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John C. Harsanyi

The Tracing Procedure:

A Bayesian Approach to Defining a Solution for n-Person Noncooperative Games. PART II

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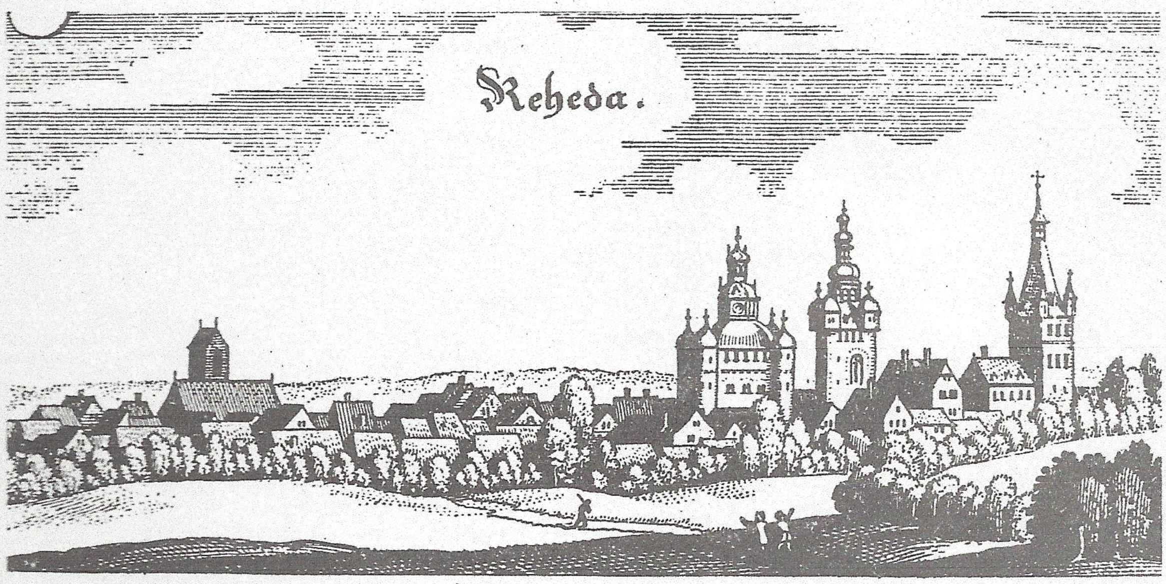


Anno 1600 nach Merian

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Schloß Rheda
484 Rheda
Bundesrepublik Deutschland
Federal Republic of Germany

THE TRACING PROCEDURE:
A BAYESIAN APPROACH TO DEFINING A SOLUTION FOR n-PERSON NONCOOPERATIVE GAMES
PART II

By John C. Harsanyi
University of California, Berkeley
and
University of Bielefeld

ABSTRACT

(For parts I and II)

The paper proposes a Bayesian approach to selecting a particular equilibrium point s* of any given finite n-person noncooperative game Γ as solution for Γ. It is assumed that each player i starts his analysis of the game situation by assigning a subjective prior probability distribution p_j to the set of all pure strategies available to each other player j. (The prior distributions p_j used by all other players i in assessing the likely strategy choice of any given player j will be identical, because all these players i will compute this prior distribution p_j from the basic parameters of game Γ in the same way.) Then, the players are assumed to modify their subjective probability distributions p_j over each other's pure strategies systematically in a continuous manner until all of these probability distributions will converge, in an appropriate sense, to a specific equilibrium point s* of Γ, which, then, will be accepted as solution.

A mathematical procedure, to be called the tracing procedure, is proposed to provide a mathematical representation for this intellectual process of convergent expectations. Two variants of this procedure are described. One, to be called the linear tracing procedure, is shown to define a unique solution in "almost all" cases but not quite in all cases. The other variant, to be called the logarithmic tracing procedure, always defines a unique solution in all possible cases. Moreover, in all cases where the linear procedure yields a unique solution at all, both procedures always yield the same solution. For any given game Γ, the solution obtained in this way heavily depends on the prior probability distributions $p_1, \ldots, p_n$ used as a starting point for the tracing procedure. In the last section, the results of the tracing procedure are given for a simple class of two-person variable - sum games, in numerical detail.

