equals, Minimal Egalitarianism for Proportional Allocations or Hansson Independence, which are, separately, compatible with indifference to inequality. And of all axioms provided here about proportional allocations, Minimal Egalitarianism for Proportional Allocations is the least constraining with respect to the focus on proportional allocations, since it allows a wide array of rankings of proportional allocations, with such rankings possibly depending on individual preferences in complex ways.

8 Conclusion

The main point of the paper is that the Pazner-Schmeidler SOF can easily be defended in the simple distribution problem, if one accepts the idea that allocations proportional to the bundle to be distributed deserve a special status in the analysis.

In addition, this paper provides a new illustration of the approach proposed by Fleurbaey and Maniquet [2] in order to construct social preferences on the basis of individual preferences. By relaxing Arrow's Independence and replacing it by Hansson Independence, it is definitely possible to find social preferences which satisfy not only Weak Pareto and minimal conditions of impartiality such as Arrow's non-dictatorship axiom, but also many appealing principles of fairness. The axiomatic method is not always well considered, in particular because of its propensity to endlessly produce axioms and impossibility results. But the impression that should be retained here is that, although there is certainly no limit to the formulation of appealing equity principles, it is reassuring that the results consistently point toward a very small (but strictly positive!) number of solutions.

Another methodological lesson that may be drawn here is that it appears simpler to defend the Pazner-Schmeidler first-best solution to the distribution problem (that is, the egalitarian-equivalent allocation rule) on the basis of the related social ordering, than directly. As explained in the introduction, it is only recently that Sprumont and Zhou [10] have axiomatically characterized the Pazner-Schmeidler first-best allocation rule, and these authors stress, in an implicit criticism of the Pazner-Schmeidler allocation rule, that the axioms involved in their result look rather restrictive (requiring solidarity of agents with respect to changes in both preferences and size of the population). That the analysis of the *ordering* is simpler than the analysis of the *allocation rule* is not very surprising, since the egalitarian-equivalent allocation is naturally constructed as the maximal element of this simple social ordering.

In contrast, one can compare the rather complex result of Fleurbaey and Maniquet [3] with respect to the Walrasian social ordering function R_W, relying in particular on an axiom that bears on the allocation rule derived from the social ordering, to the simple and elegant characterizations of the Walrasian allocation rule obtained by Gevers [5], Thomson [12], among others. For the Walrasian case it seems simpler to focus on the allocation rule rather than on the social preferences.

These comparisons suggest that the theory of fair allocation should be flexible and consider the analysis of allocation rules and the analysis of social preferences as two complementary branches of the same important enterprise: Formulating precise expressions of fairness principles and deriving appealing solutions to conflicts of interests. This ecumenical conclusion has, however, a caveat: Fine-grained social preferences are more powerful tools than first-best allocation rules when the latter cannot be implemented and one has to fall back on second-best solutions.

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Appendix: Proofs

Proof of Lemma 1: Let R be a SOF satisfying the two axioms. It is enough to show that $x\ R\ y$ implies $x\ R_{PS}\ y$, or, equivalently, that $y\ P_{PS}\ x$ implies $y\ P\ x$. Assume, then, that x and y are two allocations such that $y\ P_{PS}\ x$. Let

$$\varepsilon = \frac{\min_i v_i(y_i) - \min_i v_i(x_i)}{10}.$$

Let x' and y' be two proportional allocations such that for all i,

$$\begin{aligned} x'_i &= (v_i(x_i) + \varepsilon)\,\Omega, \\ y'_i &= (v_i(y_i) - \varepsilon)\,\Omega. \end{aligned}$$

By construction, for all i, $x'_i\ P_i\ x_i$ and $y_i\ P_i\ y'_i$. So that, by Weak Pareto, $x'\ P\ x$ and $y\ P\ y'$.

Moreover, $\min_i y'_i > \min_i x'_i$, so that by Maximin for Proportional Allocations, $y'\ P\ x'$.

