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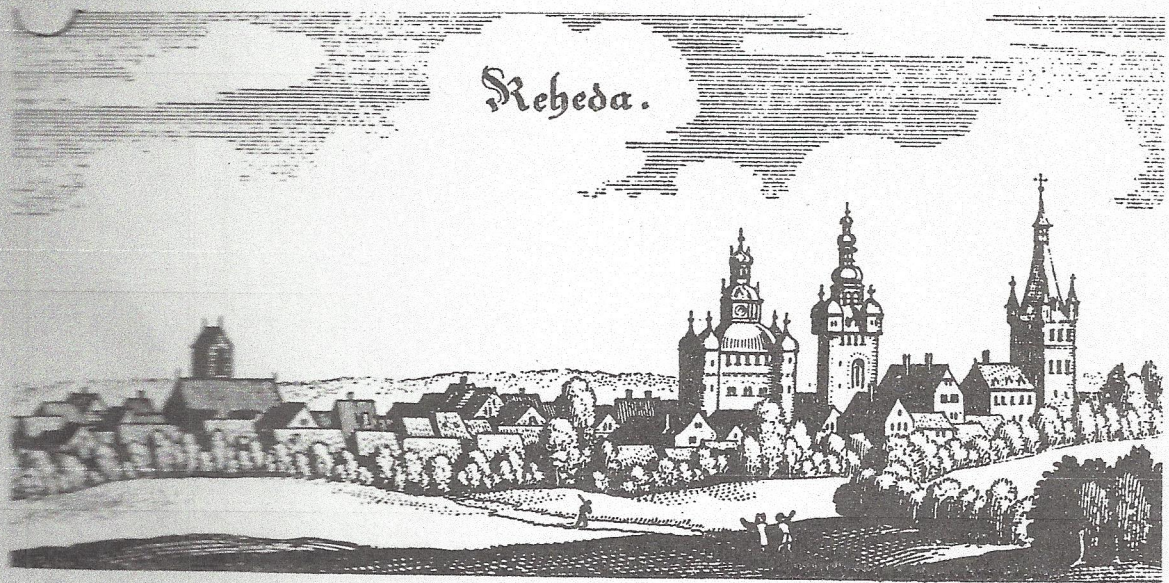
**Arbeiten aus dem
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A New Technical² Progress Function
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Editorial Remark

The following paper has been written in the fall of 1962, while I stayed at MIT with a fellowship of the "Deutsche Forschungsgemeinschaft." At that time I had produced only a dozen copies. After the paper was finished Christopher Bliss, Edmund Phelps, Robert Solow and Paul Samuelson discussed it with me. Their comments would have been taken into account in a revision of the paper. But some of the criticisms, in particular of Paul Samuelson, were so much to the point that I decided not to pursue the basic idea of the paper. Later on Kennedy [1] published a paper containing the same basic idea, whereupon a substantial number of papers were written on the same topic.

In the meantime, all copies of my paper seemed to have got lost. I am indebted to Richard Eckaus that he had kept a copy of the paper in his files. Since the main point of the paper has been discussed by others I do not think that the paper should take away scarce publication space in a professional journal. It is reproduced here to enable those to read it who are interested in the "Dogmengeschichte" of the theory of induced technical progress.

[1] Charles Kennedy: Induced Bias in Innovation and the Theory of Distribution, The Economic Journal, Volume LXXIV, 1964.

In this paper I am interested in the quality or "structure" of technical progress rather than its quantity, making this distinction although I know that it is impossible to separate totally these two different aspects of technological change. For making calculations easy I shall neglect most of the recent achievements of the new frontiers of growth economics as for instance embodied technical progress, the division of the economy into at least two sectors, etc. Since Samuelson has shown that it is at least sometimes possible to simulate a more complicated model by a simple "Surrogate Production Function" with "Labor" and "Capital" as arguments, I shall use this well known and convenient "surrogate" tool for the derivation of my propositions. Perhaps it is possible to carry over some of the results to models which are well equipped with the heavy artillery of many sectors, physical life-time constraints for all and fixed labour input coefficients for already existing capital goods and other useful complications.

Summary of Results

I want to show what the effects of a "technical progress function" are, which represents the possibility of substituting labour saving by capital saving technical progress or vice versa. One can summarise the results of the model as follows:

- 1) Under the assumption that entrepreneurs try to choose the optimum composition of labour saving and capital saving technical progress, under the further assumption of a production function with an elasticity of substitution less than one, I can show that the economy approaches an equilibrium path of growth, which is characterized by Harrod neutral technical progress (which is not very exciting) and a distribution of income, which is only determined by the technical progress function and is influenced neither by the

production function nor by the savings ratio of the economy. Although this result seems to be new it also seems to be quite plausible: if there exists a possibility of choosing different kinds of technical progress, which is in fact the possibility of choosing different kinds of production functions, we should not expect that a special production can have the same importance as in the case where the whole system relies definitely on its properties. If a special production function is inconvenient it will be transferred over time into a more convenient one. The possibilities of transformation are now the last fixed parameters, which determine the system, not the results of this transformation process.

