INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

No. 257

The Canonical Extensive Form of a Game Form
Part II – Representation

by

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June 1996

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Printed at the University of Bielefeld

ISSN 0931–6558

Abstract

This series of papers exhibits to any noncooperative game in strategic or normal form a 'canonical' game in extensive form that preserves all symmetries of the former one. The operation defined this way respects the restriction of games to subgames and yields a minimal total rank of the tree involved. Moreover, by the above requirements the 'canonical extensive game form' is uniquely defined.

Part II is devoted to the discussion representation of strategic game forms, the notion of symmetrizations of an atom, and the time-structured mapping which assigns the 'canonical representation' to a strategic game form.

Key words: Games, Extensive Form, Normal Form, Strategic Form.
AMS(MOS) Subject Classification: 90D10, 90D35, 05C05.

*The authors acknowledge the important rôle Bezalel Peleg played in the evolution of this paper. Our work is based on an idea of his and he frequently discussed the topic with us giving vital hints.

0 Introduction

This paper continues along the path described in Section 0 of Part I of PELEG, ROSENMÜLLER, and SUDHÖLTER (1996).

Section 4 is devoted to constructing and characterizing faithful representations of general and square game forms. An extensive game form γ is a *representation* of a game form g if for every choice of a vector of payoff functions u the normal form of the extensive game $\Gamma(\gamma, u)$ is (impersonally) isomorphic to the strategic game $G(g, u)$. Theorem 4.5 shows that the outcome function of a representation does not depend on chance moves.

The minimal representations of square and general game forms are called *atoms.* A representation γ of a game form g is *faithful* if for every choice of a vector of payoff functions u the games $\Gamma(\gamma, u)$ and $G(g, u)$ have the same symmetry group. Theorem 4.25 shows that symmetrizations of atoms are faithful representations of strategic game forms that are square and general. The main result of Section 4, Theorem 4.27, proves a converse result: A minimal and faithful representation of a general and square strategic game form must be a symmetrization of an atom.

An atom is *time structured* if the order of play is the same for all n-tupels of strategies. Section 5 is devoted to the proof that symmetrizations of time structured atoms yield canonical representations of finite strategic games. First, Theorem 5.4 shows that minimal and faithful representations of square and general strategic game forms must be time-structured (provided the number of strategies is at least three).

A representation (in the sense of Theorem 5.5) is a mapping which assigns to every game form (in the domain of the mapping), an extensive game form that represents it. A representation is *canonical* if it satisfies the following conditions:

(1) For each game form the assigned representation is faithful.

(2) It respects isomorphisms between game forms.

(3) It is consistent with respect to deletion of strategies.

(4) For each square game form the assigned representation is minimal.

Then Theorem 5.5, the main result of this work, proves that there exists a unique canonical representation of strategic games which is given by (generalized) time structured representations.

4 Atoms and faithful representations

Our previous results provide us with a possibility to discuss 'square games' as a first attempt to introduce the symmetric canonical extensive version. In order to approach this program we shall first of all discuss the simplest (and 'nonsymmetric') version of a representation: an atom. Let ϵ be an extensive preform. A ***pure strategy*** of player i is a mapping that selects a choice at each information set of player i. If we define for $i \in N$

$$\tilde{S}_i := \{\sigma_i \mid \sigma_i \text{ is a pure strategy of player } i\} \tag{4.1}$$

then we obtain a strategic preform

$$\nu(\epsilon) = (N, \tilde{S}) = (N, (\tilde{S}_i)_{i \in N}) \tag{4.2}$$

Next, if $v : \partial E \to I\!R^N$ is a vector of utility functions, then to any $\sigma \in \tilde{S}$ there is a corresponding random variable X^σ choosing plays in accordance with the distribution induced by σ and p. We may define the payoff at each realization of X^σ to be the one at the endpoint, thus the expectation

$$\tilde{u}_i(\sigma) = Ev_i(X^\sigma) \tag{4.3}$$

is well defined. This generates a strategic game

$$\mathcal{N}(\epsilon; v) = \mathcal{N}(\Gamma) := (N, \tilde{S}; \tilde{u}) = (\nu(\epsilon); \tilde{u}(\epsilon, v)) \tag{4.4}$$

which is the 'normalized strategic game' corresponding to Γ.

Definition 4.1 *$\nu(\epsilon)$ as defined by 4.1 and 4.2 is called the* ***normalized*** *strategic preform* ***corresponding*** *to ϵ; $\mathcal{N}(\Gamma)$ as defined by 4.2 and 4.4 is the* ***normalized*** *strategic game corresponding to Γ. (The term normal form of an extensive game is common in the literature. This we shall not employ in our present context, because the term 'form' in our context is used for a particular type of structure (strategic and extensive game* ***forms*** *and pre****forms****)).*

Our next purpose is to define the relation between isomorphisms of extensive forms and strategies. To this end, the reader should review Definition 1.1. In this context, when we are given the partitions $\mathbf{P}, \mathbf{Q}, \mathbf{C}$ of a game tree (a game form, a game), we would like to refer to the player dependent elements only. Thus we introduce the following notations. We write

$$\mathbf{P}_{-0} := \mathbf{P} - \{P_0\}, \mathbf{Q}_{-0} := \mathbf{Q} - \mathbf{Q}_0 \text{ where } \mathbf{Q}_0 := \{Q \in \mathbf{Q} | Q \subseteq P_0\}. \tag{4.5}$$

Now, if (π, ϕ) is an isomorphism between our game tree and some other game tree with partitions $\mathbf{P}', \mathbf{Q}', \mathbf{C}'$, then we want to consider the ***induced mappings*** given by

$$\phi^{\mathbf{P}} : \mathbf{P}_{-0} \longrightarrow \mathbf{P}'_{-0}, \quad \phi^{\mathbf{Q}} : \mathbf{Q}_{-0} \longrightarrow \mathbf{Q}'_{-0},$$
$$\text{where } \phi^{\mathbf{P}}(P) = P', \text{ whenever } P = \phi^{-1}(P'), \tag{4.6}$$
$$\text{and } \phi^{\mathbf{Q}}(Q) = Q', \text{ whenever } Q = \phi^{-1}(Q').$$

Note that both mappings are well-defined and bijective, because ϕ is bijective and respects the partitions. Also note that

$$\phi^{\mathbf{C}(Q)} : \mathbf{C}(Q) \longrightarrow \mathbf{C}'(Q') \tag{4.7}$$

can be defined analogously to $\phi^{\mathbf{P}}$.

The influence of isomorphism on strategies is then explained as follows.

Remark 4.2 *Note that an isomorphism (π, ϕ) between extensive preforms ϵ and ϵ' induces an isomorphism between the corresponding normalized preforms. More precisely, there exists a mapping Ξ which to any (π, ϕ) assigns a (π, φ), where φ is defined by*

$$(\varphi_i(\sigma))(\phi^{\mathbf{Q}}(Q)) = \phi^{\mathbf{C}(Q)}(\sigma(Q)) \quad (Q \in \mathbf{Q}, Q \subseteq P_i, i \in N, \sigma \in S_i). \tag{4.8}$$

The mapping Ξ can be seen to respect compositions of isomorphisms in case ϵ equals ϵ'. Therefore it exhibits the structure of a ***homomorphism*** *in this case.*

Let Γ and Γ' be games and consider an isomorphism (π, ϕ) between them. Then, of course, (π, ϕ) is an isomorphism between the underlying extensive preforms, hence $\Xi(\pi, \phi)$ is an isomorphism between the resulting normalized strategic preforms. Now we have

Lemma 4.3 *Let (π, ϕ) be an isomorphism between the extensive games $\Gamma = (\epsilon; v)$ and $\Gamma' = (\epsilon'; v')$. Then $\Xi(\pi, \phi)$ is an isomorphism between $\mathcal{N}(\Gamma)$ and $\mathcal{N}(\Gamma')$.*

The proof is rather straightforward as all mappings involved are bijective. In view of this fact we restrict ourselves to a mere sketch.

Proof: The probability at each chance move of ϵ is fully transported to the corresponding probability at the image node in ϵ', which is also a chance move. Therefore the expectations of payoffs are preserved. **q.e.d.**

With game forms the situation is much more involved. On the other hand here is the clue to the decisive role game forms play in our treatment of symmetries. Therefore we have to start with the following definition which emphazises the importance of the game form with respect to all games it may induce. To this purpose, we introduce a set of outcomes A and utilities $U : A \rightarrow \mathbb{R}^N$. Recall Remark 2.5 concerning composition of a game form g or γ and U to obtain a game $U * g$ or $U * \gamma$.

Definition 4.4

(1) Let $g = (e; A, h)$ and $\gamma = (\epsilon; A, \eta)$ be game forms (with identical N and A). Then γ is called a ***representation*** *of g if there is a familiy of bijections $\psi = (\psi_i)_{i \in N}$, $\psi_i : S_i \to \tilde{S}_i$ such that for every $U : A \to I\!R^N$ (id, ψ) is an (impersonal) isomorphism between $U * g$ and $\mathcal{N}(U * \gamma)$.*

(2) A representation γ of g is said to be ***faithful*** *if, for every $U : A \to I\!R^N$ the symmetry groups coincide, i.e., we have*

$$\mathcal{S}(U * g) = \mathcal{S}(U * \gamma) \quad (U \ : \ A \to I\!R^N) \tag{4.9}$$

Our first observation is that representations can only occur if the extensive game form is of a nature which avoids the introduction of 'lotteries' for the computation of outcomes resulting from strategies. To this end, for any pure strategy σ_0 of chance and any strategy profile (n-tuple) σ of the players the resulting play is denoted by $X^{\sigma_0, \sigma} = (X_0^{\sigma_0, \sigma}, \ldots, X_T^{\sigma_0, \sigma})$. The outcome induced is $\eta(X_T^{\sigma_0, \sigma})$. However, it turns out that, given a representation, the outcome does not depend on σ_0. More precisely, we have

Theorem 4.5 *Let γ be a representation of g. Then for all $\sigma \in \tilde{S}$ the outcome $\eta(X_T^{\sigma_0, \sigma})$ does not depend on σ_0.*

Proof: Let (id, ψ) be the isomorphism mentioned in Definition 4.4. Let $a \in A$. Let $s \in S$ be such that the outcome (in g) is a and let σ be the image under ψ^{id} of s, i.e.

$$\sigma \in \psi^{id}(h^{-1}(\{a\})).$$

We want to show that $\eta(X_T^{\sigma_0, \sigma}) = a$ for all pure strategies σ_0 of chance. To this purpose define $U : A \to I\!R^N$ by $U_i(a) = 1 \ \ (i \in N)$ and $U_i(b) = 0 \ \ (i \in N, b \in A, b \neq a)$. Consider the games $\Gamma = U * \gamma$ and $G = U * g$ which is impersonally isomorphic to $\mathcal{N}(U * \gamma)$. The first game possesses only payoffs 0 and 1 and so does the latter one. Hence it follows that $E(U_i \circ \eta)(X^\sigma) = 1 \ \ (i \in N)$ (cf. 4.3). But this necessarily implies that all plays $X^{\sigma_0, \sigma}$ yield a payoff 1. **q.e.d.**

Corollary 4.6 *Let γ be a representation of g. If g is general, then the impersonal isomorphism (id, ψ) given by Definition 4.4 is uniquely defined.*

Proof: To see this observe that 'mixing' (taking expectations) can be avoided in computing the outcomes resulting from strategy profiles in the framework of γ. Hence an outcome can be associated to any entry of g. As g is general this association defines a unique mapping. **q.e.d.**

The same consideration motivates the introduction of a *normalized strategic game form of* an extensive game form g, even if g does not happen to be general. For the above mentioned association can be performed in any case, hence the following definition is noncontradictory.