By transitivity, then, $y\ P\ x$. ■

Proof of Lemma 2: In view of the above lemma, one only has to prove that

$$\min_i v_i(x_i) = \min_i v_i(y_i) \Rightarrow x\ I\ y.$$

Let x be an allocation such that $\min_i v_i(x_i) > 0$. Let $\alpha = \min_i v_i(x_i)$. And define a sequence of allocations $(x^k)_{k\geq 1}$ by

$$x_i^k = \alpha\left(1 - \frac{1}{k}\right)\Omega$$

for all i. And let $x_i^\infty = \lim_{k\to\infty} x_i^k = \alpha\Omega$. By Weak Pareto, for all k, $x\ P\ x^k$, so that by Continuity, $x\ R\ x^\infty$.

Now, define a sequence of allocations $(y^k)_{k\geq 1}$ by

$$y_i^k = \alpha \left(1 + \frac{1}{k}\right) \Omega$$

for all i. One has $\lim_{k\to\infty} y^k = x^\infty$. By the previous lemma, for all k, $y^k \; P \; x$, so that by Continuity, $x^\infty \; R \; x$. In conclusion, $x \; I \; x^\infty$.

Now, consider another allocation x such that $\min_i v_i(x_i) = 0$. Let x^k be a sequence of allocations with $\min_i v_i(x_i^k) > 0$ for all k and such that $\lim_{k\to\infty} x^k = x$. For every k, every i, define

$$x_i^{k\infty} = \left(\min_j v_j(x_j^k)\right) \Omega.$$

By the above argument, one has $x^k \; I \; x^{k\infty}$ for all k. By continuity, one then has $x \; I \; x^{\infty\infty}$, where $x_i^{\infty\infty} = 0$ for all i.

Therefore, for every allocation x, it holds true that $x \; I \; x^\infty$, where $x_i^\infty = \left(\min_j v_j(x_j^k)\right) \Omega$ for all i. As a consequence, for any pair of allocations x, y, one has

$$\min_i v_i(x_i) = \min_i v_i(y_i) \Rightarrow x \; I \; y. \qquad \blacksquare$$

Proof of Theorem 1: In view of the lemmas, it is enough to prove that Weak Pareto and Minimal Equity for Proportional Allocations (I or II) imply Maximin for Proportional Allocations. Let x and y be two allocations, such that $\min_i x_i > \min_i y_i$.

Minimal Equity I: Let $\alpha = \min_i x_i - \min_i y_i$. Define a new allocation y' as follows. For all $i \in W(y)$, $y'_i = y_i + \alpha/10$. For all $i \notin W(y)$, $y'_i = \max_j x_j + \max_j y_j$. By Weak Pareto, $y' \; P \; y$.

One has $W(y') = W(y)$ and $B(y') = \{1, ..., n\} \setminus W(y)$. Construct allocation z as follows. For all $i \in W(y)$, $z_i = y_i + \alpha/5$. For all $i \notin W(y)$, $z_i = y_i + \alpha/2$. By Minimal Equity I, $z \; R \; y'$, so that by transitivity, $z \; P \; y$. Now, by Weak Pareto, $x \; P \; z$. As a consequence, $x \; P \; y$.

Minimal Equity II: Define a new allocation y' such that for all i, $y'_i \; P_i \; y_i$ and there is m such that $W(y') = \{m\}$, and M such that $W(y') = \{M\}$. Now, consider y'' such that for all $k \neq m, M$, $y''_k = y'_k$, while

$$y'_M > y''_M > \frac{1}{n} \sum_i y'_i \geq \frac{1}{n} \sum_i y''_i \geq y''_m > y'_m.$$

By Minimal Equity II, $y'' \; R \; y'$, and by Weak Pareto $y' \; P \; y$, so that $y'' \; P \; y$. Notice that, by suitably choosing y' and y'', it is possible to have y'' arbitrarily close to the allocation y^1 such that $y_i^1 = y_i$ for all i except one j such that $y_j = \max_i y_i$ and $y_j^1 = \frac{1}{n}\sum_i y_i$.