2) It can also be shown that there is a discrepancy between the private optimal composition of technical progress and the social optimal composition. In the case of the optimum of the firm there exists a cycle such that capital saving and labour saving technical progress (in the sense of Harrod) alternate over time. But the amplitudes of the cycle become smaller and smaller until the economy reaches the equilibrium state, in which technical progress is (Harrod-) neutral. In the case of the social optimum, technical progress is always almost Harrod neutral, having only a very small, constant capital saving bias, if the initial distribution of income was more in favour of profits than on the equilibrium path and a small labour saving bias, if in the beginning the profit share was smaller than in equilibrium. The discrepancy between the optimum behaviour of the firm and the social optimum coincides with the discrepancy of maximizing the rate of growth of output in a relatively short period (private optimum) and the rate of growth of output in a considerably longer period (social optimum) under the assumption of a given savings ratio.

3) The independence of the equilibrium income distribution of the savings ratio reminds one of the unity elasticity of substitution case, i.e. the Cobb-Douglas-Function.

It turns out to be possible to simulate by a Cobb-Douglas-Function the different social optimum equilibrium paths, which correspond to different savings ratios. The exponents of this "Surrogate - Surrogate - Function" are equal to the share of profits respectively to the share of wages in the equilibrium path.

4) I do not pretend to explain by this model the long-run stability or income distribution which is sometimes claimed to be evident out of the existing statistical data. Since the income distribution in this model is a function of the special shape of the technical progress function, this would be possible only if the technical progress function is constant in time. But why should it be constant? Nobody can stop the relevant parameters to change over time. The technical progress function does not represent natural or anthropological constants.

The Model

Assume an economy with a macroeconomic production function

$$Y = f(X_1, X_2) = X_2 \cdot \varphi\left(\frac{X_1}{X_2}\right) \quad (1)$$

which is homogeneous of degree one, where

$$X_1 = AK : X_2 = BL \quad (2)$$

K representing capital and L labour, A and B are parameters, which we shall use for the introduction of technological progress. We define

$$X = X_1/X_2 \text{ and } p(X) = \frac{\delta f}{\delta X} \cdot \frac{X_1}{Y} = \varphi'(X) \frac{X}{\varphi(X)} \quad (3)$$

Further let us assume (this is an assumption on f)

$$\left. \begin{array}{l} 0 < p(X) < 1; \quad p'(X) < 0 \\ \lim_{X \rightarrow 0} p(X) = 1; \quad \lim_{X \rightarrow \infty} p(X) = 0 \end{array} \right\} \text{ for } X > 0 \quad (4)$$

(4) implies that the elasticity of substitution is smaller than one (in absolute value).

One example for a production function, which obeys conditions (4) is the Arrow-Solow constant elasticity of substitution function

$$Y = [\gamma \dot{X}_1^{-\beta} + (1 - \gamma)X_2^{-\beta}]^{-\frac{1}{\beta}}; \quad 0 < \gamma < 1$$

in the case that β is positive.

The next thing I want to introduce is a "technical progress function", which in this model is simply a relationship between capital augmenting (A increasing) and labour augmenting (B increasing) technical progress.

Assume that

$$\frac{\dot{A}}{A} = \varphi\left(\frac{\dot{B}}{B}\right) \quad (5)$$

holds, where φ is characterized by

$$\varphi(0) > 0; \quad \varphi'\left(\frac{\dot{B}}{B}\right) < 0 \quad \text{and} \quad \varphi''\left(\frac{\dot{B}}{B}\right) < 0 \quad (5a)$$

Since φ is a monotonously decreasing function of $\frac{\dot{B}}{B}$, we also can write $\frac{\dot{B}}{B}$ as a function of $\frac{\dot{A}}{A}$

$$\frac{\dot{B}}{B} = \varphi^{-1}\left(\frac{\dot{A}}{A}\right) \quad (6)$$

where

$$\varphi^{-1}(0) > 0; \quad (\varphi^{-1})' < 0 \quad \text{and} \quad (\varphi^{-1})'' < 0 \quad (7)$$

In the following I shall use the symbols a and b defined by

$$a = \frac{\dot{A}}{A}, \quad b = \frac{\dot{B}}{B}$$

Since the elasticity of substitution is less than one the total effect of technological progress is labour saving in the sense of Hicks, if $b > a$, and capital saving, if $b < a$.

The first question which I would like to ask and to answer is: how to achieve a maximum current rate of growth of output for given rates of growth of capital and labour?

Differentiating Y with respect to time

$$\dot{Y} = \frac{\delta f}{\delta X_1} \dot{X}_1 + \frac{\delta f}{\delta X_2} \dot{X}_2$$

yields

$$\begin{aligned} \frac{\dot{Y}}{Y} \frac{\delta f}{\delta X_1 Y} \frac{X_1}{X_1} \cdot \frac{\dot{X}_1}{X_1} + \frac{\delta f}{\delta X_2} \cdot \frac{X_2}{Y} \frac{\dot{X}_2}{X_2} \\ = p(X) \left[a + \frac{\dot{K}}{K} \right] + [1 - p(X)] \left[b + \frac{\dot{L}}{L} \right] \end{aligned}$$

Since $\dot{K}/K, \dot{L}/L$ and X are given, we have to maximize the expression

$$Z = p(X)a + [1 - p(X)] b = p(X) \varphi(b) + [1 - p(X)] b$$

I differentiate with respect to b and put $Z'(b)$ to zero

$$Z'(b) = p(X) \varphi'(b) + 1 - p(X) = 0$$

or

$$\varphi'(b) = - \frac{1 - p(X)}{p(X)} \quad (8)$$

Since $Z''(b) = p(X)\varphi''(b)$ is negative we know that (8) represents the value of b which leads to a maximum rate of growth of output in the current period. The "marginal rate of substitution" between a and b has to be equal to the ratio between wages income and profits income.