SYNOPSIS

Part I

Part II

7. Proof of Theorem 5 [9)]

Let $s=(s_1,\ldots,s_n)$ be a strategy n-tuple belonging to a given carrier class Q^*, i.e., having set $Q^*=Q(s)$ as its carrier set. Suppose that the carrier sets $Q_1(s_1),\ldots,Q_n(s_n)$ of the component strategies $s_1,\ldots,s_n$ consist of $\gamma_1,\ldots,\gamma_n$ pure strategies, respectively. Accordingly, the carrier $Q(s)=Q^*$ of the whole strategy n-tuple s will consist of γ^* pure strategies, where

$$\gamma^* = \sum_{i=1}^{n} \gamma_i \tag{58}$$

We can assume, without loss of generality, that the pure strategies of each player i have been renumbered in such a way that his mixed strategy s_i will now contain his <u>first</u> γ_i pure strategies $a_i^1,\ldots,a_i^{\gamma_i}$. To characterize this mixed strategy s_i, it will be sufficient to specify the (γ_i-1) probabilities $s_i^2,\ldots,s_i^{\gamma_i}$ since

$$s_i^1 = 1 - \sum_{k=2}^{\gamma_i} s_i^k, \tag{59}$$

and

$$s_i^k=0, \quad \text{for} \quad k=\gamma_i + 1,\ldots, K_i. \tag{60}$$

The vector listing these (γ_i-1) probabilities characterizing the mixed strategy s_i will be written as [10)]

$$\sigma_i = \sigma_i(s_i) = (s_i^2,\ldots,s_i^{\gamma_i}), \tag{61}$$

for $i = 1,\ldots,n$.

and will be called the <u>main vector</u> for strategy s_i.
The whole strategy n-tuple s can be characterized by

$$\gamma^{**} = \sum_{i=1}^{n} (\gamma_i - 1) = \gamma^{*} - n \tag{62}$$

probabilities. The vector listing these γ^{**} probabilities will be written as

$$\sigma = \sigma(s) = (\sigma_1, \ldots, \sigma_n) \tag{63}$$

and will be called the <u>main vector</u> for the strategy n-tuple s.

The set Σ_i of all possible main vectors $\sigma_i(s_i)$ for strategy n-tuples s belonging to the carrier class Ω^* is an open simplex of $(\gamma_i - 1)$ dimensions, defined by the strong inequalities

$$s_i^k > 0, \quad \text{for } k=2, \ldots, \gamma_i , \tag{64}$$

and

$$\sum_{k=2}^{\gamma_i} s_i^k < 1. \tag{65}$$

On the other hand, the set Σ of all possible main vectors $\sigma(s)$ for strategy n-tuples s belonging to the carrier class Q^* is an open polyhedral set, defined as $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$.

Now, suppose that this strategy n-tuple s is an equilibrium point. For each player i, let $\check{Q}_i$ be the set of all pure-strategy best replies that player i has to the other (n-1) players' strategy combination $\overline{s_i}$. Moreover, let

$$\check{Q}=\check{Q}(s) = \bigcup_{i=1}^{n} \check{Q}_i. \tag{66}$$

If s is a <u>quasi-strong</u> equilibrium point (which is the usual situation), then we shall have

$$\check{Q}(s) = Q(s). \tag{67}$$

But if s is an <u>extra-weak</u> equilibrium point, then we shall have

$$\check{Q}(s) \supset Q(s), \tag{68}$$

where $\supset$ is used in the sense of <u>proper</u> set inclusion (which is meant to exclude equality between the two sets). Suppose that this set $\check{Q}=\check{Q}(s)$ consists of $\check{\gamma}^*$ pure strategies. Then, in view of (68), we shall have