Definition 4.7 *Let $\gamma = (\epsilon; A, \eta)$ be a representation of g. For any $\sigma \in \tilde{S}$ and arbitrary pure strategy σ_0 of chance define*

$$h^\eta(\sigma) := \eta(X_T^{\sigma_0,\sigma}) \tag{4.10}$$

(independently of σ_0). Then

$$\mu(\gamma) := (\nu(\epsilon); A, h^\eta)$$

is the ***normalized strategic game form*** *of γ.*

Naturally the question arises which kind of extensive game forms admits of a normalized strategic game form. Implicitly the answer is provided by the requirement that formula (4.10) holds true independently of σ_0. An extensive game form with this property should be referred to as ***nonmixing***. The normalized game form of a nonmixing extensive game form is described in Definition 4.7. Intuitively within the framework of a nonmixing extensive game form taking lotteries (or expectations for that matter) is avoided. Every play which chance can generate yields the outcome determined by the strategy profile of the players.

Remark 4.8 *Let γ be an extensive game form. Then the following are equivalent.*

(1) γ is nonmixing.

(2) γ is a representation of some strategic game form g.

Every representation of a strategic game form is clearly nonmixing. Conversely, if γ is nonmixing, then it is straightforward to verify that it represents $\mu(\gamma)$.

Corollary 4.9 *Let γ be a nonmixing extensive game form and g be a a strategic game form.*

(1) γ represents g, if and only if there is an ***impersonal outcome preserving (IOP)*** *isomorphism (id, ψ, id) between g and $\mu(\gamma)$.*

(2) If γ represents g and g is general, then the IOP isomorphism mentioned above is uniquely determined.

So far we have elaborated upon the topic of representation in general. Let us now turn to faithful representations. This will be done in the context of games that in principle allow for symmetries, i.e., games with an equal number of strategies for each player. Since for two persons the strategic versions of such games resemble square matrices, we call such versions square as well, more precisely

Definition 4.10 *A strategic preform e, game form $(e; A, h)$, or game $(e; u)$ respectively is called* ***square*** *if, for some $r \in I\!N, r \geq 2$ we have $\mid S_i \mid = r \quad (i \in N)$.*

Lemma 4.11 *Let $\gamma = (\epsilon; A, \eta)$ be a nonmixing extensive game form and let $U : A \to \mathbb{R}^N$ be a utility profile. Also, let (π, ϕ) be an automorphism of ϵ. Then $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ if and only if $\Xi(\pi, \phi) \in \mathcal{M}(U * \mu(\gamma))$.*

The proof is easy and shall be omitted.

Remark 4.12 *The situation is essentially the same if the normalized game form $\mu(\gamma)$ is replaced by a strategic game form g of which γ is a representation; however we have to observe the IOP isomorphism (id, ψ, id) (cf. Corollary 4.9).*

Indeed, for fixed γ and g, Ξ induces a mapping $\Theta = \Theta^{\epsilon, e}$ which carries automorphisms of ϵ into automorphisms of e via

$$\Theta(\pi, \phi) = (\pi, \psi^{-1} \otimes \varphi \otimes \psi), \text{ where } \Xi(\pi, \phi) = (\pi, \varphi). \tag{4.11}$$

Theorem 4.13 *Let γ be a faithful representation of the square general strategic game form g and let $U : A \to \mathbb{R}^N$ be a utility profile. Then*

$$\Theta : \mathcal{M}(U * \gamma) \longrightarrow \mathcal{M}(U * g) \tag{4.12}$$

is surjective.

Proof:

1st Step: Let $\gamma = (\epsilon; A, \eta)$ and $g = (N, S; A, h)$ be game forms with the desired properties. We can assume without loss of generality that $S_i = S_j = \{1, \ldots, r\} \quad (i, j \in N = \{1, \ldots, n\})$ holds true. Indeed, we are going to show that

$$\Theta : Aut(\epsilon) \longrightarrow Aut(e) \tag{4.13}$$

(here Aut denotes the group of automorphisms) is surjective. On first sight this might seem to be a more comprehensive statement, however in view of our subsequent proof it will become clear that every automorphism can occur as a motion of a suitable game; hence both claims are in fact equivalent.

2nd Step: First of all consider a utility profile $U : A \to \mathbb{R}^n$ specified as follows. We take

$$U_i(s) = i \cdot r + s_i \quad (i \geq 3, s \in S)$$

in order to avoid any symmetries between players $i, j \geq 3$. Furthermore put

$$U_1(s) = U_1(s_1, s_2) \text{ and } U_2(s) = U_2(s_1, s_2)$$

(meaning that U_1, U_2 depend on the first two coordinates only). Then U_1, U_2 can be specified by

$$U_1(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{array} \qquad U_2(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \tag{4.14}$$

for $(\cdot,\cdot) \in \{1,2\} \times \{1,2\}$ and

$$U_1(\cdot,\cdot) = 0 = U_2(\cdot,\cdot)$$

otherwise.

Now we are going to discuss the group of motions corresponding to $U * g$. To this end let $\pi = (1,2)$ be the transposition of the first two players. Also let $\tau^{ij} : S_i \to S_j$ be defined by $\tau^{ij} : 1 \to 2 \to 1$ and let $id^{ij} : S_i \to S_j$ be the identity mapping for $i, j \in N$. Then we have

$$\mathcal{M}(U * g) = \left\{ \begin{array}{ll} (id, (id^{11}, id^{22}, id^{(n-2)})), & (\pi, (\tau^{12}, id^{21}, id^{(n-2)})), \\ (\pi, (id^{12}, \tau^{21}, id^{(n-2)})), & (id, (\tau^{11}, \tau^{22}, id^{(n-2)})) \end{array} \right\} = \left\{ \begin{array}{ll} c^0, & c, \\ c^3, & c^2 \end{array} \right\} \tag{4.15}$$

where id^{n-2} is self-explaining.

As γ is faithful, there exists ϕ such that $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ and Ξ throws (π, ϕ) on either c or c^3. As the group is cyclic, the powers of (π, ϕ) are thrown onto all of $\mathcal{M}(U * g)$.

3rd Step: The next utility profile we have to consider is indicated by

$$U_1(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \end{array} \qquad U_2(\cdot,\cdot) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \end{array} \tag{4.16}$$

(using the convention established in the 1st Step). Here the group of motions can easily be computed as

$$\mathcal{M}(U * g) = \{e, (\pi, (id^{12}, id^{21}, id^{(n-2)}))\} = \{d^0, d\}. \tag{4.17}$$

Again using faithfulness it is at once established that d necessarily has to be the image of some $(\pi, \phi) \in \mathcal{M}(U * \gamma)$ under Θ.

4th Step: Now c^0, c, c^2, c^3, and d occured as motions in a suitable context but, of course, they are automorphisms of e as well. We may generate similar automorphisms as images under Θ by exchanging any two strategies of players 1 and 2 or, for that matter, of any two players. The reader has now to convince himself that the family of automorphisms created this way generates the full group of automorphisms of e. **q.e.d.**

For a strategic general game form $g = (e; A, h)$ every automorphism (π, φ) of e induces an automorphism $(\pi, \varphi, h \circ \varphi^\pi \circ h^{-1})$ of g. Analogously, for any nonmixing extensive game form $\gamma = (\epsilon; A, \eta)$ every automorphism (π, ϕ) of ϵ induces an automorphism (π, ϕ, ρ) of γ, where ρ is essentially given by $\eta \circ \phi \circ \eta^{-1}$); meaning that $\rho(a) = \eta(\phi(\xi))$ for all $\xi \in \eta^{-1}(a)$

is welldefined independently of $\xi \quad (a \in A)$. This fact and the last proof enables us to reformulate Theorem 4.13.

Corollary 4.14 *Let $\gamma = (\epsilon; A, \eta)$ be a representation of the general square strategic game form $g = (e; A, h)$ and let*

$$\tilde{\Theta} : Aut(\gamma) \rightarrow Aut(g)$$

be defined by

$$\tilde{\Theta}(\pi, \phi, \rho) = (\Theta^{\epsilon,e}(\pi, \phi), \rho) \quad ((\pi, \phi, \rho) \in Aut(\gamma)).$$

Then γ is a faithful representation of g, if and only if $\tilde{\Theta}$ is surjective.

Indeed, note that $\tilde{\Theta}(\pi, \phi, \rho)$ is an automorphism, because $\rho = h \circ \varphi^{\pi} \circ h^{-1}$, where $\Theta^{\epsilon,e}(\pi, \phi) = (\pi, \varphi)$, is satisfied.

The above development suggests to briefly consider automorphisms of extensive preforms that leave the corresponding normalized preforms untouched. This kind of authomorphisms is described by the following definition.