Behaviour of Entrepreneurs

One reasonable assumption about the behaviour of entrepreneurs in this economy is that they believe the prevailing relative shares of profits and wages to remain constant in the future. If they want to minimize costs under this assumption they will choose that kind of technical progress which maximizes the absolute value of the (negative) rate of change of costs per unit of output at existing factor prices. This is because they assume that factor prices will change inversely to the factor inputs, which

is another way of saying that they assume that the relative weights of labour and capital costs remain unchanged over time. If entrepreneurs behave in this way, they will choose precisely that technical progress structure, which maximizes the current rate of growth of output.

The result of this behaviour of the firms would be that the economy tends towards an equilibrium path of growth which is characterized by a constant distribution of income and by a constant capital output ratio. This is at least true if the investment ratio of the economy is constant. I shall try to characterize this equilibrium path. The constancy of the distribution of income requires a constant X and hence an equal rate of growth for X_1 and X_2 . This leads to a rate of growth of output, which has also the same value. If the capital output ratio is constant, the stock of capital grows with the same rate as output does. But the growth rate of X_1 is equal to the sum of the growth rate of capital and the growth rate of A , which is a . This implies that a has to be zero in the equilibrium path. But a is zero only if the distribution of income is such that the entrepreneurs decide to make a equal to zero. This they will do only at one special ratio between wages income and profits income, which we can characterize by the equation

$$\varphi'(\beta) = - \frac{1-p(X)}{p(X)} \quad (9)$$

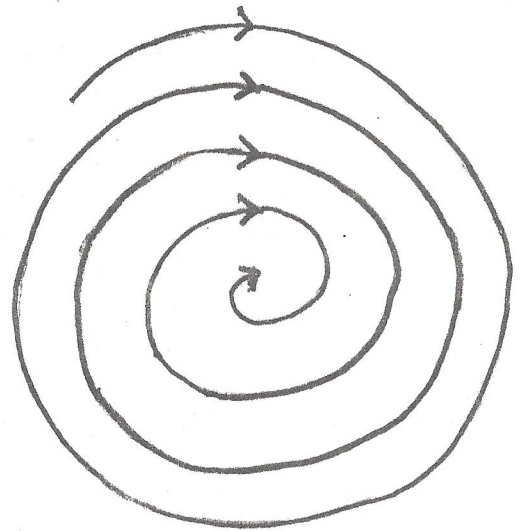
where β is defined by

$$\varphi(\beta) = 0 \quad (9a)$$

Hence the stable income distribution is determined only by the function φ and it is independent of the shape of the production function and of the savings ratio. There remains of course the question whether the economy approaches this equilibrium path or not. The equilibrium can be proved to be stable if the share of profits is a decreasing function of x , i.e. if (4) holds. Verbally the mechanism, which leads the economy to the equilibrium path, can be described as follows.

If X is smaller than its equilibrium value \bar{X} , profits are higher than in the equilibrium. Therefore a is positive. Since in the long run capital and output are growing at almost the same rate, X_1 is growing faster than Y . This implies that X_2 is growing not as fast as Y or even X_1 . Therefore X must grow until it reaches its equilibrium value. Now it might be possible that X grows still when \bar{X} is reached. This will happen if A is relatively high in the moment when X is equal to \bar{X} and therefore the capital output ratio is low compared with the given savings ratio. If a becomes higher than \bar{X} , profits become smaller than in equilibrium and therefore a will be negative. This lowers A and hence in the long run also X , until it reaches again its equilibrium value \bar{X} . Now presumably A will be too low, i.e. the capital output ratio will be too high for the equilibrium conditions and X will again fall below its equilibrium value. It can be shown mathematically that A and X approach eventually their equilibrium values in a way similar to Fig. 1. The convergence can be made plausible by the following arguments: sometimes A as well as X tend towards their stable values, but if one of them withdraws from its equilibrium value, the other approaches it or is at least constant.