By a similar reasoning it is possible to construct an allocation strictly preferred to y and arbitrarily close to y^2 defined by $y_i^2 = y_i^1$ for all i except one j such that $y_j^1 = \max_i y_i^1$ and

$$\begin{aligned} y_j^2 &= \frac{1}{n}\sum_i y_i^1 \\ &= \frac{1}{n}\sum_i y_i - \frac{1}{n}\left(\max_i y_i - \frac{1}{n}\sum_i y_i\right). \end{aligned}$$

By iteration, one can construct an allocation strictly preferred to y and with all individual bundles arbitrarily close to $\min_i y_i$. Such an allocation can then be chosen with all bundles less than $\min_i x_i$. Then, by Weak Pareto and transitivity, one concludes that $x\ P\ y$. ■

Proof of Theorem 2: Let x and y be two allocations proportional to Ω, such that

$$\min_i x_i > \min_i y_i.$$

Step 1. We will rely on the following fact: One can go from y to x by a sequence of proportional allocations $z^1, ..., z^T$ such that $z^1 = y$, $z^T = x$, and for all $t = 1, ..., T-1$, either (case 1) $z_i^{t+1}\ P_i\ z_i^t$ for all i, or (case 2) for two agents i and j,

$$z_i^t > z_i^{t+1} > z_j^{t+1} > z_j^t,$$

with $z_i^t - z_i^{t+1} > z_j^{t+1} - z_j^t$, whereas for all other agents k, $z_k^{t+1} > z_k^t$.

The proof of this fact is as follows. Let m be such that $y_m = \min_i y_i$, $S = \{i \mid y_i > y_m\}$, $\bar{y} = \min_{i\in S} y_i$ and $\varepsilon \in \mathbb{R}_{++}$ be such that

$$\varepsilon\Omega < \frac{1}{n+1}\left(\min\{\bar{y}, \min_i x_i\} - y_m\right).$$

Let $T = |S| + 2$. At every step $t = 1, ..., T-2$, let $z_i^{t+1} = z_m^t + 2\varepsilon\Omega$ for some agent $i \in S$, while for all $k \neq i$, $z_k^{t+1} = z_k^t + \varepsilon\Omega$. In particular, $z_m^{t+1} = y_m + t\varepsilon\Omega$. This corresponds to case 2, since

$$\begin{aligned} z_i^{t+1} &= z_m^t + 2\varepsilon\Omega \\ &= y_m + (t-1)\varepsilon\Omega + 2\varepsilon\Omega \\ &= y_m + (t+1)\varepsilon\Omega \\ &< y_m + \frac{t+1}{n+1}(\bar{y} - y_m) \leq \bar{y} \leq z_i^t. \end{aligned}$$

Let us check that $z_i^t - z_i^{t+1} > z_j^{t+1} - z_j^t$. One has

$$\begin{aligned} z_i^t - z_i^{t+1} &= z_i^t - (z_m^t + 2\varepsilon\Omega) \\ &\geq \bar{y} - (y_m + (t+1)\varepsilon\Omega) \end{aligned}$$

and

$$\bar{y} - (y_m + (t+1)\varepsilon\Omega) \geq z_m^t - z_m^{t-1} = \varepsilon\Omega$$

is true if

$$\varepsilon\Omega < \frac{1}{t+2}(\bar{y} - y_m),$$

which is indeed the case since $t + 2 \leq T \leq n + 1$.

One has $z_i^{T-1} < \min_i x_i$ for all i. Indeed, by construction, for all i,

$$z_i^{T-1} \leq z_m^{T-2} + 2\varepsilon\Omega = y_m + (T-1)\varepsilon\Omega \leq y_m + n\varepsilon\Omega < \min_i x_i.$$

The last step from z^{T-1} to $z^T = x$ then corresponds to case 1. This achieves the proof of the fact.

Step 2. We now proceed with the proof of the theorem. In case 1, Weak Pareto implies $z^{t+1}\ P\ z^t$. Let us show that $z^{t+1}\ P\ z^t$ also holds in case 2. Assume first that i and j have the same preferences, and that these preferences are such that there exists a bundle w satisfying: $w \gg z_j^{t+1}$, $z_i^{t+1}\ P_i\ w$ and $w + (z_j^{t+1} - z_j^t)/2\ P_i\ z_i^t$.