Definition 4.15 *An automorphism (π, ϕ_0) of an extensive preform ϵ is said to be* ***chance related*** *if the following holds true:*

(1) $\pi = id$

(2) $\phi_0^{\mathbf{Q}}(Q) = Q \quad (Q \in \mathbf{Q}_{-0})$ *(cf. Definition 1.1)*

(3) $\phi_0^{\mathbf{C}(Q)}(C) = C \quad (C \in \mathbf{C}(Q), Q \in \mathbf{Q}_{-0})$

A motion (π, ϕ_0) of a game Γ is ***chance related*** *if conditions (1), (2), and (3) are satisfied. $\mathcal{C}(\Gamma)$ denotes the subgroup of chance related motions of $\mathcal{M}(\Gamma)$. Note that formula 4.8 of Remark 4.2 implies that $\Xi(\pi, \phi_0)$ is the identity, i.e., the strategies of the corresponding normalized preform are not disturbed.*

Theorem 4.16 *The chance related automorphisms of an extensive preform constitute a normal subgroup. $\mathcal{C}(\Gamma) \subseteq \mathcal{I}(\Gamma)$ is a normal subgroup.*

Proof: We have to show that for any automorphism (π, ϕ) and any chance related automorphism (id, ϕ_0) we can find a chance related automorphism (id, ϕ_0') such that

$$(\pi, \phi)(id, \phi_0) = (id, \phi_0\prime)(\pi, \phi)$$

holds true. To this end it suffices to show that

$$\phi_0' = \phi\ \phi_0\ \phi^{-1}$$

is chance related. Indeed, we have for $i \in N$ and $Q \subseteq P_{\pi(i)}$

$$\phi^{-1}(Q) \subseteq P_i$$

that is

$$\phi_0(\phi^{-1}(Q)) = \phi^{-1}(Q)$$

and hence

$$\phi(\phi_0(\phi^{-1}(Q)) = Q,$$

and analogously for (3) of Definition 4.15 **q.e.d.**

Theorem 4.17 *Let γ be a faithful representation of the square general strategic game form g. Then, for any utility profile $U : A \rightarrow I\!R^N$ it follows that*

$$\mathcal{M}(U \star \gamma)/\mathcal{C}(U \star \gamma) = \mathcal{M}(U \star g) \tag{4.18}$$

holds true.

Proof: By Theorem 4.13 Θ is a surjective mapping which respects composition (Remark 4.2). It suffices to show that $\mathcal{C}(U \star \gamma)$ is the kernel of this mapping.

Clearly, if $(\pi, \phi) \in \mathcal{C}$, then $\Theta(\pi, \phi) = (id, id)$. On the other hand if $\Theta(\pi, \phi) = (id, id)$, then ϕ has to satisfy conditions (2) and (3) of Definition 4.15 for otherwise we can construct a strategy $\sigma_i \in \tilde{S}_i$ that suffers under the influence of ϕ as defined in (4.5). **q.e.d**

The simplest way of representing a square game form g is described as follows.

Definition 4.18

(1) A game tree $(E, \prec, \mathbf{P}, \mathbf{C})$ (i.e. $P_0 = \emptyset$ and $\mathbf{Q} = \mathbf{P}$, cf. Section 1) is ***atomic,*** *if $|C(\xi)| = |C(\xi')| = r \geq 2$ for $\xi, \xi' \in E - \partial E$.*

(2) A preform $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota)$ is ***atomic,*** *if $(E, \prec, \mathbf{P}, \mathbf{C})$ is an atomic game tree.*

(3) An extensive game form $\alpha = (\epsilon; A, \eta)$ is an ***atom,*** *if ϵ is an atomic preform and $\eta : \partial E \rightarrow A$ is bijective.*

(4) An atom α and its preform and game tree is ***time structured*** *if every nonvoid level $\mathcal{L}(E, \prec, t)$ coincides with one player set $P \in \mathbf{P}$. In this case α is called* ***T-atom.***

Remark 4.19 *Let $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota)$ be a preform of an atom and let $e = (N, S)$ be a strategic preform of a square game form such that $r = |S_i| = |\tilde{S}_i|$ $(i \in N)$ is satisfied, where $\nu(\epsilon) = (N, \tilde{S})$. Moreover, let (id, ψ) be an isomorphism between e and $\nu(\epsilon)$ (which exists because the stratgy sets have the same size).*

(1) If $g = (e; A, h)$ is a general strategic game form, then there exists a unique mapping $\eta : \partial E \rightarrow A$ such that (id, ψ, id) is an IOP isomorphism between g and $\mu(\alpha)$ (where $\alpha = (\epsilon; A, \eta)$), i.e. α is a representation of g. An atom which represents g is said to be an ***atom of*** *g.*

(2) If $\alpha = (\epsilon; A, \eta)$ is an atom (i.e. $\eta : \partial E \rightarrow A$ is bijective), then there exists a unique mapping $h : S \rightarrow A$ such that (id, ψ, id) is an isomorphism between $g = (e; A, h)$ and $\mu(\alpha)$, i.e. α is a representation of the general game form g.

Example 4.20

(1) For two persons and $s = 2$ consider g as indicated by

$$\begin{array}{cc} & \begin{matrix} l & r \end{matrix} \\ \begin{matrix} t \\ b \end{matrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \tag{4.19}$$

There are two atoms as indicated by Figure 4.1.

Figure 4.1: Atoms for a $(2; 2 \times 2)$ game

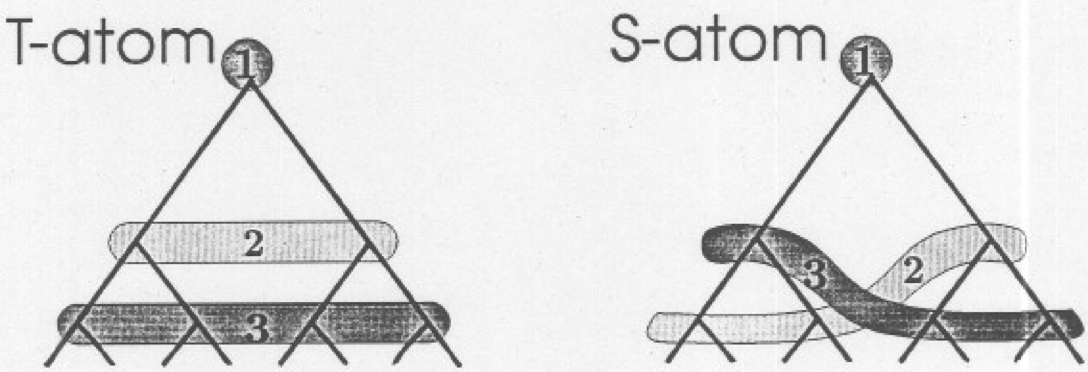

Figure 4.2: Atomic preforms for a $(3; 2 \times 2 \times 2)$ game

(2) For 3 persons and $r = 2$ consider the preforms in Figure 4.2 which may be augmented to game forms representing appropriate strategic forms.

The T-atom is 'time structured'. Assuming that the game is 'Common Knowledge', player i is aware that he moves 'at instant i'. The S−atom seems to exhibit some symmetry between players 2 and 3.

(3) For 4 persons and $r = 2$, examples of atoms can be seen in Figure 4.3.

Remark 4.21

(1) Figure 4.3 suggests that the underlying **tree** $(E, \prec)$ *of an atom α of $g = (N, S; A, h)$ is essentially (i.e. up to order respecting bijective mappings, cf. Section 1) uniquely*

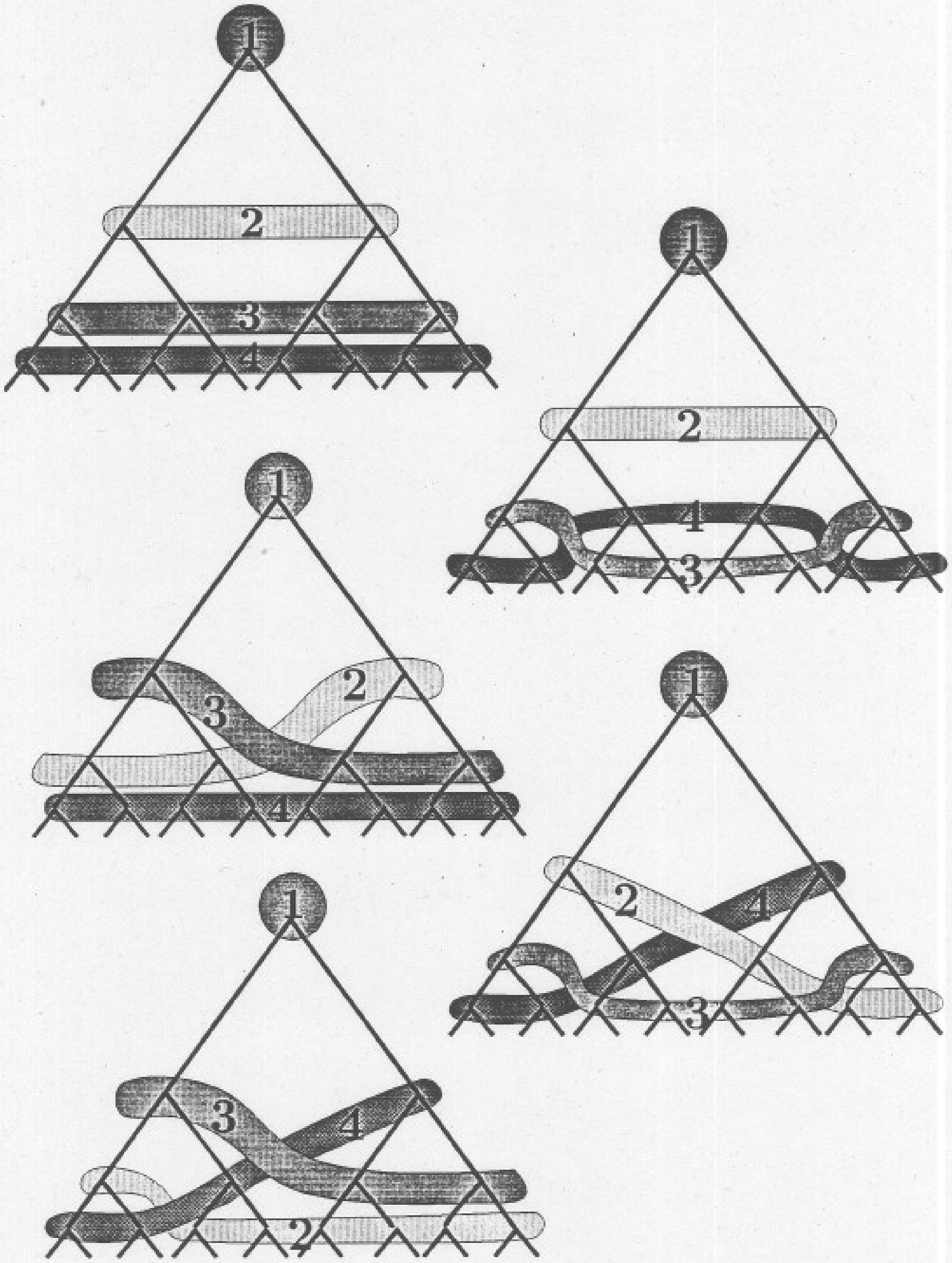

Figure 4.3: Atomic preforms for a $(4; 2 \times 2 \times 2 \times 2)$ game

determined. Clearly, the pair $(E, \prec)$ *is a tree of an atom* $\alpha = (E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$ *of* $\mu(\alpha)$ *which is general (in the sense of Definition 2.3) iff the maximal rank coincides with the number* $\mid N \mid$ *of players and at every node* $\xi \in E - \partial E$ *there are exactly* $r = \mid S_i \mid_{i \in N}$ *alternatives.*

