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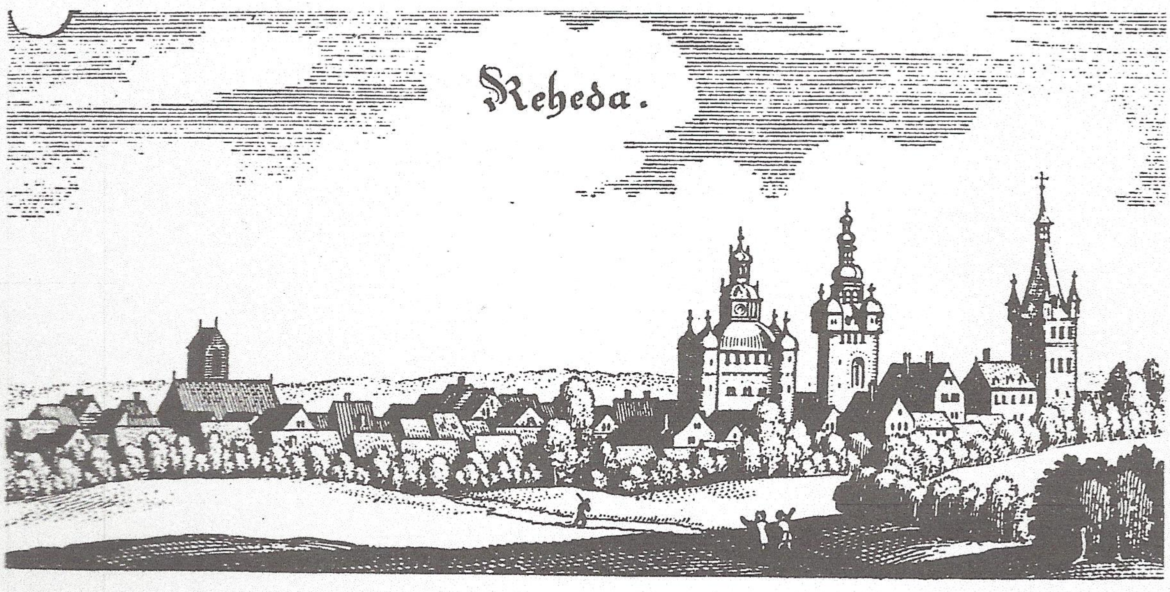

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ABSTRACT.

DECOMPOSITION FOR STOCHASTIC DYNAMIC SYSTEMS

Hans W. Gottinger

A decomposition for a stochastic non-linear dynamic system is attempted in which stochastic controls are applied to N coupled non-linear subsystems. Such problems arise frequently in large-scale models of economic organizations (in particular stochastic dynamic teams) and in dynamic multi-person control problems with different information structures for each controller. A computational procedure (regarding off-line computations in such systems) for optimization is suggested and decomposition procedures for both the deterministic and the stochastic problem are derived.

Decomposition for Stochastic Dynamic Systems - Part I.

0. Preliminary Considerations and Definitions

There has been recent interest in investigating the structures of complex systems that occur everywhere in economics, management science, engineering and biology. Also there have been new developments in the mathematical tools for describing such systems. These tools aim at a comprehensive treatment of many interrelated concepts in complex systems such as decision, control, information and reward [7a]. The overall structure of these systems comprise the following elements:

(1) a set of agents (decision-makers, controllers),

(2) each agent reveals a preference ordering represented by various sorts of utility functions, loss or cost functions, performance indices, pay-off or reward functions,

(3) each agent knows his permissible decisions that he controls, therefore we have a set of permissible decisions,

(4) each agent observes his environment and acts in response to it, this environment may be specified as the state of the world set, and the mapping from this set into the set of observations may constitute his information structure.

All of these elements have been the subject of study in decision theory or the theory of games but we are looking at them as elements of complex systems.

The systems which are of immediate interest here are described in the literature as teams, competitive economic systems and hierarchical systems. To some extent such systems are overlapping but they often use different mathematical techniques. For a brief description and review of such systems see P. Varaiya /14/ and also /10/.

Many of such systems are presented in a static framework, such as team theory /9/, or decomposition theory of large-scale systems /8/. In the latter case uncertainty has not been satisfactorily introduced. In order to come closer to the complexity of such systems it is absolutely necessary that these systems are put in a dynamic framework, with explicit consideration of time and furthermore that we have to cope with uncertainties on various levels of the system's performance. A hierarchical decomposition of a static system in which uncertainties may affect the system's performance has been discussed in /6/. Some of the results obtained so far can also be used for dynamic systems. A classification of stochastic systems can be found in the book by M. Aoki /1a/, the most useful distinction is that between a purely stochastic system where the probability distributions of a parameter or moments are known and an adaptive stochastic system where some of the key statistical information is lacking, or incomplete. In the latter case Bayesian procedures will gain considerable importance. Particular forms of stochastic systems are constituted by learning automata /12/ or stochastic control systems /5/. In what follows, we will restrict ourselves to purely stochastic systems. We shall also assume that the same set of preference orderings my be shared by all the decision agents and that these preference orderings altogether can be represented by a cost functional (performance index). All the agents choose their controls (decisions) to optiminize (minimize) this cost functional. Hence, in general, the cost functional will be additive. There are two essential constraints which have to be coped with in large-scale systems. First, establishing communication links between tne decision agents might be expensive or even technically infeasible.