$$\check{\gamma}^* > \gamma^*. \tag{69}$$

When s is extra-weak, then in some ways it will behave as if it were an equilibrium point belonging to the carrier class $\check{Q}$, even though in fact it belongs to the carrier class Q^*. More particularly, in general, an equilibrium point of carrier class Q^* will satisfy only $\gamma^{**} = \gamma^*-n$ equations of form (23). In contrast, an extra-weak equilibrium point s will satisfy $\check{\gamma}^{**} = \check{\gamma}^*-n > \gamma^{**}$ equations of this form, in the same way as an equilibrium point actually belonging to carrier class $\check{Q}$ would do. Intuitively speaking, we can say that such an extra-weak equilibrium point s results when, for an equilibrium point $\check{s}$ of carrier class $\check{Q}$, certain probabilities $\check{s}_i^k$ go to zero - - viz. those probabilities $\check{s}_i^k$ which are associated with pure strategies a_i^k that do belong to set $\check{Q}$ but do not belong to set Q^*.

Accordingly, we shall say that such an extra-weak equilibrium point s is an <u>improper member</u> of carrier class $\check{Q}$ - - though of course, at the same time, by the definition of a carrier class, s is also a <u>(proper) member</u> of carrier class Q^*. We also define a main vector $\sigma' = \sigma' (s, \check{Q})$ which has the same mathematical form as a main vector $\check{\sigma} = \sigma(\check{s})$ of a strategy n-tuple $\check{s}$ belonging to carrier class $\check{Q}$ has, but which assigns to every pure strategy a_i^k the probability s_i^k that the equilibrium point s itself associates with a_i^k - - if a_i^k belongs to those pure strategies to which main vectors of this mathematical form assign a probability at all. (Naturally, this means that σ' will always violate some of the strong inequalities of form (64) and/or (65), and will satisfy them only if the strong inequality sign is replaced by a weak inequality sign.) We shall call $\sigma' = \sigma'(s,\check{Q})$ the <u>improper main vector</u> associated with equilibrium point s for carrier class $\check{Q}$.

More generally, for every carrier class $\hat{Q}$ such that

$$Q^* \subset \hat{Q} \subseteq \check{Q}, \tag{70}$$

we shall say that s is an improper member of carrier class $\hat{Q}$ and, for every carrier class $\hat{Q}$ of this kind, we shall associate an improper main vector $\sigma' = \sigma'(s,\hat{Q})$ with s.

Now, suppose that s is an equilibrium point of a given game Γ^t. In view of (23) and (25), the main vector $\sigma = \sigma(s)$ of s must satisfy γ^{**} equations of the form

$$t\left[U_i(a_i^k,\overline{s_i})-U_i(a_i^1,\overline{s_i})\right] + (1-t)\left[U_i(a_i^k,\overline{p_i})-U_i(a_i^1,\overline{p_i})\right]=0 \tag{71}$$

for $i=1,\ldots,n$; and for $k=2,\ldots,\gamma_i$.

We shall write the left-hand side of any given equation of this form as $\omega_i^k(t,\sigma)$. Let M be the $\gamma^{**} \times \gamma^{**}$ matrix

$$M = \left(\frac{\partial\omega_i^k}{\partial\sigma_j^m}\right) \quad \begin{matrix} i,j=1,\ldots,n \\ k,m=2,\ldots,\gamma_i, \end{matrix} \tag{72}$$

where all partial derivatives are evaluated at point s itself. Let M^o be a row vector of γ^{**} components, of the form

$$M^o = \left(\frac{\partial\omega_i^k}{\partial t}\right) \quad \begin{matrix} i = 1,\ldots,n \\ k = 2,\ldots,\gamma_i. \end{matrix} \tag{73}$$

Again, all partial derivatives are to be evaluated at point s itself. Finally, let M be the $(\gamma^{**}+1) \times \gamma^{**}$ matrix

$$M^* = \begin{pmatrix} M \\ M^o \end{pmatrix} \quad (74)$$

As s is an equilibrium point of game Γ^t, the graph $P = P(\Gamma,p)$ will have a point of the form $x=(t,s)$. In view of this fact, we shall write

$$M^* = M^*(t,s), \quad (75)$$

and shall call M^* the proper M^*-matrix associated with this point $x=(t,s)$.