A



X

Of course the assumption (8) is not crucial for our result. Entrepreneurs might foresee changes in the relative weight of capital and labour costs and might therefore behave somewhat different. If they decide to install machines with an expected economic life time of τ years, they are interested to minimize costs over this whole period. They may expect for

instance that the rate of change of income distribution remains the same in the future as it is now. Then we would get the following formulas. Let us define

$$\Delta(X) = \frac{1 - p(X)}{p(X)}$$

Entrepreneurs will equalize the marginal rate of substitution of \underline{a} and \underline{b} and a kind of expected weighted average of the income distribution index $\Delta(X)$ over the next τ years, that is

$$\varphi'(b) = -\tilde{\Delta}(X, \tau) = -\Delta(X) \exp \left[\left(\frac{\dot{\Delta}(X)}{\Delta(X)} \right) \theta^\tau \right] \quad (10)$$

where $0 < \theta < 1$. The value of θ is determined by several economic factors, one of them being the relative importance of the earlier in comparison to the later years of life of the capital goods. Since the nearer future is much more important than the more remote years are (because of psychological reasons, because of problems of the risk calculations, because of the higher gross rents, which can be earned out of new machines compared with old machines, etc.) a realistic value of θ would probably lie between .15 and .35.

Perhaps entrepreneurs are even more sophisticated and take also into account the rate of change of the rate of change of the distribution of income. All these behaviour patterns would in general have stabilizing effects on the movements of X and Δ . This can be made plausible for formula (10). It is easy to see that X and therefore $\Delta(X)$ would move in the same direction as they would according to (8). But the movement would be damped: if e.g. $\dot{\Delta}(X)$ is positive, the absolute value of $\varphi'(b)$ would be greater than in the case of (8). But a high absolute value of $\varphi'(b)$ is equivalent to a small (perhaps even negative) value of $\varphi(b) = a$. This means that the rate of growth of X and $\Delta(X)$ would become smaller. Thus we see that a behaviour of entrepreneurs, which takes into account the changes in the near future, would have a stabilizing effect

on the economy. The equilibrium path itself will be characterized by the same properties as in the case of (8).

The Social Optimum Technical Progress

I turn now to the question, whether there is a difference between the social optimum composition of technological progress and the so far investigated kind of technical progress, which we may call the private optimum composition. The first problem is to find the maximum growth rate of output, which can last forever.

It is easy to verify that $\beta + \gamma$ is a sustainable rate of growth of output, where β is the value of b for which a becomes zero, and γ is the rate of growth of the labour force. Because, if a is zero, technical progress is Harrod neutral and one knows that in that case the "natural" rate of growth of the system is equal to $\beta + \gamma$. Is it possible to maintain a higher rate than $\beta + \gamma$?

First we see that Y cannot grow faster in the long run than the slower growing of its two arguments X_1 and X_2 . If e.g. X_1 is always growing faster than X_2 , X will become larger and larger and therefore $p(X)$ will approach zero. This has the consequence, as we can infer from

$$\frac{\dot{Y}}{Y} = p(X) \frac{\dot{X}_1}{X_1} + [1-p(X)] \frac{\dot{X}_2}{X_2}$$

that output will grow only so fast as X_2 . If we call the long-run rate of growth of a magnitude Z , $R(Z)$, i.e.

$$R(Z) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\frac{Z(t)}{Z(0)} \right)$$

we can write

$$R(Y) = \text{Min} [R(X_1), R(X_2)]$$

So we have to maximize $\text{Min} [R(X_1), R(X_2)]$. $R(X_2)$ is only greater than $\beta + \gamma$, if $R(B) > \beta$, which is equivalent to $R(A) < 0$.

The consequence of this policy becomes clear after we have calculated $R(K)$ for positive $R(Y)$.

$$\begin{aligned} R(K) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{K(t)}{K(0)} = \lim_{t \rightarrow \infty} \frac{1}{t} \log \left[\frac{S \int_0^t Y du}{K(0)} + 1 \right] \\ &= \lim_{t \rightarrow \infty} \log \left(\frac{S}{K(0)} \right)^{\frac{1}{t}} \left(\int_0^t Y du + \frac{K(0)}{S} \right)^{\frac{1}{t}} = \log \lim_{t \rightarrow \infty} \left(\int_0^t Y du \right)^{\frac{1}{t}} \\ &= \log \lim_{t \rightarrow \infty} \left[\int_0^t e^{R(Y)u} N(u) du \right], \text{ where } N(t) \text{ is a} \end{aligned}$$

function of t with $R(N) = 0$. Since

$$\int_0^t e^{R(Y)u} N(u) du = M(t) \frac{e^{R(Y)t} - 1}{R(Y)}, \text{ where}$$

$$\begin{aligned} \text{Min}_{0 \leq u \leq t} N(u) &\leq M(t) \leq \text{Max}_{0 \leq u \leq t} N(u) \end{aligned}$$

we know also that $R(M) = 0$. Hence

$$\begin{aligned} R(K) &= \log \lim_{t \rightarrow \infty} \left[\frac{M(t)}{R(Y)} \right]^{\frac{1}{t}} \left[e^{R(Y)t} - 1 \right]^{\frac{1}{t}} \\ &= \log \lim_{t \rightarrow \infty} \left[e^{R(Y)t} \right]^{\frac{1}{t}} = \log e^{R(Y)} = R(Y) \end{aligned}$$

Therefore

$$R(K) = R(Y) \leq R(X) = R(K) + R(A) \text{ for } R(Y) > 0.$$

which proves that no long-run positive rate of growth of γ is compatible with $R(A) < 0$. So we have shown that the maximum sustainable rate of growth of Y is equal to $\beta + \gamma$ and that $R(A) = 0$ in the case where this maximum rate of growth is achieved.