(2) Also, α *is an atom of* g *if and only if it represents* g *and its total rank is minimal.*

(3) *In addition note that to any atom α we can at once construct isomorphic ones by permuting the players and renaming the outcomes accordingly. E.g., there are at once 4! different but isomorphic atoms corresponding to each one suggested by Figure 4.3; all of them being obtained by permuting the players arbitrarily and renaming the outcomes accordingly.*

(4) *An atom of a general square game form g cannot be a faithful representation of g, because there is no automorphism of the preform which replaces the 'owner of the root' by any other player. In order to obtain a version which allows for symmetries as necessary, we will now construct 'symmetrizations' of atoms.*

To this end we shall shortly touch a further operation acting on game trees called ***restriction.*** Given $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$, most operations on the data are defined canonically. Thus, if $E \subseteq E^*$, then the restriction of $\prec$ on E^* is $\prec^* := \prec|_{E^*} := \prec \cap (E^* \cap E^*)$ and the partitions are given by

$$\begin{aligned} \mathbf{P}^* : &= \{P \cap E^* \mid P \in \mathbf{P}\} \\ \mathbf{Q}^* : &= \{Q \cap E^* \mid Q \in \mathbf{Q}\} \\ \mathbf{C}^*(Q \cap E^*) : &= \{S \cap E^* \mid S \in \mathbf{C}(Q)\}. \end{aligned}$$

We will always assume that the restriction results in a tree, thus we require

$$(E^*, \prec^*) \text{ is a tree.} \tag{4.20}$$

Also, no path should end outside the boundary after restriction (for, later on the definition of payoffs and outcomes will hinge on endpoints); hence we want

$$\partial E^* \subseteq \partial E. \tag{4.21}$$

Generally, the probabilities p^* should be the conditional probabilities given E^*. It is not necessary to dwell on the intricacies of this notion since in Definition 4.22 the notion is obvious and in Section 5 we shall restrict ourselves to proper restrictions in which chance moves are not disturbed. This way we have defined the restriction $(E^*, \prec^*, \mathbf{P}^*, \mathbf{Q}^*, \mathbf{C}^*, p^*)$ of $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ to E^*. The restriction of game forms and games is then defined in an obvious way.

Definition 4.22 *An extensive game form $\gamma = (N, E, \prec, \mathbf{P}, \mathbf{C}, p; \iota; A, \eta)$ (i.e. $\mathbf{P} = \mathbf{Q}$) is a* ***symmetrization of*** *the atom α, if the following conditions are satisfied.*

(1) *γ is nonmixing.*

(2) *The root x_0 of γ is the only chance move and p^{x_0} is uniform distribution, i.e., every edge at x_0 has the same probability.*

(3) *For every $\xi \in C(x_0)$ the restricted game form $\gamma^\xi := (N, E^\xi, \prec^\xi, \mathbf{P}^\xi, \mathbf{C}^\xi; \iota^\xi; A, \eta^\xi)$ of γ obtained by restricting γ to the subtree with root ξ generated by the edge (x_0, ξ) is isomorphic to α.*

(4) For every atom β which represents $\mu(\alpha)$ and is isomorphic to α there exists a unique $\xi \in C(x_0)$ such that β is IOP isomorphic to γ^ξ.

Example 4.23 *Consider the case of two persons each of them having two strategies. Two atoms of the general game form, i.e., of g represented by*

$$\begin{array}{cc} & \begin{array}{cc} l & r \end{array} \\ \begin{array}{c} t \\ b \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \tag{4.22}$$

have been indicated in Figure 4.1. Clearly, they are isomorphic. A symmetrization is indicated as follows.

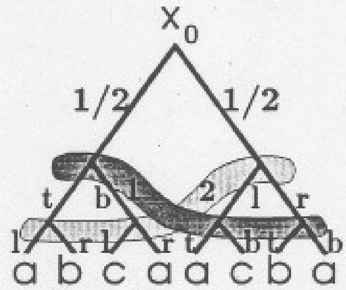

Figure 4.4: The symmetrization

Thus for the simple $(2; 2 \times 2)$-case, the symmetrization described in Figure 4.4 suggests the structure of the 'canonical' representation we have in mind.

Already for 3 persons, this is not so obvious. As Figure 4.2 suggests, there are essentially 2 nonisomorphic atoms: the 'time structured' or 'T-atom' and the 'S-atom' which seems to exhibit more symmetry with respect to the players not called upon in the first move, i.e., players 2 and 3 in 4.2.

Both allow for symmetrizations and at this stage it is not clear which of them will be a candidate for the canonical version.

Example 4.24 *Figure 4.5 shows a symmetrization of the T-atom in Figure 4.2, that could be called $TSYM^3_{2\times2\times2}$.*

Figure 4.6 is the analogous version with respect to the S-atom of Figure 4.2. At this state of affairs it may become conceivable that there is a problem arising from the question as to which version of an extensive game represents the $(3; 2 \times 2 \times 2)$ case 'appropriately' in view of symmetry considerations.

Note also Figure 4.7, which is not the symmetrization of an atom in the sense of Definition 4.22 but nevertheless looks 'rather symmetric'. However, it is not a faithful representation. (Augment the sketch by a suitable set of outcomes at the endpoints such that you obtain a representation of 'the' corresponding general square 3-person game form - for the definition

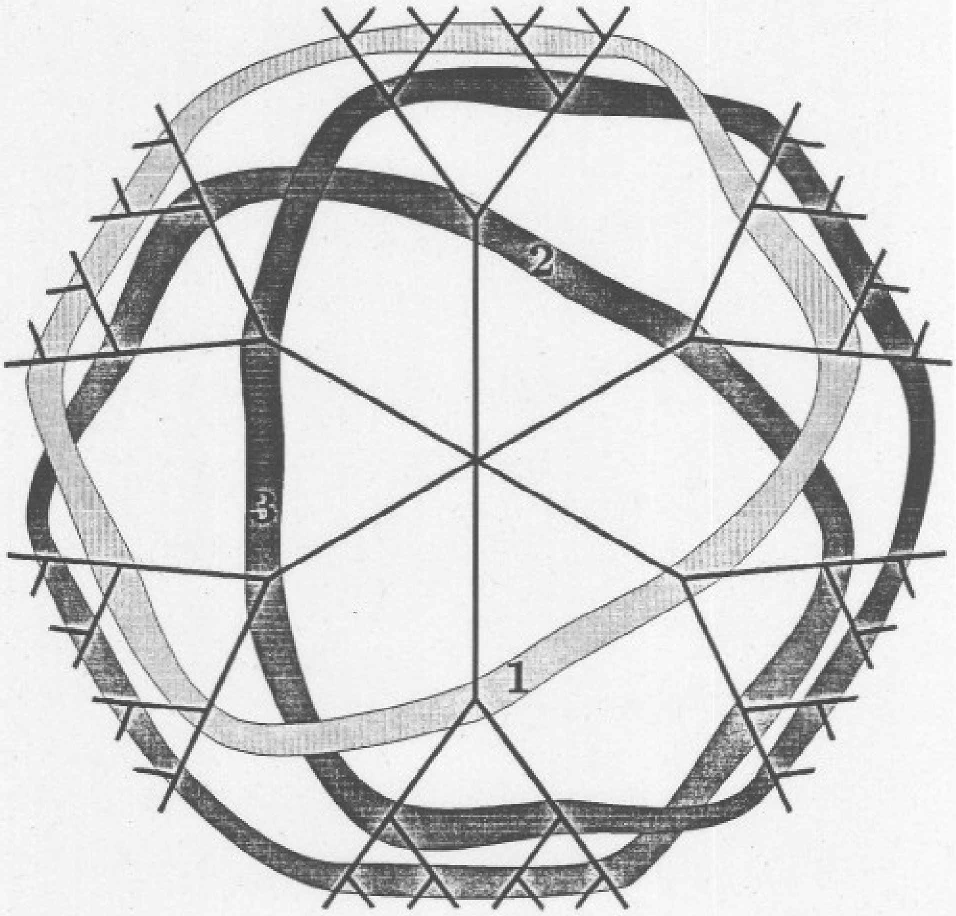

Figure 4.5: The symmetrization of a T-atom ($TSYM^3_{2\times2\times2}$)

of a particularly suited game G within this context the proof of the following theorem is referred to.)

The next result shows that symmetrizations exist and are faithful.

Theorem 4.25 *If α is an atom, then the following assertions are valid.*

(1) α possesses a symmetrization.

(2) Every two symmetrizations of α are IOP isomorphic.

(3) A symmetrization of α is a faithful representation of every strategic game form g represented by α.