Now, suppose that s is an extra-weak equilibrium point of game Γ^t. In this case, we shall define an M^*-matrix also for each improper main vector σ' associated with s, i.e., for each carrier class $\hat{Q}$ of which s is an improper member. Any M^*-matrix of this kind will be called an improper M^*-matrix associated with the point $x=(t,s)$. We now introduce the following definition:

Any given point $x=(t,s)$ of graph P is called regular if all matrixes M^* associated with x, both proper and improper (if any), have the highest possible rank. If s is an n-tuple of pure strategies, then M^* will have no elements at all. In this case, we shall always consider M^* to be of the highest possible rank. (Of course, if s is an extra-weak equilibrium point in pure strategies, then some of the improper M^*-matrixes associated with it might very well have less-then-maximum rank.)

Let T be a set of possible t values. We shall say that graph P is regular in T if every point $x=(t,s)$ with $t \in T$ is regular. If no set T is specified then the set $T=I$ of all possible t values will be meant.

<u>Lemma 19.</u> Game Γ^{o} will have only one pure-strategy equilibrium point if and only if graph P is regular in the region $t=0$.

<u>Proof.</u> By Lemmas 1 and 2, the set E^{o} of all equilibrium points in Γ^{o} will always be a closed and convex set. Therefore, by the Implicit Function Theorem, all points of form $(0,s)$ in P will be irregular, unless E^{o} consists of a unique point. On the other hand, if E^{o} consists of a unique point s^{o}, then this will always be a strong equilibrium point in pure strategies, so that the matrix $M^{*}(0,s^{o})$ will have no elements and, therefore, will be of the highest possible rank. Moreover, the point $x^{o}=(0,s^{o})$ will have no improper M^{*}-matrixes associated with it, owing to the strongness of s^{o}.

<u>Lemma 20.</u> If graph P is regular in the region $0\leq t<1$, then the limit curve L^{**} will be the only feasible path.

<u>Proof.</u> By Lemma 19, every feasible path L would have to start at the same point $x^{o}=(0,s^{o})$. Therefore, if P contained a feasible path $L \neq L^{**}$, then L would have to branch off from L^{**} at point $x^{o}=(0,s^{o})$ itself, or at some later point $x=(t,s)$ of L^{**} with $0<t<1$. But the first-mentioned situation is impossible because s^{o} is a strong equilibrium point of game Γ^{o}. Therefore, by continuity, for all sufficiently small positive values of t, the strategy part s of each point (t,s) of L^{**} is likewise a strong equilibrium point and is, consequently, also an <u>isolated</u> equilibrium point, of the relevant game Γ^{t}. Therefore for sufficiently small values of t, L^{**} is a locally unique branch of graph P.

Moreover, by the Implicit Function Theorem, it is equally impossible that L should branch off from L^{**} at a point $x=(s,t)$ with $0<t<1$, because every point x of L^{**} is regular. Hence, no feasible path L different from L^{**} can exist.

<u>Note 1</u>. Even if a point $x=(t,s)$ of graph P is regular, it can still happen that the submatrix M [see equation (72)] of its matrix M^{*} is singular: but, then, M^{*} must contain at least one other $\gamma^{**} \times \gamma^{**}$ submatrix that is nonsingular. For example, even if all points x of graph P are regular, it can happen that the limit curve L^{**} will contain a whole one-dimensional segment Λ along which t has a constant value. In this case, by the Implicit Function Theorem, M must be singular at all points x of Λ. But, then, since one of the other relevant submatrixes of M^{*} will be nonsingular, we can infer, again by the Implicit Function Theorem, that the strategy part s of the points $x=(t,s)$ of Λ will be a locally unique function - - of course, not of t, but at least of another coordinate, say, of s_i^k. This suffices to ensure that no other feasible path L can branch off from L^{**} at any point x.

<u>Note 2.</u> In defining a regular point $x=(t,s)$, it was necessary to consider, not only its proper M^{*}-matrix, but also the improper M^{*}-matrixes (if any) associated with it, in order to exclude degeneracies at those points x of L^{**} where, as we follow L^{**}, the strategy part s of x moves from one carrier class Q^{*} to another carrier class $Q^{**} \neq Q^{*}$. The strategy part s of any such point x is, of course, always an extra-weak equilibrium point of the relevant game Γ^{t}.