Such particular preferences can be constructed, for instance, by taking $w = (z_{i1}^t, z_{i2}^{t+1} - \varepsilon, ..., z_{i\ell}^{t+1})$, for $\varepsilon > 0$ sufficiently small so that $w \gg z_j^{t+1}$ and $w + (z_j^{t+1} - z_j^t)/2 \gg z_i^{t+1}$, and letting

$$\begin{aligned} \{q &\in \mathbb{R}_+^\ell \mid q\ R_i\ z_i^{t+1}\} = \{q \in \mathbb{R}_+^\ell \mid q \geq z_i^{t+1}\} \\ \{q &\in \mathbb{R}_+^\ell \mid q\ R_i\ w + (z_j^{t+1} - z_j^t)/2\} = \{q \in \mathbb{R}_+^\ell \mid q \geq w + (z_j^{t+1} - z_j^t)/2\}. \end{aligned}$$

One then has $z_i^{t+1}\ P_i\ w$ since $w_2 < z_{i2}^{t+1}$. And $w + (z_j^{t+1} - z_j^t)/2\ P_i\ z_i^t$ because

$$z_{i1}^t < w_1 + \frac{1}{2}(z_{j1}^{t+1} - z_{j1}^t).$$

Let z' be defined by $z_i' = w + (z_j^{t+1} - z_j^t)/2$, $z_j' = z_j^t + (z_j^{t+1} - z_j^t)/4$, and $z_k' = (z_k^t + z_k^{t+1})/2$ for all other k. By Weak Pareto, $z'\ P\ z^t$. Then, letting $\delta = (z_j^{t+1} - z_j^t)/2$, one can apply Transfer Principle for Equals to i and j and obtain that $z''\ R\ z'$, where $z_i'' = w$, $z_j'' = z_j^t + 3(z_j^{t+1} - z_j^t)/4$, and $z_k'' = z_k'$ for all other k. Now, by Weak Pareto, one has $z^{t+1} \; P \; z''$, so that, by transitivity, $z^{t+1} \; P \; z^t$.

By Independence of Preferences for Proportional Allocations, this still holds for any preferences of i and j.

Step 3. We have therefore proved that $x \; P \; y$, which means that the SOF satisfies Maximin for Proportional Allocations. The result then follows from the two lemmas. ■

Proof of Theorem 3: The proof consists in showing that Maximin for Proportional Allocations must be satisfied as well. Let x and y be two allocations proportional to Ω, such that $\min_i x_i > \min_i y_i$. Let a be an integer large enough so that

$$\frac{1}{n+a}\left(\sum_i y_i + a \min_i y_i\right) < \min_i x_i.$$

Then find a bundle ε proportional to Ω, being small enough so that

$$\frac{1}{n+a}\left(\sum_i y_i + a \min_i y_i + [na + a(a+1)/2]\,\varepsilon\right) < \min_i x_i.$$

Let m be an agent such that $y_m = \min_i y_i$. Let R be the social ordering in the initial economy with n agents, and R' the social ordering in the economy with $n+a$ agents, where m is cloned a times.

Assume that $y \; R \; x$. Then, by repeated applications of Incremental Cloning,

$$(y_1 + a\varepsilon, ..., y_n + a\varepsilon, \underbrace{y_m + \varepsilon, ..., y_m + a\varepsilon}_{a}) \; R \; (x_1, ..., x_n, \underbrace{x_m, ..., x_m}_{a}).$$

By Minimal Egalitarianism for Proportional Allocations, one has

$$y' \; R \; (y_1 + a\varepsilon, ..., y_n + a\varepsilon, \underbrace{y_m + \varepsilon, ..., y_m + a\varepsilon}_{a}),$$

where y' is defined by, for all i :

$$y_i' = \frac{1}{n+a}\left(\sum_i y_i + na\varepsilon + a \min_i y_i + \frac{a(a+1)}{2}\varepsilon\right).$$

By transitivity one has

$$y' \; R \; (x_1, ..., x_n, \underbrace{x_m, ..., x_m}_{a})$$

but this contradicts the fact that $y'_i < \min_i x_i$ for all i, so that by Weak Pareto, one actually has

$$(x_1, ..., x_n, \underbrace{x_m, ..., x_m}_{a}) \; P \; y'.$$

■

Proof of Theorem 4: For any particular individual preference relation R_i, let

$$uc_{R_i}(x_i) = \left\{q \in \mathbb{R}^{\ell}_{+} \mid q \; R_i \; x_i\right\}.$$