Proof:

(1) Let $\alpha = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$ and $\mu(\alpha) = g$. Furthermore, define

$$\mathcal{B} = \{\beta | \beta \text{ is an atom of g isomorphic to } \alpha \text{ with tree } (E, \prec)\}.$$

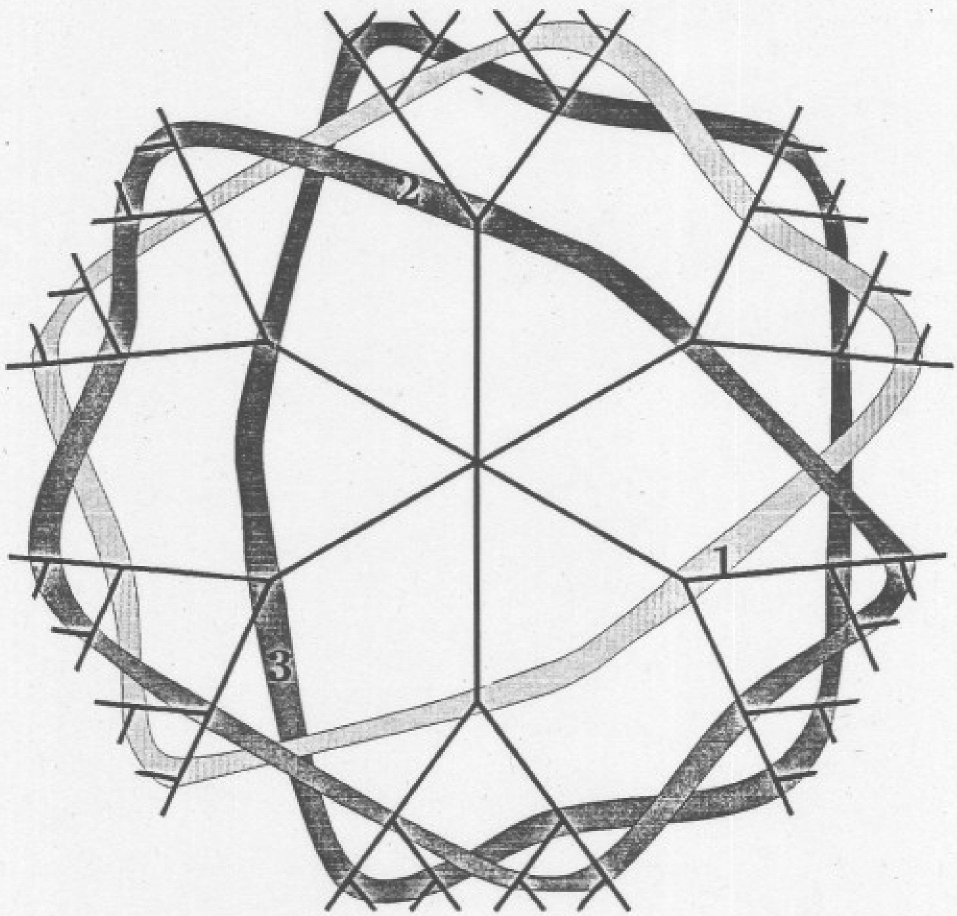

Figure 4.6: The symmetrization of an 'S-atom'

Every atom of g which is isomorphic to α is isomorphic to some atom of the finite set $\mathcal{B}$. Choose a maximal subset $\mathcal{A} \subseteq \mathcal{B}$ of atoms which are not IOP isomorphic. Indeed, $\mathcal{B}$ can be partitioned into the equivalence classes of IOP isomorphic atoms. The set $\mathcal{A}$ contains precisely one representative of each equivalence class. Moreover, for every $\beta = (N, E, \prec, \mathbf{P}^\beta, \mathbf{C}^\beta; \iota^\beta; A, \eta^\beta) \in \mathcal{A}$ take an IOP isomorphism (id, ψ_β, id) between g and $\mu(\beta)$. In view of Corollary 4.9 ψ_β exists. The extensive game form

$$\gamma = (N, \breve{E}, \breve{\prec}, \breve{\mathbf{P}}, \breve{\mathbf{C}}, p; \breve{\iota}; A, \breve{\eta})$$

is defined as follows.

(a) $\breve{E} = \{0\} \cup (E \times \mathcal{A})$,

(b) $0 \breve{\prec} (x_0, \beta), p^0(x_0, \beta) = |\mathcal{A}|^{-1}$ $(\beta \in \mathcal{A})$, where x_0 is the root of $(E, \prec)$,

(c) $(\xi, \beta) \breve{\prec} (\xi', \beta')$, iff $\beta = \beta'$ and $\xi \prec \xi'$ $(\beta, \beta' \in \mathcal{A}, \xi, \xi' \in E)$.

(d) $\breve{P}_i = \bigcup_{\beta \in \mathcal{A}} P_i^\beta \times \beta, \breve{P}_0 = \{0\}$ $(i \in N)$,

(e) $\breve{C}(\breve{P}_i) = \{\bigcup_{\beta \in \mathcal{A}} (\psi_{\beta,i}(s_i), \beta) | s_i \in S_i\}$ $(i \in N)$,

(f) $\breve{\eta}(\xi, \beta) = \eta^\beta(\xi)$ $(\xi \in \partial E, \beta \in \mathcal{A})$.

By construction γ is a symmetrization of α.

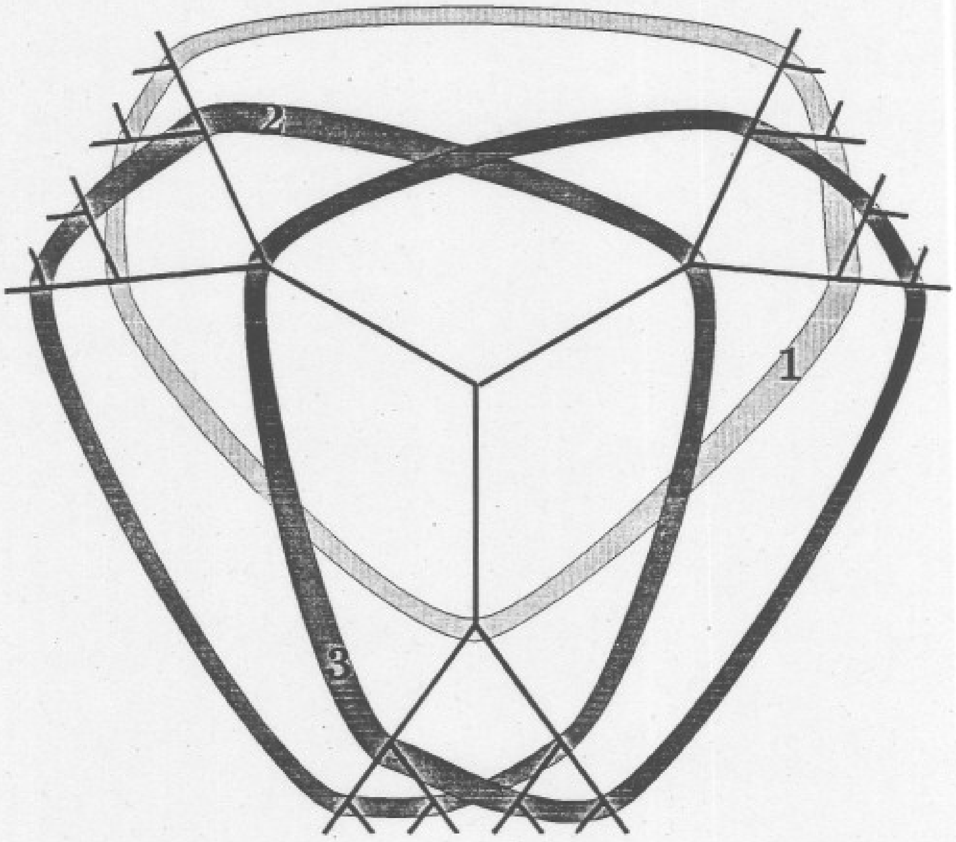

Figure 4.7: A further extensive preform

(2) Let γ and δ be two symmetrizations of α. By Definition 4.22 (3) there is a bijection between the atoms in γ and δ which maps every atom in γ to an IOP isomorphic atom in δ. These IOP isomorphisms together induce an IOP isomorphism between γ and δ in a straightforward manner.

(3) Let γ be a symmetrization of α which represents g. The extensive game form γ represents g, because γ is nonmixing and every atom in γ represents g. Let $\beta = (\epsilon; A, \eta)$, where $\epsilon = (N, E, \prec, \mathbf{P}, \mathbf{C}, \iota)$, be an atom of $g = (N, S; A, h) = (e; A, h)$ such that β is isomorphic to α. Assume without loss of generality that $S_i = \{1, \ldots, r\}$ $(i \in N)$ holds true. Moreover, let (π, φ) be an automorphism of e and (id, ψ, id) the IOP isomorphism between g and $\mu(\beta)$. Then ρ, defined by $\rho = h \circ \varphi \circ h^{-1}$, generates an automorphism (π, φ, ρ) of g. Define $\beta' = (N, E, \prec, \mathbf{P}', \mathbf{C}', \iota'; A, \eta')$ and $\phi : E \to E$ as follows.

(a) Define $\phi(x_0) = x_0$ and assume that $\phi(\xi) \in \mathcal{L}(E, \prec, t)$ $(\xi \in \mathcal{L}(E, \prec, t))$ is already defined for some $0 < T \leq n = |N|$ and $0 \leq t < T$. If $\xi \in \mathcal{L}(E, \prec, T)$, let us say $\xi \in C(\xi')$ and $\xi' \in P_i$ for some $i \in N$, then take the unique strategy $s_i \in S_i$ such that $\xi' \in \psi_i(s_i)$ and determine $\zeta \in C(\phi(\xi'))$ which satisfies $\zeta \in \psi_k(\varphi(s_i))$, where $\phi(\xi') \in P_k$. Define $\phi(\xi) = \zeta$ and observe that ϕ is bijective and respects $(\prec, \prec)$.

(b) Put $P'_{\pi(i)} = \phi(P_i)$ $(i \in N)$.

(c) Put $\mathbf{C}'(P'_{\pi(i)}) = \phi(\mathbf{C}(P_i))$ $(i \in N)$.

(d) Put $\eta'(\xi) = (\rho \circ \eta \circ \phi^{-1})(\xi)$ $(\xi \in \partial E)$, observe that (π, ϕ, ρ) is an isomorphism between β and β', and that β' represents g. Indeed, with $\psi'_i(s_i) = \{\psi_j(s_i) | P_j \cap P'_i \neq \emptyset\} \cap C(P'_i)$ $(i \in N, s_i \in S_i)$ the triple (id, ψ', id) is an IOP isomorphism between g and $\mu(\beta')$.