Geometrically, these points x are characterized by the fact, at these points, L^{**} moves from the interior of set R to its boundary $\bar{R}$, or conversely; or by the fact that, at these points, L^{**} moves from a given face of $\bar{R}$ to another face with a higher or a lower dimensionality. Moreover, they are also the points where different "pieces" of the piecewise algebraic curve L^{**} join each other.

Lemma 21. For any given choice of p, within any particular set $\mathcal{G}$ of a given size, the set $\mathcal{G}^o$ of all those games Γ which give rise to a graph $P=P(\Gamma,p)$ not regular in the region $t=0$, is a set of measure zero.

Proof. The lemma directly follows from Lemmas 8^* and 19.

Lemma 22. For any given choice of p, within any particular set $\mathcal{G}$ of a given size, the set $\bar{\mathcal{G}}$ of all those games Γ which give rise to a graph $P=P(\Gamma,p)$ having any irregular point(s) at all, is a set of measure zero.

Proof. In view of Lemma 21, we have to prove the measure-zero property only for the set $\hat{\mathcal{G}}$ of games that give rise to graphs P not regular in the region $0<t\leq 1$.

In Harsanyi [1973, p.248], I have defined, for all equilibrium points s of a given carrier class Ω^*, a mapping ρ^* which in my present notation can be written as ρ^*: $(\sigma,u^{**}) \rightarrow (u^*, u^{**}) = u$. Here $\sigma=\sigma(s)$ is the main vector for each equilibrium point s, whereas u^* is a vector consisting of γ^{**} appropriately chosen components of the vector $u=u(\Gamma)$ of possible payoffs in game Γ, and u^{**} is a vector consisting of the remaining $(nK-\gamma^{**})$ components of u.

I shall now apply this mapping ρ^* to equilibrium points s of carrier class Q^* in a given game Γ^t. I shall write $u(t) = u(\Gamma^t)$, and shall write the two subvectors u^* and u^{**} of $u(t)$ as $u^*(t)$ and $u^{**}(t)$. The mapping ρ^* as applied to game Γ^t will be called ρ^t. Thus, I shall write $\rho^t: (\sigma, u^{**}(t)) \rightarrow (u^*(t), u^{**}(t)) = u(t)$.

In the paper quoted, I have proposed a specific way of selecting the γ^{**} components of vector u^* [now called $u^*(t)$]. But of course there are many other, equally admissible, ways of choosing them. The only requirements are as follows. Out of the γ_i pure strategies of player i which are included in the carrier set Q^*, one has to choose (γ_i-1) pure strategies. Then, for each pure strategy a_i^k chosen in this way, one must choose one pure-strategy n-tuple b^m such that $b^m(i)=a_i^k$ (i.e., b^m must have this pure strategy a_i^k as its ith component). This must be done in such a way that no strategy n-tuple b^m is associated with two different pure strategies a_i^k and $a_i^{k'}$, $k \neq k'$, of the same player i. Then, for each strategy n-tuple b^m chosen, one selects the payoff $u_i^m(t) = V_i(b^m; p,t)$-- that the strategy combination b^m would yield to player i in game Γ^t-- as a component of vector $u^*(t)$. In this way, one obtains (γ_i-1) components $u_i^m(t)$ for a given player i, i.e., all together $\sum_i(\gamma_i-1) = \gamma^{**}$ components, as required. Clearly, subject to these, rather mild, restrictions, the γ^{**} components of vector $u^*(t)$ can be chosen in a great many different ways.

As can be seen from equation (50) of my earlier paper, this mapping ρ^t will always be well defined for every main vector σ satisfying conditions (64) and (65), which means that it will always be well defined for every proper main vector σ, and therefore for every equilibrium point s that is a proper member of the relevant carrier class Q^*.

But ρ^t will be undefined for some improper main vectors σ' and, therefore, for some equilibrium points s that are improper members of this carrier class Q^*. However, for other improper main vectors σ', and for some other improper members s of this carrier class Q^*, ρ^t will be completely well defined. This is so because the denominator in equation (50) of my paper mentioned will never vanish for any proper main vector σ; whereas, for an improper main vector σ', it may or may not vanish.