We first prove the following fact: Weak Pareto, Transfer Principle for Equals and Hansson Independence imply the following property, named P for further reference. Let x and y be two allocations proportional to Ω, and i and j two agents with identical preferences denoted R_0, such that

$$y_i > x_i > x_j > y_j,$$

whereas for all other agents k, $x_k \gg y_k$. Let us assume moreover that

$$\begin{aligned} uc_{R_0}(x_i) &= \left\{q \in \mathbb{R}^{\ell}_{+} \mid q \geq x_i\right\} \\ uc_{R_0}(y_i) &= \left\{q \in \mathbb{R}^{\ell}_{+} \mid q \geq y_i\right\}. \end{aligned}$$

Then $x \; R \; y$.

The property is obvious if $y_i - x_i \leq x_j - y_j$, and we focus on the hard case $y_i - x_i > x_j - y_j$. Let

$$\alpha = \frac{1}{3}(x_j - y_j), \;\; \beta = \frac{1}{4}\alpha.$$

Let $z = (y_{i1} - \alpha_1, x_{i2} + \beta_2, ..., x_{i\ell} + \beta_\ell)$, and R'_0 be an individual preference relation such that for all $k = i, j$, $uc_{R'_k}(x_k) = uc_{R_0}(x_k)$ and $uc_{R'_k}(y_k) = uc_{R_0}(y_k)$, while

$$uc_{R'_0}(z) = \left\{q \in \mathbb{R}^{\ell}_{+} \mid q \geq z\right\},$$

which is possible since $z \gg x_i$. Let R' be the social ordering when agents i's and j's preferences are changed to R'_0. By Hansson Independence, $x \; R \; y$ if and only if $x \; R' \; y$.

Let y' be defined by $y'_i = y_i - \alpha$, $y'_j = y_j + \alpha$, and $y'_k = y_k$ for all $k \neq i, j$. By Transfer Principle for Equals, $y' \, R' \, y$. Let y'' be defined by $y''_i = z + \beta$, $y''_j = y'_j + \beta$, and $y''_k = y_k + \frac{1}{2}(x_k - y_k)$ for all $k \neq i, j$. Notice that $z \, I'_0 \, y'_i$, so that $y''_i \, P_i \, y'_i$. By Weak Pareto, $y'' \, P' \, y'$.

Let x' be defined by $x'_i = y''_i - \alpha = z + \beta - \alpha$, $x'_j = y''_j + \alpha = y_j + 2\alpha + \beta$, and $x'_k = y''_k$ for all $k \neq i, j$. By Transfer Principle for Equals, $x' \, R' \, y''$. But

$$\begin{aligned} x'_i &= (y_{i1} - \alpha_1, x_{i2} + \beta_2, ..., x_{i\ell} + \beta_\ell) + \beta - \alpha \\ &= (y_{i1} + \beta_1 - 2\alpha_1, x_{i2} + 2\beta_2 - \alpha_2, ..., x_{i\ell} + 2\beta_\ell - \alpha_\ell) \end{aligned}$$

and since $x_{i2} + 2\beta_2 - \alpha_2 < x_{i2}$, one has $x_i \, P'_0 \, x'_i$. Similarly, $x_j \, P'_0 \, x'_j$, since

$$x'_j = y_j + 2\alpha + \beta = y_j + \frac{3}{4}(x_j - y_j) \ll x_j.$$

And $x'_k \ll x_k$ for all $k \neq i, j$. Therefore, by Weak Pareto, one has $x \, P' \, x'$. Summarizing, by transitivity, one has $x \, R' \, y$, and therefore $x \, R \, y$, as was to be proved. This achieves the proof of Property P.