This procedure applied to every restricted game γ^ξ (where ξ is a successor of the root of γ) yields an automorphism $(\pi, \tilde{\phi}, \rho)$ of γ (note that β' is, up to an IOP isomorphism, a restricted game of γ). Clearly $\Theta(\pi, \tilde{\phi}) = (\pi, \varphi)$ (cf. Remark 4.12 for the definition of Θ), thus the proof is finished.

q.e.d.

Corollary 4.26 *Let γ be a symmetrization of some atom which represents g. Then every restriction of γ which faithfully represents g coincides with γ.*

Proof: To verify this assertion a part of the last proof (3) has to be repeated. Clearly at least one of the atoms in γ, say β , has to occur in the faithful restriction (otherwise the restriction is no representation of g). Moreover, for another atom β' which occurs in γ there is an isomorphism (π, ϕ) between both atoms. Applying Ξ and and the IOP isomorphisms between the normalizations of the atoms and g yields an automorphism (π, φ, ρ) of g. In view of the proof of Theorem 4.13 there must be an automorphism of γ which is mapped to this automorphism of g. Therefore Definition 4.22 (3) completes the proof. **q.e.d.**

Different atoms of a square general game form possess isomorphic trees (e.g., the number of nodes coincides). This is no longer true for 'the' symmetrizations as shown in the next section, i.e. symmetrizations of different atoms of a square general game form may have different numbers of plays and, thus, endpoints. Hence the total ranks may differ. Nevertheless the following result holds true.

Theorem 4.27 *Let $g = (e; A, h)$ be a general square strategic game form and let γ be a faithful representation of g with minimal total rank. Then γ is the symmetrization of an atom of g. Moreover, the symmetrization of a T-atom of g possesses minimal total rank.*

Proof:
1st Step: Let $r := |S_i|$ $(i \in N)$. First of all consider the case that γ is the symmetrization of a T-atom of g. Clearly γ possesses exactly $n!$ (where $n = |N|$) atoms. Now all plays have the same length (i.e. rank of the endpoint) which is $n+1$. In each of the $n!$ atoms there are r^n such paths. Hence the total rank of any of these atoms is $r^n n$. With respect to γ, the corresponding rank originating from each atom is $r^n(n+1)$, because there is an additional edge joining the atom to the root of γ. There are $n!$ atoms, hence the total rank of the graph $(E, \prec)$ of γ is $r^n(n+1)n! = r^n(n+1)!$.
2nd Step: Next, we are going to show that a representation which is faithful has total rank which is at least $r^n(n+1)!$. To this end let γ be a faithful representation of g which has

minimal total rank. Without loss of generality it can be assumed that $\mu(\gamma) = g$ hols true. For every $s \in S$ and every $\pi \in \Sigma(N)$ choose $\varphi(\pi, s) = \varphi$ and $\rho(\pi, s) = \rho$ such that (π, φ, ρ) is an automorphism of g and $\varphi^\pi(s) = s$ is satisfied. Let $\tilde{\Theta}$ be defined as in Corollary 4.14. Let $\phi = \phi(\pi, s)$ be an automorphism of the game tree of γ such that (π, ϕ, ρ) is in the inverse image of (π, φ, ρ), i.e., $\tilde{\Theta}(\pi, \phi, \rho) = (\pi, \varphi, \rho)$. The existence of ϕ is guaranteed by faithfulness. Fix a pure strategy σ_0 of chance and let $X^{\sigma_0, s} = (x_0, x_1^s, \dots, x_{T(s)}^s)$ be the play generated by (σ_0, s). For every permutation π the outcome $\eta(\phi(\pi, s)(x_{T(s)}^s))$ coincides with $h(s)$, because $\varphi(\pi, s)^\pi$ keeps the strategy profile s. Different strategy profiles lead to different outcomes, because g is assumed to be general. Counting the number of strategy profiles and the number of permutations yields $r^n n!$ different plays with endpoints $\phi(\pi, s)(x_{T(s)}^s)$.
The length $T(s)$ of every play is at least $n+1$, because every play intersects an information set of every player and of chance. Indeed, if a player is not involved, then a 'row' of g does not depend on the player's strategy (which is impossible, because g is general). Moreover, the root of γ cannot belong to the information set of some player.
3rd Step: The total rank of $(E, \prec)$ is therefore at least $n!r^n(n+1)$ (recall that the length of each play is at least $n+1$ due to the 2nd step). By minimality of the total rank and the 1st step it follows that the total rank is exactly equal to this number and the root is the only chance move. Clearly γ has to be a symmetrization of an atom. **q.e.d.**

Remark 4.28 *Clearly the notion of an atom is not restricted to the square case. Generally speaking an* **atom** *of a (not necessarily square) general game form g (with n players) is a representation of g without chance moves such that every $\xi \in \partial E$ has rank $r(\xi) = n$ and at every $\xi \in P_i$ player i has exactly $| S_i |$ choices. An atom α is* **time structured** *or a* **T-atom** *if, for every $0 \leq t \leq n-1$, the level $\mathcal{L}(\alpha, t)$ is an information set of a player.*

We will not define symmetrizations in general for atoms of non- square game forms. However, the **symmetrization of a T-atom** *can be defined in close resemblance to Definition 4.22. To this end repeat item (1) of Definition 4.22 and replace items (2),(3) by*

(2') For every $\xi \in C(x_0)$ the restricted game form γ^ξ is a T-atom.

(3') For every T-atom β which represents $\mu(\alpha)$ there is a unique $\xi \in C(x_0)$ such that β is IOP isomorphic to γ^ξ.

5 Restriction and the general case

Within the previous section we have characterized the symmetrizations of atoms as the only representations of square strategic games that respect the symmetries and satisfy a minimality condition. Apart from the fact that the result holds true only in the case that all players have the same number of strategies, the assignment of an extensive game form to a given strategic game form is not unique. For the class of atoms (and their symmetrizations) is still remarkably large: compare e.g. Figure 4.3; here we see various non isomorphic atoms that are capable of representing a $2 \times 2 \times 2 \times 2$ -game.

Of course we will have to accept that a representation can only be defined up to outcome preserving impersonal isomorphisms. On the other hand the variety offered by all atoms is too large. And moreover we should have representations in the general case, not just the symmetric one.

Clearly the preservation of symmetries as formulated so far cannot help in a general non square game, for even in the case of two players there are no symmetries of a general game at all since there are no bijective mappings of the strategy sets. However, as our discussion in Section 0 shows, there are symmetries of restricted versions which should be preserved. Verbally, if two strategies / actions of a player result in the same payoff no matter what his opponents choose to do, then this game is in a well defined sense reducible and the restricted version may well have symmetries the preservation of which should be satisfied by a 'canonical' representation. And if we construct nongeneral game forms with the above property, then the symmetries obtained this way may indeed be used to further reduce the family of representations and hence result in a canonical representation.

Thus it will be the interplay of restriction and symmetries that characterizes the canonical representation of a strategic game form (minimality assumed). Therefore we shall add the notion of 'consistency' (with respect to restriction) to our requirements concerning represention.

Arbitrary restrictions however, as defined in Section 4 (see e.g. formulas 4.20 and 4.21) cannot be admitted. We shall call a restriction of a game tree $(E, \prec, \mathbf{P}, \mathbf{Q}, \mathbf{C}, p)$ to a game tree $(E^{*}, \prec^{*}, \mathbf{P}^{*}, \mathbf{Q}^{*}, \mathbf{C}^{*}, p^{*})$ ***proper*** if the root is preserved and all chance moves together with their choices (and the probabilities) are either fully preserved or completely disappear. The notion is at once extended to game forms and games.

There is a further, more formal obstacle to be tackled before we can reach a rigorous formulation of the 'canonical' representation. This is presented by the aim to precisely define a mapping which represents the choice of a canonical representation. Mappings should be defined on a nice domain - of game forms in our present context. However, if we speak about the 'set of all game forms' we might encounter unpleasant surprises common in elementary set theory, for game forms so far are defined with arbitrary (finite) outcome sets.

More than that, if we look closer, we made no restrictions on the underlying sets of strategies (in a strategic form) neither concerning the elements of the underlying graph (in an extensive form). Thus, when speaking about the set of e.g. strategic games, at the present state of affairs, we will be forced to speak about the set of all finite sets several times.

In order to avoid such footangels we should restrict ourselves to a fixed at least countable ***alphabet*** or ***universe*** $\boldsymbol{U}$ of *letters* or *outcomes* which intuitively first of all is a list of all possible outcomes admitted for game forms (strategic and extensive). I.e., we shall always tacitly assume that for any game form mentioned, the outcome set satisfies $A \subseteq \boldsymbol{U}$; thus the admissible outcome sets are subsets of $\boldsymbol{U}$.

It is no loss of generality to assume in addition that any strategy set S_i mentioned as well as the set of nodes E of a graph involved in our consideration is also a subset of $\boldsymbol{U}$. For the present section we set out under this additional hypothesis.

We feel that this kind of intricacies should be mentioned but not overstressed. Thus, we fix the *set of game forms* say $\boldsymbol{G}$ or $\boldsymbol{\Gamma}$ (strategic or extensive respectively) and define a mapping $\mathcal{F} : \boldsymbol{G} \to \boldsymbol{\Gamma}$ always assuming that the outcomes, nodes, strategies ... involved are given by subsets of $\boldsymbol{U}$.