In fact, it is easy to verify that, by appropriate choice of the γ^{**} components of vector $u^*(t)$, we can always construct a mapping ρ^t that is well defined for any specific improper main vector σ'.

For any given mapping ρ^t, based on a particular choice of the γ^{**} components of vector $u^*(t)$, let Σ^* be the set of all main vectors σ, both proper and improper, for which ρ^t is well defined. Σ^* will always include all points σ of the open polyhedral set Σ defined by (64) and (65): It will also include some parts of the boundary $\overline{\Sigma}$ of Σ, but will exclude other parts. Let $\mathcal{G}^* = \{u^*(t)\}$ be a γ^{**}-dimensional Euclidean space, and let $\mathcal{G}^{**} = \{u^{**}(t)\}$ be an $(nk-\gamma^{**})$-dimensional Euclidean space. Then, mapping ρ^t will be from set $\Sigma^* \times \mathcal{G}^{**}$ to set $\mathcal{G}^* \times \mathcal{G}^{**} = \mathcal{G}$.

As can be seen from equation (50) of my earlier paper, any mapping ρ^t will be analytic, and therefore will be continuously differentiable any number of times, at any point $(\sigma, u^{**}(t))$ where it is defined at all.

In view of (25), for every element $u_i^k(t)$ of vector u^{**}, we can write

$$u_i^k(t) = tu_i^k + (1-t)\ U_i(a_i^k, \overline{p_i}). \tag{76}$$

Equation (76) defines a mapping $\mu^{**}: (t,u^{**}) \to u^{**}(t)$. This is a linear mapping from set $I' \times \mathcal{G}^{**}$ to $\mathcal{G}^{**}$, where I' is the half-open unit interval $I' = \{t | 0<t\leq 1\}$. (We define μ^{**} in this particular way because we are now concerned only with that part of graph P which lies in the region $0<t\leq 1$.)

Again, in view of (25), for every element u_i^k of vector u we can write

$$u_i^k = \left[u_i^k(t) - (1-t)\ U_i(a_i^k\ ,\ \overline{p_i})\right]/t. \tag{77}$$

For all values of t such that $0<t\leq 1$, equation (77) defines a mapping $\mu^{*}: (t,u(t)) \to u$. This is a linear mapping from set $I' \times \mathcal{G}$ to $\mathcal{G}$. On the other hand, the mappings μ^{**}, ρ^t (for all $t \in I'$), and μ^{*} together define a fourth mapping $\mu: (t, \sigma, u^{**}) \to u$, which is an analytic mapping from set $I' \times \sum^{*} \times \mathcal{G}^{**}$ to $\mathcal{G}$. Of course, there exists a different mapping μ for each carrier class Q^{*} in game Γ, and for each possible way of choosing the γ^{**} components of vector u^{*} with respect to any given carrier class Q^{*}.

By Sard's [1942] theorem, the existence of these mappings μ (together with the fact that the various mappings μ collectively cover all main vectors σ, both proper and improper, for all carrier classes Q^* in the game) implies that, for every possible set $\mathcal{G}$ of games of a given size, the set $\hat{\mathcal{G}}$ of all games Γ associated with graphs P not regular in the region I', is a set of measure zero in $\mathcal{G}$. This completes the proof of Lemma 22.

<u>Lemma 23</u>. Set $\overline{\mathcal{G}}$ defined by Lemma 22 is a closed set.

<u>Proof</u>. Let $\Gamma_1, \Gamma_2, \ldots$ be a sequence of a games within a given set $\mathcal{G}$, such that their u-vectors $u^1=u(\Gamma_1)$, $u^2=u(\Gamma_2), \ldots$ converge to the u-vector $u^o=u(\Gamma_o)$ of a particular game Γ_o. Suppose that all games $\Gamma_1, \Gamma_2, \ldots$ belong to set $\overline{\mathcal{G}}$. We have to prove that the limit game Γ_o will likewise belong to set $\overline{\mathcal{G}}$.