The rest of the proof consists in showing that Property P combined with Weak Pareto, Minimal Egalitarianism for Proportional Allocations and Hansson Independence imply Maximin for Proportional Allocations. Let x and y be two allocations proportional to Ω, such that $\min_i x_i > \min_i y_i$. Assume that, contrary to Pazner-Schmeidler preferences, one has $y \, R \, x$. Let m be an agent such that $y_m = \min_i y_i$. Let

$$\alpha = \min_i x_i - \min_i y_i$$

and

$$\varepsilon = \frac{1}{n+3}\alpha.$$

Let x' be defined by $x'_i = \min_i x_i - \varepsilon$ for all i. Let y' be proportional to Ω and chosen so that for all i, $y'_i > y_i$ and for all $i \neq m$, $y'_i > \min_i x_i$ while $y'_m = y_i + \frac{1}{2}\varepsilon$. By Weak Pareto, $y' \, P \, y$ and $x \, P \, x'$. Then, by transitivity, $y' \, P \, x'$.

Let y'' be defined by $y''_i = y'_i + \varepsilon$ for all $i \neq m$, and $y''_m = y_i + \varepsilon$. Now, let $(R'_1, ..., R'_n)$ be a profile such that for all $q \geq y''_i$, and all $i \neq m$,

$$uc_{R'_i}(q) = \left\{ q' \in \mathbb{R}^\ell_+ \mid q' \geq q \right\},$$

while $uc_{R'_m}(y'_m) = uc_{R_m}(y'_m)$, and for all i, $uc_{R'_i}(x'_i) = uc_{R_i}(x'_i)$ and $uc_{R'_i}(y'_i) = uc_{R_i}(y'_i)$. Let R' be the corresponding social ordering. By Hansson Independence, one still has $y'\ R'\ x'$. And by Weak Pareto, $y''\ P'\ y'$, so that $y''\ P'\ x'$.

Let $(R''_1, ..., R''_n)$ be a profile such that for all $q \geq \min_i x_i$, and all $i \neq m$,

$$uc_{R''_i}(q) = \left\{q' \in \mathbb{R}^\ell_+ \mid q' \geq q\right\},$$

while $uc_{R''_m}(y''_m) = uc_{R'_m}(y''_m)$, and for all i, $uc_{R''_i}(x'_i) = uc_{R'_i}(x'_i)$. Notice that for all i, one has $uc_{R''_i}(y''_i) = uc_{R'_i}(y''_i)$.Let R'' be the corresponding social ordering. By Hansson Independence, one still has $y''\ R''\ x'$.

Let R_0 be an individual preference relation such that for all $q \geq \min_i x_i$,

$$uc_{R_0}(q) = \left\{q' \in \mathbb{R}^\ell_+ \mid q' \geq q\right\},$$

while

$$\begin{aligned} uc_{R_0}(y''_m) &= uc_{R''_m}(y''_m) \\ uc_{R_0}(y''_m + \varepsilon) &= uc_{R''_m}(y''_m + \varepsilon). \end{aligned}$$

Let R^0 be the social ordering for the uniform profile $(R_0, ..., R_0)$. Consider the allocation x'' defined by $x''_m = y''_m + \varepsilon = y_m + 2\varepsilon$, and for all $i \neq m$, $x''_i = x'_i + \varepsilon = \min_i x_i$. Notice that for all i, $uc_{R''_i}(y''_i) = uc_{R_0}(y''_i)$ and $uc_{R''_i}(x''_i) = uc_{R_0}(x''_i)$. Therefore, by Hansson Independence, $x''\ R^0\ y''$ if and only if $x''\ R''\ y''$.

By a repeated application of Property P (similarly as in step 1 of the proof of Th. 2), one does have $x''\ R^0\ y''$, so that $x''\ R''\ y''$. By transitivity, $y''\ R''\ x'$. But consider the egalitarian allocation y^* such that for all i, $y^*_i = \frac{1}{n}\sum_i y''_i$. By Minimal Egalitarianism for Proportional Allocations, $y^*\ R''\ y''$ and by transitivity, $y^*\ R'\ x'$. But since, by construction of ε,

$$\frac{1}{n}\sum_i y''_i = \frac{1}{n}\left[y_m + 2\varepsilon + (n-1)\min_i x_i\right] < \min_i x_i - \varepsilon,$$

one has $x'_i\ P'_i\ y^*_i$ for all i, yielding a contradiction with Weak Pareto.

Therefore one must have necessarily $x\ R\ y$ (actually $x\ P\ y$), proving Maximin for Proportional Allocations. ∎