Definition 5.1 *Let $\mathcal{F} : \boldsymbol{G} \to \boldsymbol{\Gamma}$ be a mapping.*

(1) $\mathcal{F}$ is called a **representation** *(of strategic game forms) if, for any $g \in \boldsymbol{G}$ it follows that $\mathcal{F}(g)$ is a representation of g (cf. Definition 4.4 (1)).*

(2) A representation $\mathcal{F}$ is said to be **faithful** *if it preserves symmetries and respects isomorphisms. More precisely, for any $g \in \boldsymbol{G}$, it should follow that $\mathcal{F}(g)$ is a faithful representation (cf. Definition 4.4 (2).) and whenever g and g' are isomorphic, then so are $\mathcal{F}(g)$ and $\mathcal{F}(g')$.*

(3) A faithful representation $\mathcal{F}$ is said to be **consistent** *if it respects proper restriction up to impersonal isomorphisms. More precisely, for any $g \in \boldsymbol{G}$ and any extensive game form $\tilde{\gamma} \in \boldsymbol{\Gamma}$ resulting from $\mathcal{F}(g)$ via proper restriction there exists a strategic game form $\tilde{g} \in \boldsymbol{G}$ resulting from g via restriction such that $\mathcal{F}(\tilde{g})$ is impersonally outcome preserving isomorphic to $\tilde{\gamma}$.*

(4) A faithful representation $\mathcal{F}$ is said to be **minimal**, *if for every square $g \in \boldsymbol{G}$, the total rank of $\mathcal{F}(g)$ is minimal.*

(5) A faithful, consistent, and minimal representation is said to be **canonical.**

Remark 5.2 *Given our present state of development, we are in the position to construct a canonical representation. To this end, assign to every $g \in \boldsymbol{G}$ the symmetrization of a time structured atom (cf. Definition 4.18 and Remark 4.28). This mapping is not uniquely defined; given $g \in \boldsymbol{G}$, we may apply an impersonal and outcome preserving isomorphism to $\mathcal{F}(g)$ without 'essentially' changing the nature of the mapping thus defined. In this sense a 'time structured' canonical representation is defined uniquely 'up to impersonal outcome preserving isomorphisms'.*

Definition 5.3 *The* **time structured** *canonical representation as described by Remark 5.2 is denoted by $\mathcal{T}$.*

Clearly our next aim is to show that $\mathcal{T}$ is 'the' only canonical representation. As it stands now the development in Section 4 and in particular Theorem 4.27 point to symmetrizations of atoms but not necessarily to the time structured version. As a first result we shall now prove that the time structure appears necessarily for general square game forms with at least three strategies for each player.

Theorem 5.4 *Let $g = (N, S; A, h)$ be a square general game form such that $|S_i| = r \geq 3$ $(i \in N)$. Let α be an atom of g. Then the symmetrization of α is a totally rank minimal faithful representation of g, if and only if α is a T-atom.*

Proof: Without loss of generality we assume that $S_i = \{1, \ldots, r\}$ $(i \in N)$ holds true. The atom is denoted by $\alpha = (N, E, \prec, \mathbf{P}, \mathbf{C}; \iota; A, \eta)$, the symmetrization is γ; we may assume without loss of generality that α occurs as some γ^ξ in the sense of Definition 4.22 (3).

1st Step: Now we attach labels according to strategies at all nodes of γ except the root and its successors. To this end observe first that ψ_i identifies elements of S_i and of $\tilde{S}_i$ as explained in Definition 4.4. Therefore, if player i is in command at node ξ and chooses $s_i \in S_i$, this leads to a well specified successor ζ of ξ which now carries the label s_i.

2nd Step: This kind of labeling induces an identification of plays in γ as well as of all the atoms in γ as follows. First of all any $s \in S$ corresponds to a unique play in α (just follow the labels). Next consider the automorphism (π, id^*) of the preform of g. Here id^* is the natural family of 'identities' $id_i^* : S_i \to S_{\pi(i)}$. To this automorphism there corresponds a unique automorphism (π, ϕ) of the preform of γ (cf. Corollary 4.14). ϕ transforms the play in α labeled by s into some other play carrying the same label. In particular consider $\pi \neq id$ and $s = (1, \ldots, 1)$ or $s = (2, \ldots, 2)$ or etc. Then the second play cannot run through α, because it leads to the same outcome as the first one - but there is exactly one play carrying an outcome in each atom. From this we see immediately that ϕ carries α bijectively to some other atom in γ, say α^π, and that, indeed, $n!$ atoms can be identified by the permutations (id corresponding to α). As the faithful representation of g by γ is totally rank minimal, we conclude in view of Theorem 4.27 that γ has exactly the $n!$ atoms α^π $(\pi \in \Sigma(N))$.

3rd Step: Next, let $\mathcal{L}^\zeta(t)$ denote the set of nodes on level t that have ζ as a commom ancestor (with respect to $\prec$, this takes place in α). Next we introduce

$$\bar{t} = 1 + \max\{t | \mathcal{L}(E, \prec, t) = P_i \text{ for some } i \in N\}.$$

We have to show that $\bar{t}$ equals n. Let ζ be such that

(1) $\mathcal{L}^\zeta(\bar{t}) \neq P_i$ $(i \in N)$ and $r(\zeta) < \bar{t}$ (cf. Section 1 for the definition of $r(\cdot)$),

(2) the rank $r(\zeta)$ is maximal with respect to (1).

Then every successor $\xi \in C(\zeta)$ of ζ is the common ancestor of some group of nodes $\mathcal{L}^\xi(\bar{t})$ which belongs to one player set (unless $\bar{t} = n$, this case will be excluded henceforth). We now want to show that $\mathcal{L}^\xi(\bar{t})$ $(\xi \in C(\zeta))$ belong to different player sets, hence at least r players are in command on level $\bar{t}$. Figure 5.1 indicates the procedure to be followed during the remaining steps of the proof.

4th Step: To this end let $\xi_1, \xi_2, \xi_3 \in C(\zeta)$ be different successors of ζ (recall that $r = |S_i| \geq 3$). Assume without loss of generality that $\mathcal{L}^{\xi_1}(\bar{t}) \subseteq P_1$ and $\mathcal{L}^{\xi_2}(\bar{t}) \subseteq P_2$. It suffices to show that $\mathcal{L}^{\xi_3}(\bar{t})$ is not contained in P_1. Let $\mathcal{L}^{\xi_3}(\bar{t})$ be contained in P_i. The player

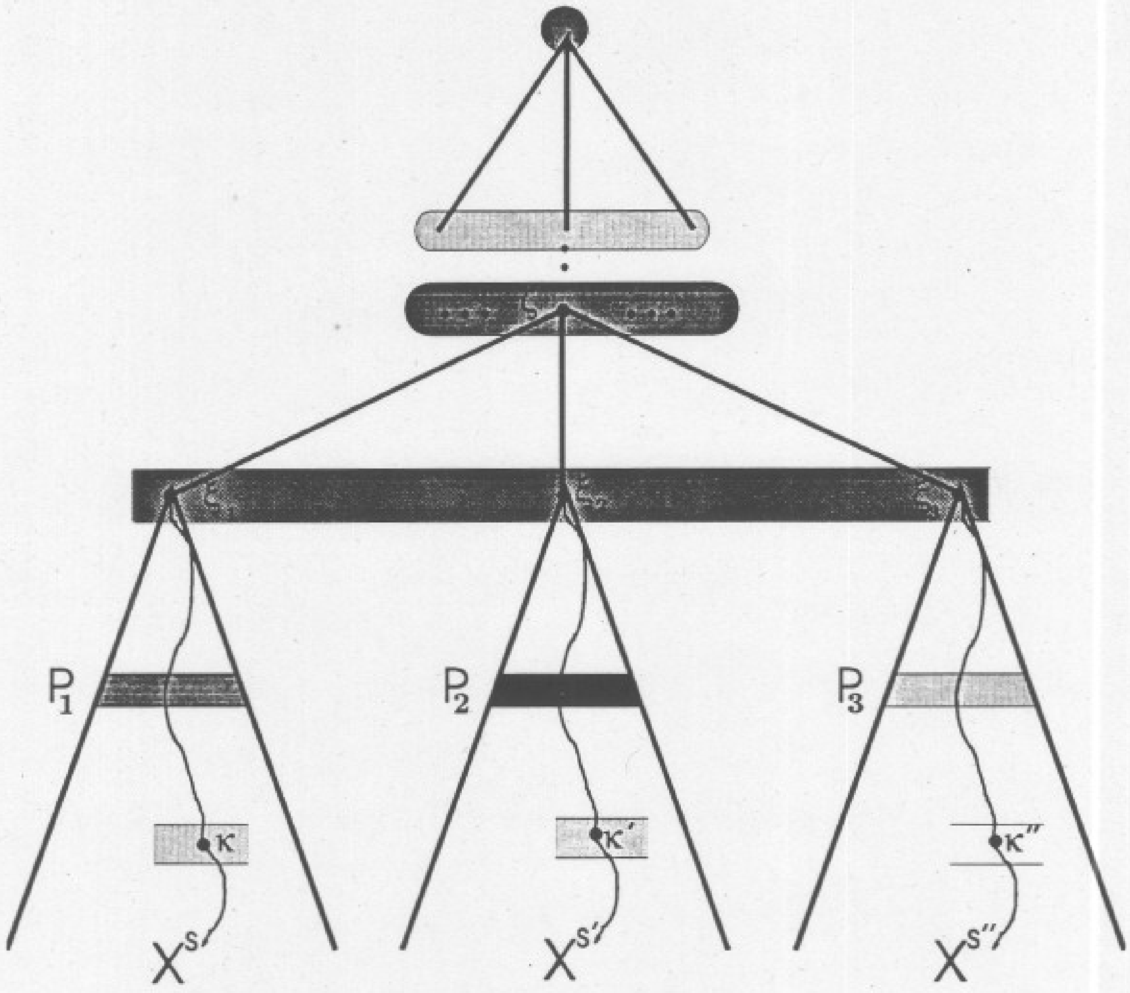

Figure 5.1: The preform of α

who is in command at node ζ assigns two labels to ξ_2 and ξ_3 as described in the first step; let these labels be s^2 and s^3 respectively. There is an impersonal automorphism of the preform of g which just transposes s^2 and s^3. To this automorphism we find the corresponding automorphism (id, ϕ^{23}) of the preform of γ with the aid of Corollary 4.14. In view of the 2nd Step we can single out an atom α^π which is the image of α under ϕ^{23}. We denote the levels etc. within α^π by subscript $_\pi$.
Next fix the successor ξ_3 of ζ. We have

$$\mathcal{L}_\pi^{\phi^{23}(\xi_3)}(\bar{t}) = \phi^{23}\mathcal{L}^{\xi_3}(\bar{t}) \subseteq \phi^{23}P_i = P_i,$$

because (id, ϕ^{23}) is impersonal. On the other hand, $\phi^{23}(\xi_3)$ carries the label s^2 (by construction of ϕ^{23}), hence $\mathcal{L}_\pi^{\phi^{23}(\xi_3)}(\bar{t})$ has to be a subset of $P_{\pi(2)}$. From this we conclude that $\pi(2) = i$. Next perform the same operation for the successor ξ_2 of ζ in order to show that $\pi(i) = 2$ holds true as well. And, if ξ_1 is fixed, it follows that $\pi(1) = 1$. Clearly this shows that $i \neq 1$ is true. Hence we have closed the argument starting at the end of the 3rd Step.