The proof of this statement is based on the fact that, from this sequence $\{\Gamma_k\}$, we can always select a subsequence $\{\Gamma_{k_m}\} = \{\Gamma_\kappa\}$, with the following properties:

1. For each game Γ_κ, the graph $P_\kappa = P(\Gamma_\kappa, p)$ has an irregular point $x(\kappa) = (t(\kappa), s(\kappa))$ such that the sequence $\{x(\kappa)\}$ converges to a given point $x(o) = (t(o), s(o))$.

2. All points $s(1), s(2), \ldots$ belong to the <u>same</u> carrier class Q^*. (In proving that requirement 2 can always be met, we can use the fact that the number of different carrier classes Q^* in any game Γ_κ is finite.)

3. All points $s(1), s(2), \ldots$ are improper members of the <u>same</u> carrier classes $\hat{Q}$ (if any).

4. For the proper M^*-matrixes $M^*(1), M^*(2), \ldots$ associated with $x(1), x(2), \ldots$, the <u>same</u> $\gamma^{**} \times \gamma^{**}$ submatrixes are singular.

5. For the improper M^*-matrixes $M^*(1,\hat{Q}), M^*(2,\hat{Q}), \ldots$ associated with $x(1), x(2), \ldots$ for any given carrier class $\hat{Q}$, the <u>same</u> $\hat{\gamma}^{**} \times \hat{\gamma}^{**}$ submatrixes are singular.

Since, for a constant p, the correspondence $u \rightarrow P(\Gamma, p)$ is upper semi-continuous, and since the determinant of any given submatrix of any particular matrix M^* is a continuous function of the vector u, properties 1 to 5 imply that:

(A) $x(o) \in P_o = P(\Gamma_o, p)$; and

(B) If a particular submatrix is singular in the M^*-matrix defined for any given carrier class Q^* or $\hat{Q}$ with respect to all points $x(1), x(2), \ldots$, then the same submatrix will be singular also in the M^*-matrix defined for the same carrier class with respect to the point $x(o)$.

Consequently, x(o) will be an irregular point of P_o and, therefore, $\Gamma_o \, \varepsilon \, \overline{\mathcal{G}}$, as desired.

Lemmas 20,22 and 23, however, imply:

Theorem 5. For any given n-tuple p of prior distributions, almost all games Γ will give rise to a well-defined linear tracing procedure.

8. Example: the tracing map for a simple class of games

For a given game Γ, let Z(s) be the set of all n-tuples p which yield a particular equilibrium point s as the solution $s=T(\Gamma,s)$ for Γ-- whether by means of the linear tracing procuedure (if this is well defined), or by means of the logarithmic tracing procedure. (In view of Theorem 4, this definition will not give rise to any ambiguity.) This set Z(s) will be called the source set for this equilibrium point s. Of course, for some equilibrium points s, this source set Z(s) may be empty. A map showing the source sets for the various equilibrium points of a given game Γ, or for a class of such games, will be called a tracing map. In this section, we shall consider the tracing map for a simple class of two-person variable-sum games. We shall consider games Γ of the following form

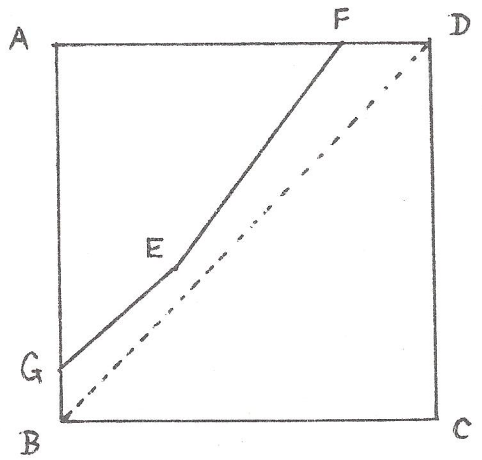
	a_2^1	a_2^2
a_1^1	α,β	0,0
a_1^2	0,0	γ,δ