Starting from the valuation axiom that for a given savings ratio it is always better to realize the maximum $R(Y) = \gamma + \beta$ than not to do this, we come to our next question: which is the optimal $A(t)$?

I have to define more precisely, what I mean by the optimal $A(t)$.

Certainly we can write output as a function of time in the form

$$Y(t) = e^{(\gamma+\beta)t} D(t)$$

It might be possible to find an upper limit of $D(t)$; for $\gamma+\beta$ is the maximum sustainable growth rate of output. If such an upper limit exists, it seems to be a reasonable long-run optimum policy to maximize $\lim_{t \rightarrow \infty} D(t) \leq B(0)L(0) \left(\frac{\bar{A}}{A(0)} \right) \frac{1}{\varphi'(\beta)} \varphi(\bar{X}) = \bar{D}$ (12)

where \bar{X} is determined by the equation

$$\varphi'(\beta) = - \frac{1-p(\bar{X})}{p(\bar{X})} \quad (12a)$$

and \bar{A} is defined by

$$\bar{A} = \frac{(\beta+\gamma)\bar{X}}{\varphi(\bar{X})} \quad (12b)$$

I first prove the proposition in mathematical terms and then interpret it economically.

Proof of (12): Consider first the behaviour of the capital stock, if

$$\lim_{t \rightarrow \infty} e^{-(\beta+\gamma)t} Y(t) = D$$

holds, where D is a positive number. We can write

$$Y(t) = e^{(\beta+\gamma)t} [D + G(t)]$$

where

$$\lim_{t \rightarrow \infty} G(t) = 0$$

Therefore the amount of capital, discounted by the rate $\beta+\gamma$ is

$$e^{-(\beta+\gamma)t} K(t) = e^{-(\beta+\gamma)t} K(0) + e^{-(\beta+\gamma)t} \int_0^t [e^{(\beta+\gamma)u} (D+G(u))] du$$

Define H by

$$H = e^{-(\beta+\gamma)t} \int_0^t e^{(\beta+\gamma)u} G(u) du$$

hence

$$\dot{H} = G(t) - (\beta + \gamma) e^{-(\beta + \gamma)t} \int_0^t e^{(\beta + \gamma)u} G(u) du = G(t) - (\beta + \gamma)H(t)$$

This we can write

$$\frac{d|H|}{dt} = \pm |G(t)| - (\beta + \gamma)|H|$$

and therefore

$$\lim_{t \rightarrow \infty} \frac{d|H|}{dt} + (\beta + \gamma)|H| = 0$$

which is possible only if

$$\lim_{t \rightarrow \infty} H = 0$$

Therefore

$$\lim_{t \rightarrow \infty} e^{-(\beta + \gamma)t} K(t) = \lim_{t \rightarrow \infty} e^{-(\beta + \gamma)t} S \int_0^t e^{(\beta + \gamma)u} D du = \frac{S}{\beta + \gamma} D$$

Hence the capital coefficient in the limit equals $\frac{S}{\beta + \gamma}$. If

$$A^* = \lim_{t \rightarrow \infty} A$$

we have for the limit of X

$$X^* = \lim_{t \rightarrow \infty} X = \frac{A^* \frac{S}{\gamma + \beta} Y}{X_2} = A^* \frac{S}{\gamma + \beta} \varphi(X^*)$$

$$\text{or } X^* - A^* \frac{S}{\gamma + \beta} \varphi(X^*) = 0$$

From this we get by the implicit function theorem

$$\frac{\delta X^*}{\delta A^*} = \frac{-\frac{S}{\gamma + \beta} \varphi(X^*)}{1 - A^* \frac{S}{\gamma + \beta} \varphi'(X^*)} = \frac{X^*/A^*}{1 - X^* \frac{\varphi'(X^*)}{\varphi(X^*)}} = \frac{1}{A^*} \frac{X^*}{1 - p(X^*)}$$

Since $\varphi(b)$ is a concave function of b and β is defined by $\varphi(\beta) = 0$ we can write

$$a = \varphi'(\beta) (b - \beta) - \delta(b)$$

where

$$\delta(b) > 0 \text{ for } b \neq \beta$$

$$\delta(b) = 0 \text{ for } b = \beta$$

From this we infer by integration

$$\log \frac{A}{A(0)} = \varphi'(\beta) \log \frac{B}{B(0)} - \varphi'(\beta) \beta t - \int_0^t \delta(b(u)) du$$

or

$$\frac{B}{B(0)} \leq e^{\beta t} \left(\frac{A}{A(0)} \right)^{\frac{1}{\varphi'(\beta)}}$$

Hence

$$Y(t) = BL\varphi(X) \leq e^{\beta t} B(0) \left(\frac{A}{A(0)} \right)^{\frac{1}{\varphi'(\beta)}} e^{\gamma t} L(0) \varphi(X)$$