5th Step: Without loss of generality we assume $i = 3$. As player 3 appears the first time on level $\bar{t}$ (in α), ξ_1 is the ancestor of some $\kappa \in P_3$ with $r(\kappa) > \bar{t}$. Let ϕ^{12} be generated by the exchange of ξ_1 and ξ_2 analogously to the construction of ϕ^{23} by the exchange of ξ_2 and

ξ_3 in the 4th Step. The corresponding atom is $\alpha^{\pi^{12}}$ (as $\alpha^{\pi} = \alpha^{\pi^{23}}$ was specified above). Analogously to the 4th Step we conclude that $\pi^{12}(1) = 2, \pi^{12}(2) = 1$, and $\pi^{12}(3) = 3$. Let $s \in S$ be a strategy profile which generates a play X^s in α passing through ξ_1 and κ (use the labeling of the 2nd Step). The automorphism (id, φ^{12}) that induces (id, ϕ^{12}) via Θ throws s into some s' and, hence, specifies a play $X^{s'}$ in α. On level $r(\kappa)$ we find exactly one node κ' on $X^{s'}$. The labeling s corresponds to $\phi(X^{s'})$ in $\alpha^{\pi^{12}}$. As $\pi^{12}(3) = 3$, the play corresponding to label s in $\alpha^{\pi^{12}}$ will pass through P_3 on level $r(\kappa)$, i.e. $\phi(\kappa') \in P_3$. Therefore $\kappa' \in P_3$ holds as well. Note that $\kappa' \in \mathcal{L}^{\xi_2}(r(\kappa))$ due to the construction of s'.

6th Step: The same procedure argued with the automorphism ϕ^{13} (constructed analogously again) is now applied to the play X^s in α. Because κ on X^s, $\kappa \in P_3$ follows ξ_1 (hence X^s passes through P_1 first and reaches P_3 at κ) we conclude that the play $X^{s''}$ (obtained by using φ^{13} passes the level $r(\kappa)$ at some node κ'' which is an element of P_1. By symmetry reasons (application of ϕ^{23} to κ' or $X^{s'}$ respectively) we have to conclude that $\kappa'' \in P_2$. This is a contradiction which shows that the assumption $\bar{t} < n$ raised in the 3rd Step cannot be true. Hence α is time structured. **q.e.d.**

The main theorem can now be stated as follows.

Theorem 5.5 *There is a unique (i.e. up to impersonal outcome preserving isomorphisms) canonical representation of strategic games (over a given universal alphabet) and this is the time structured mapping $\mathcal{T}$.*

Proof:

As we have seen in Remark 5.2 the mapping $\mathcal{T}$ has the desired properties (of course $\mathcal{T}$ again is only defined up to impersonal outcome preserving isomorphisms). Thus it remains to show uniquenes.

To this end, let $\mathcal{F}$ be a representation enjoying the desired properties. Fix a strategic game $g = (N, S; A, h) \in \boldsymbol{G}$. Let $S^* = \prod_{i \in N} S_i^*$ be such that $S_i^* \supseteq S_i$ yields $|S_i^*| = r \geq \max_{j \in N} |S_j|$ $(i \in N)$ for some $r \geq 3$ and let $g^* = (N, S^*; A^*, h^*) \in \boldsymbol{G}$ be such that g is a restriction of g^*. The existence of $g^* \in \boldsymbol{G}$ is ensured by the choice of $\boldsymbol{U}$ which renders $\boldsymbol{G}$ to be sufficiently large. By Theorem 5.4 it follows that $\mathcal{F}(g^*) =: \gamma^* = \mathcal{T}(g^*)$ holds true (up to an IOP isomorphism). Let (id, ψ^*, id) be the IOP isomorphism between g^* and $\mu(\mathcal{T}(g^*))$. This automorphism in particular carries the subset S of strategies available in g into the strategies available for $\mu(\mathcal{T}(g^*))$, called $\tilde{S}^*$. We now define an extensive game form γ with the aid of γ^* and ψ^* : All we have to do is to take all plays of γ^* that are images of strategies $s \in S$ under ψ^*, i.e. all plays generated by $\psi^{id}(S) \subseteq \tilde{S}^*$. (This amounts to taking all plays X^s in all atoms α^{π} of γ^* as discussed in the proof of Theorem 5.4.)

The nodes of γ^* obtained by persuing all these plays together with the obvious binary relation constitute a tree to which all further data of γ^* may be restricted in the obvious way. The time structured nature of a symmetrization which is characteristically for γ^* allows for an easy verification of the fact that the restriction is, indeed, proper. Call the resulting extensive game form γ. As $\mathcal{T}$ respects prper restriction it follows that γ is a faithful representation of g. However, consistency applies as well for $\mathcal{F}$, hence $\mathcal{F}(g)$ is IOP isomorphic to $\mathcal{T}(g)$. **q.e.d.**

Remark 5.6 *Definition 5.1 can be generalized to mappings $\mathcal{F} : \boldsymbol{H} \to \boldsymbol{\Gamma}$ for any subset $\boldsymbol{H} \subseteq \boldsymbol{G}$ of general strategic game forms without changes; however $\boldsymbol{H}$ has to comply with a few additional requirements. This is so because a set $\boldsymbol{H}$ which is too small may not allow for sufficiently many games, thus the existence requirement of Definition 5.1 (3) could be damaged. To avoid this possibility, call $\boldsymbol{H}$* ***hereditary****, if every restriction of a game form of $\boldsymbol{H}$ belongs to $\boldsymbol{H}$. For a hereditary $\boldsymbol{H}$ the time structured representation $\mathcal{T}$ restricted to $\boldsymbol{H}$ is clearly canonical. Uniqueness can be guaranteed (repeat the proofs of Theorems 5.4 and 5.5) provided the following condition is satisfied.*

$$\begin{array}{l} \textit{For any } g = (N, S; A, h) \in \boldsymbol{H} \textit{ there is } r \geq 3 \textit{ and a strategic game form} \\ g^* = (N, S^*; A^*, h^*) \in \boldsymbol{H} \textit{ such that } |S_i^*| = r \geq \max_{j \in N} |S_j| \quad (i \in N). \end{array} \tag{5.1}$$

The remainder of this section is devoted to an example which shows that (5.1) cannot be dropped as a prerequisite of uniqueness.

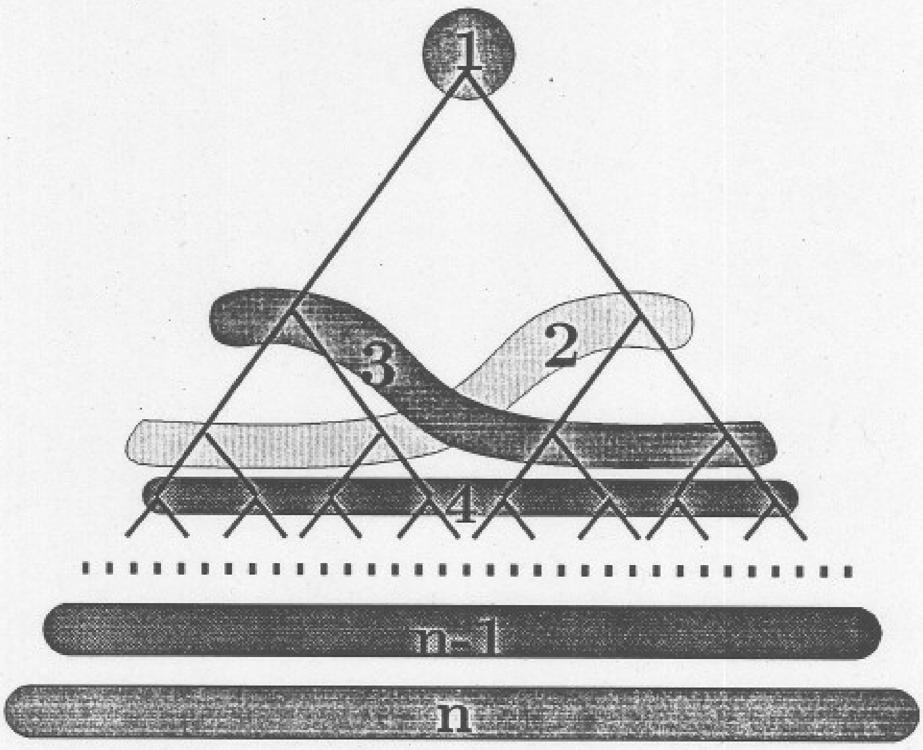

Figure 5.2: The cross over example

Example 5.7 *Let $n \geq 3$ and $\boldsymbol{H}$ be a hereditary subset of $\boldsymbol{G}$ consisting of game forms with strategy sets of cardinality 1 or 2. For $N = \{1, \ldots, n\}$ and $(N, S; A, h) \in \boldsymbol{H}$ satisfying $|S_i| = 2 \quad (i \in N)$, define the atom α as follows. The nodes of E are given by $\{(t, l) | 0 \leq$*

$t \leq n, 1 \leq l \leq 2^t\}$. The player sets are specified via $(t,l) \in P_{t+1}$ $(t = 0$ or $3 \leq t < n)$; $(1,l) \in P_{l+1}$; $(2,l) \in P_3$ $(l \leq 2)$ and $(2,l) \in P_2$ $(l \leq 3)$. The choices are indicated in Figure 5.2.

Let $\mathcal{F} : \boldsymbol{H} \rightarrow \boldsymbol{\Gamma}$ be the mapping that assigns the symmetrization of α to g and is arranged consistently otherwise. This mapping is canonical. The clue is found by an inspection of Figure 5.2 and of the proof of Theorem 5.4. As the 3rd strategy is missing, the overcrossing of player sets P_2 and P_3 cannot be avoided by the construction supplied in the 5th Step.

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