with $\alpha, \beta, \gamma, \delta > o$

Games of this class have three equilibrium points, viz. the two pure-strategy equilibrium points $b^1=(a_1^1, a_2^1)$ and $b^2=(a_1^2, a_2^2)$, as well as a mixed-strategy equilibrium point $\hat{s}=(\hat{s}_1,\hat{s}_2) = (\hat{s}_1^1, \hat{s}_1^2; \hat{s}_2^1, \hat{s}_2^2)$, where

$$\hat{s}_1^1 = \delta/(\beta+\delta) \qquad \hat{s}_1^2 = 1-\hat{s}_1^1$$

$$\hat{s}_1^2 = \gamma/(\alpha+\gamma) \qquad \hat{s}_2^2 = 1-\hat{s}_1^2$$



The strategy space of such a game can be represented by a square such as ABCD. Any point like E will represent a pair of mixed strategies of the form $s=(s_1,s_2)=(s_1^1,s_1^2;\ s_2^1,s_2^2)$. The distances of E from AD, BC, AB and DC will represent the probabilities s_1^1, s_1^2, s_2^1 and s_2^2, respectively.

We shall assume that, in fact, this point E represents the mixed-strategy equilibrium point $\hat{s}$. The pure strategy equilibrium points $b^1=(1,0;\ 1,0)$ and $b^2=(0,1;\ 0,1)$ are represented by the corner points A and C, respectively. The line FE has been constructed by extending the line BE (which is not shown). Likewise, the line GE has been obtained by extending the line DE. The dotted line is the diagonal BD.

Since, mathematically, any vector $p=(p_1,p_2) = (p_1^1, p_1^2; p_2^1, p_2^2)$ is simply a pair of mixed strategies, any vector p can also be represented by a point of the square ABCD. Simple computation will show that the linear tracing procedure is always well defined if p does not lie on the broken line GEF. More particularly, any point p lying on the north-west side of GEF will yield the equilibrium point b^1(=A) as solution; whereas any point p lying on the south-east side of GEF will yield the equilibrium point b^2(=C) as solution.

Points p lying on the boundary line GEF itself will behave as follows:

If $\alpha\beta > \gamma\delta$, then all points p lying on GEF will yield b^1 as solution.

If $\alpha\beta < \gamma\delta$, then all these points p will yield b^2 as solution.

If $\alpha\beta = \gamma\delta$, then all points p lying on the boundary will yield the mixed-strategy equilibrium point $\hat{s}$ as solution. (Note the importance of the two Nash products $\alpha\beta$ and $\gamma\delta$.)

Geometrically, this means that:

Whenever the point E (and, therefore, the whole broken line GEF) lies on the south-east side of the diagonal BD, then any point p on GEF will yield b^1 as solution.

Whenever the point E(as well as the broken line GEF) lies on the north-west side of the diagonal BD, then any point p on GEF will yield b^2 as solution.

Finally, when E lies on the diagonal BD (in which case the boundary line GEF will coincide with BD), then any point p on GEF (or on BD) will yield the mixed-strategy equilibrium point $\hat{s}$ as solution.

Thus, in most cases, the whole strategy space S will be covered by the source sets $Z(b^1)$ and $Z(b^2)$, since one of them will include also the boundary GEF. Only when game Γ is completely symmetric with respect to b^1 and b^2, do we find a nonempty source set $Z(\hat{s})$ also for the mixed-strategy equilibrium point $\hat{s}$, which in this case will correspond to the boundary line between $Z(b^1)$ and $Z(b^2)$.

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Footnotes:

9) In Part II of this paper, the numbering of sections, footnotes, equations, lemmas and theorems will be consecutive to their numbering in Part I. The author wishes to express his thanks to the National Science Foundation for its support of this research through Grant GS-3222 to the University of California, administered through the University's Center for Research in Management Science, Berkeley. Thanks are due also to the Institute of Mathematical Economics, the University of Bielefeld, in Rheda, Westphalia.

10) In equations (9) to (13) as well as (27) to (29), the symbol σ_i was used in a different sense, viz. to denote a probability distribution over set C_i. But no confusion will arise because from now on it will always be used in the sense specified by equation (61).