Going to the limit for any $A^* = \lim_{t \rightarrow \infty} A$

$$D = \lim_{t \rightarrow \infty} e^{-(\beta+\gamma)t} Y(t) \leq B(0)L(0) \left(\frac{A^*}{A(0)} \right)^{\frac{1}{\varphi'(\beta)}} \varphi(X^*(A^*))$$

holds. I call the right hand Q and maximize it with respect to the only free variable A^* .

$$0 = \frac{\delta Q}{\delta A^*} = \frac{Q}{\varphi'(\beta) A^*} + \frac{Q}{\varphi(X^*)} \cdot \varphi'(X^*) \frac{X^*}{1-p(X^*)} \cdot \frac{1}{A^*}$$

This leads to

$$\frac{1}{\varphi'(\beta)} + \frac{1(X^*)}{1-p(X^*)} = 0$$

or

$$X^* = \bar{X} \text{ (see (12a))}$$

Since the second derivative at that point is negative, we know that Q and therefore D are nowhere greater than is indicated by (12). Q.E.D.

By a similar reasoning it is also easy to see that $D(t)$ will eventually not be higher than \bar{D} , if D has no limit. Further we know now that A must have a limit in order to get a limit of $D(t)$. The interesting question which remains is: how to achieve a D as close as possible to \bar{D} ? In the special case, where $A(0) = \bar{A}$, the economy can reach \bar{D} itself, for in that case a can always be put to zero with the effect that the term

$$\int_0^t \delta(b) du$$

is vanishing for any t . In general, the discrepancy between D and \bar{D} results from a positive

$$\int_0^t \delta(b) du$$

or in a formula

$$D = e^{-\int_0^{\infty} \delta(b) du} \bar{D}, \text{ if } \lim_{t \rightarrow \infty} A = \bar{A}$$

The theorem of Taylor tells us that

$$a = \varphi'(\beta) (b - \beta) + \varphi''(\beta + \theta b) \frac{(b - \beta)^2}{2}; \quad 0 \leq \theta \leq 1$$

and hence

$$\lim_{b \rightarrow \beta} \frac{a - \varphi'(\beta) (b - \beta)}{a} = 0$$

The percentage "loss of friction" by shifting A can be made arbitrarily small if one moves A very cautiously. Thus every value $D < \bar{D}$ can be achieved. Our mistake is therefore negligible, if we assume that

$$\text{Max}_{A(t)} \lim_{t \rightarrow \infty} e^{-(\beta + \gamma)t} Y(t) = B(0)L(0) \left(\frac{\bar{A}}{A(0)} \right)^{\frac{1}{\varphi'(\beta)}} \varphi(\bar{X}) \quad (13)$$

where \bar{X} and \bar{A} are defined in (12b).

The interesting although not too surprising result of the analysis is that in the social optimum case the economy approaches the same equilibrium path as it does in the laissez faire state of affairs.

The social optimum equilibrium income distribution is described by (12a) and it is equal to the laissez faire equilibrium income distribution.

The discrepancy between the optimum of the firm and the social optimum lies in their different manner to approach the equilibrium path. The "friction losses" are higher in the laissez faire economy, because A does not change slowly here and it even oscillates around its long-run equilibrium value.

The consequence of these losses is that the economy eventually lags some years behind the social optimum economy. The order of magnitude of this lag depends mainly on two factors: it will be the greater the larger the difference between $A(0)$ and \bar{A} is, and it will be the smaller the better entrepreneurs anticipate the future development of the income distribution. For a good anticipation of the future reduces the absolute value of a and dampen the amplitude of fluctuations of A , as we have seen earlier.

But it should be underlined that the most important effect on the lag is due to the initial condition $A(0)$. Since it is improbable that the technical progress function of an economy changes very suddenly it is fair to assume A being relatively close to its equilibrium value \bar{A} after some decades of laissez faire history. If the government of a developed capitalistic country decides to take care of the structure of technological progress, it will find that A already moved in the past very close to its optimum value. Thus an intervention, which could have a substantial improving effect on productivity, comes too late. This statement is not true, if s changes.

Although it is tempting to construct a bridge from business cycle theory to the fluctuations of A and of the capital output ratio in this model, I am rather sceptical about the di-

rect applicability of my technical progress function to problems of the real world. This is a too simple tool for an isolated use in the attempt to explain anything in reality.

The Surrogate Surrogate Production Function

As mentioned before the equilibrium distribution of income is not only independent of the special shape of the production function but it also does not depend on the average rate of savings. This can be verified from (12a). The result brings us into the neighbourhood of the Cobb-Douglas Function, where savings also do not have an influence on the income distribution.

Consider the Cobb-Douglas Function

$$Y = e^{\alpha t} K^m; \quad 0 < m < 1, \quad \alpha > 0$$

The equilibrium path is defined by the constancy of the capital output ratio and of the interest rate.

If $\frac{K}{Y}$ is constant, we have $\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y}$ and therefore

$$\frac{\dot{Y}}{Y} = \alpha + m \frac{\dot{K}}{K} = \alpha + m \frac{\dot{Y}}{Y} = \frac{\alpha}{1-m}$$

The value of the constant capital output ratio is given by

$$\frac{K}{Y} = \frac{\dot{K}}{\dot{Y}} = \frac{sY}{\frac{\alpha}{1-m}Y} = (1-m) \frac{s}{\alpha} \quad (14)$$

where s is the savings ratio. Substituting the K of the production function by the corresponding expression in (14) yields

$$Y = e^{\alpha t} \left((1-m) \frac{s}{\alpha} \right)^m Y^m$$

and hence

$$Y^{1-m} = e^{\alpha t} \left(\frac{1-m}{\alpha} \right)^m s^m \quad \text{or} \quad Y = e^{\frac{\alpha}{1-m} t} \left(\frac{1-m}{\alpha} \right)^{\frac{m}{1-m}} s^{\frac{m}{1-m}}$$

Y is proportional to $s^{\frac{m}{1-m}}$

Consider now (13) together with (12a) and (12b). Since \bar{X} is independent of s , we get \bar{A} as a proportional of the inverse of s

$$\bar{A} = \frac{1}{s} \cdot \frac{(\beta + \gamma)\bar{X}}{\varphi(\bar{X})}$$

Therefore

$$e^{-(\beta + \gamma)t} \bar{Y} = \text{Max} \lim_{t \rightarrow \infty} e^{-(\beta + \gamma)t} Y = B(0)L(0) \left(\frac{A(0)s\varphi(\bar{X})}{(\beta + \gamma)\bar{X}} \right)^{-\frac{1}{\varphi'(\cdot)}} \cdot \varphi(\bar{X})$$

or abbreviated, respecting (12a)

$$e^{-(\beta + \gamma)t} \bar{Y} = G(\bar{X})s^{-\frac{1}{\varphi'(\bar{X})}} = Gs^{\frac{p(\bar{X})}{1-p(\bar{X})}} \quad (15)$$

where \bar{X} is independent of s .

The social optimum equilibrium value of output is the same function of the savings ratio as the equilibrium value of output is in the case of the Cobb-Douglas Function. The exponent $\frac{p(\bar{X})}{1-p(\bar{X})}$ has the same meaning as in the Cobb-Douglas

Function: it is the ratio between capital income and labour income. Therefore I call (15) the Surrogate Surrogate Production Function, referring to the fact that the production function itself is a surrogate for more complicated productivity relations. The surrogate of the production function is no more a real production function, since it does not tell us what happens at a given state of technical knowledge. But if technical knowledge can be influenced by the society, this also does not seem to be the interesting question. In this case it is more interesting to know what the output would be under some resource restrictions and under the assumption that the society influences technical knowledge in an optimal way. The Surrogate Surrogate Function answers this last question. It is in some sense the envelope of all production functions (in the normal sense), which are available to the society, as in some sense the normal macroeconomic production function is the envelope of more special microeconomic pro-

duction possibilities.

We might proceed in this game and try to look for a Surrogate Surrogate Surrogate Function by making the assumption that my technical progress function itself can be shifted by the society. I shall not do that in this paper.

A Pseudo Marxistic Conjecture

I am far away from being able to prove what I want to suggest in this section. It is certainly a question of taste, whether the reader will follow me in my conjectures. I want to combine the Golden Rule with certain aspects of Marxian economics.

Assume for the simplicity of the argument that the labour force is constant in our economy. It is easy to see that we can apply the Golden Rule, using equation (15). If we denote by \bar{C} the amount of consumption, which corresponds to \bar{Y} , we get the equation

$$\bar{C} = e^{-\gamma t} G(1-s) s^{\frac{p}{1-p}} \quad (16)$$

Maximization of \bar{C} with respect to s leads to

$$0 = \frac{\delta \bar{C}}{\delta s} = e^{-\gamma t} G \left[\frac{p}{1-p} s^{\frac{p}{1-p} - 1} - \frac{1}{1-p} \right] s^{\frac{p}{1-p}} \quad \text{or} \quad s = p$$

The second derivative is negative and therefore $p(\bar{X})$ is the "optimal" savings ratio.

Now split up the economy into several sectors with several distinct commodities and assume that the system moves along the Golden Rule path, where the interest rate is equal to the rate of growth of output and therefore (labour force and income distribution being constant) also equal to the rate of growth of the real wage rate. My proposition is that the prices of the different commodities are proportional

to the total amount of labour used up to produce one unit of them, the proportionality factor being equal to the current wage rate.

This proposition can be made plausible in the following way: the direct inputs for the production of a certain commodity consist of labour and certain capital goods. Capital goods themselves have been produced by labour and other capital goods in earlier periods. These former capital goods consist of more remote inputs of labour and capital goods etc. In this way it is possible to transform the necessary direct inputs of capital goods and labour into mere labour inputs, which are spread over time. The price of the commodity today is equal to the sum of the "present" values of the different labour inputs. Hence

$$P_i(t) = \int_0^{\infty} e^{r\tau} a_i(t, \tau) W(t-\tau) d\tau$$

where $P_i(t)$ represents the price of the i -th commodity, r the interest rate, $a_i(t, \tau)$ the amount of indirect labour inputs τ periods before t , which were specifically necessary to produce one unit of the good i at period t , and $W(t-\tau)$ the wage rate τ periods before t . If the rate of interest is equal to the rate of growth of the labour force we can write

$$W(t-\tau) = e^{-r\tau} W(t)$$

and therefore

$$P_i(t) = \int_0^{\infty} e^{r\tau} a_i(t, \tau) W(t) e^{-r\tau} d\tau = W(t) \int_0^{\infty} a_i(t, \tau) d\tau$$

which proves our statement that the commodity prices at any moment of time are equal to the total amount of labour used up per unit of product times the prevailing wage rate.

It would not be very interesting for the appraisal of the Marxian labour theory of value, if there would exist just one equilibrium path of growth with the correct labour-value price system, as long as this path is not the optimal

path. But (at least according to the theory of the Golden Rule) this path coincides with the optimum. The meaning of this coincidence cannot be that the mistakes of Marx and the Marxists are no mistakes. Most of the better arguments against the theory of labour value still hold. But, on the other hand, modern growth economics shows that Marx and others were right in treating labour and capital in an asymmetric way, since the amount of capital is adjustable, which is not true for the "natural" factor labour. If the adjustment of capital is optimal, the prices of the goods represent the amount of the real scarce factor embodied in one unit of them; as in the stationary Walrasian equilibrium prices measure the costs in terms of the scarce factors of economy.

This kind of reasoning, which tries to defend the starting point of the Marxian system, is of course not only pseudo-Marxistic but even anti-Marxistic in its consequence. A growing economy must have a positive rate of interest, if the labour value theory makes any sense. In a growing system the interest rate has a function as a guide in the allocation of resources even if the optimum amount of capital is supplied. Thus we find the labour theory of value compatible with and even based on a theory of interest, which has nothing to do with the Marxian theory of exploitation. This also should lead to the conclusion that the old problems of interest theory ought to be reinvestigated on the background of the modern economics of growth.

This digression into more or less ideological parts of economics was the preparation for the formulation of a conjecture about the optimal price structure of an economy with a technical progress function of the type discussed in this paper. Since the model of the previous sections is essentially a one-sector model, I cannot prove propositions about price structures, which presuppose at least two sectors. We are constrained to mere conjectures.

I have pointed out that there exists a discrepancy between the behaviour of the firms with respect to technical progress and the social optimum technical progress for any given savings ratio s . Especially this would also be true, if the economy implements the "optimum" savings ratio $p(\bar{X})$, which is a function of the technical progress function. The discrepancy lies in the different way of approaching the identical final equilibrium path. How to influence entrepreneurs to introduce technical progress in a socially more efficient way? This is a question, which is too difficult to be answered here. But I have one suggestion for a planned socialist economy, which uses prices and a "shadow" rate of interest for organizational purposes. Let me assume that the factories are free to choose their production processes in a system of prices, which are prescribed by the government or central planning commission. The task of the planning commission is to decree an optimal set of prices.

If technical progress could not be influenced, the optimum price structure undoubtedly would coincide with the perfect competition price structure, regardless of the savings behaviour of the economy. This has been proved again and again. But with our kind of a technical progress function the competitive price structure is no more the optimum. For the maximization of the very long-run consumption it would be better, as I believe, to set up a price system, which corresponds to the Golden Rule equilibrium path, even if this path is not yet reached. This means, the planning commission should introduce from the very beginning a correct labour value theory price system with an interest rate equal to the equilibrium rate of growth.

This would have the following effect: the single "firm" cannot distinguish the economic situation from the situation on the final Golden Rule path. It will behave as if this path had been reached already. This would mean that it introduces a Harrod neutral kind of technical progress; and hence the economy is prevented from jumping around between (Harrod)-labour saving and (Harrod)-capital saving technical

progress and losing a lot of "technical progress energy." The disadvantage of this labour value price system would be that in the beginning there exists either a capital- or a labour-shortage. Assume the more realistic case: that there is too little real capital in the beginning. Since the firms are using capital as if the (more capital intensive) Golden Rule path were reached, there will be a kind of structural unemployment. Production today also would not be so high as it would be in a competitive equilibrium. The rate of interest is equal to the natural rate of growth and capital intensity is growing at this same rate; thus unemployment would last forever, if only profits would be reinvested as it is the case on the Golden Rule path. A certain amount of the wage bill has to be taxed away by the government for investment purposes in order to achieve full employment and to take full advantage of the production possibilities in that country. If this is done, finally the real Golden Rule path will be reached.

Since normally the initial value $A(0)$ is different from the optimum value \bar{A} , my statement has to be modified a bit. The planning commission will try to induce the factories to change A , but to change it very, very slowly. Therefore it will set up a price system, which is slightly different from the labour value price system. But the difference is small in comparison with the difference between the competitive and the labour value price system. When the optimum value \bar{A} is reached, the planning commission will return to the accurate labour value system, which coincides with the competitive price system as soon as the economy moves along the Golden Rule path.

This theory presupposes a constant labour force. If population and labour supply are growing exponentially, the optimum rate of interest would be higher than the growth rate of real wages and the ordinary labour value theory fails.

But this should not be a very serious trouble for a well trained dialectician; what is still true is that on the G.R. path the price of a commodity is proportional to the sum of the relative portions of the labour force, which are used up to produce the commodity.