Second, the size of some typical large-scale system may make the control problem bigger than that can be handled by the fastest computers available. For the control of dynamic systems, we need actually to distinguish between two kinds of computation, off-line and on-line. Off-line computation is what can be computed before the system starts running, e.g., the computation of the optimal strategies. On-line computation has to be done in real time while the system is actually running, e.g., transforming the data received in real time into decision (controls) using the optimal strategies computed off-line. In general, on-line computation presents bigger problems than off-line computation since it has to be done in real time.

In Part I and II of this paper we will exclusively deal with off-line computation of dynamic (stochastic) systems.

1. Introduction

In this paper we consider the stochastic control of N coupled nonlinear subsystems. Each system has a controller who has noisy measurements on his subsystem. There is no communication between the controllers. The overall objective of the system consisting of all subsystems is the sum of individual objectives of the subsystems.

Because of the dynamic nature of the problem, the difficulties encountered here are different from those in static systems. Generally speaking, since the controls have to be applied in real time, on-line computation requirements for implementation of the optimal control strategy become important. The class of problems with different information patterns for the different controllers have been studied under the topic of dynamic teams [1, 3, 4, 7]. So far, the results have not been very satisfactory in several respects. First, the optimal solution for even a linear-quadratic-Gaussian team is not known yet although there are indications of what the optimal solution should look like. Second, although the information structure in team decision problems is decentralized, often this is accompanied by an increase in both on-line and off-line computation. To give an example, let us consider the linear-quadratic-Gaussian problem. If information is centralized, then the optimal control strategy is given by the "separation" theorem and consists of the optimal deterministic control law acting on the estimate generated by the Kalman-Bucy filter [2, 11]. The on-line computation can be replaced by building a finite-dimensional filter.

However, if information is decentralized, then the on-line computation is extremely involved since each controller has to remember all his past observations or an "infinite-dimensional filter" is required. For a discussion of this, see Willman [15]. As for the off-line computation, little is known since the optimal solution is not available. However, the computation involved in finding a suboptimal solution to the dynamic team problem has been shown to be relatively complicated [3].

Since the computation and implementation of a control strategy is as important as the optimality resulting from the strategy itself, we will formulate in this paper an optimization problem which is computionally more feasible as well as informationally efficient. The special coupled structure of the system and the form of the cost functional will be exploited. The concept of information structure is extended to include a priori information as well as a posteriori information. Thus the local controllers will not only have measurements on their subsystems alone, but will also be ignorant about the structure of the other subsystems. The coupled nature of the subsystems is taken care of by a coordinator who sees that certain constraints are satisfied. In this paper we study the case when the coordinator has only a priori information, i.e. he does not make any measurements, in other words, he does not take any observations as the process goes on.

The dynamic team problem is stated in the next section. A decomposition for the deterministic problem is then stated. This will be used to motivate the formulation of the stochastic decomposition problem in Section 3. In Section 4 we formulate a constrained stochastic optimal control problem as a mathematical programming problem. In Section 5, the problem formulated in Section 3 is decomposed.

2. Statement of the Problem

We consider a discrete-time system consisting of N subsystems coupled together.

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \xi_i(k)) \qquad i=1,\ldots, N \qquad (2.1)$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k)) \qquad (2.2)$$

where

$\underline{x}_i(k) \in R^{n_i}$ is the "state" of the ith subsystem.

$\underline{v}_i(k) \in R^{q_i}$ is the action on the ith subsystem due to the other N-1 subsystems.

$\underline{u}_i(k) \in R^{p_i}$ is the control on the ith subsystem.

$\underline{\xi}_i(k) \in R^{r_i}$ is the driving noise on the ith subsystem.

$\underline{f}_i$ is the state transition function.

$$\text{Let} \quad \underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \vdots \\ \underline{x}_N(k) \end{bmatrix} \quad \underline{u}(k) = \begin{bmatrix} \underline{u}_1(k) \\ \vdots \\ \underline{u}_N(k) \end{bmatrix} \quad \underline{\xi}(k) = \begin{bmatrix} \underline{\xi}_1(k) \\ \vdots \\ \underline{\xi}_N(k) \end{bmatrix} \qquad (2.3)$$

Then $\underline{v}_i(k)$, $i=1,\ldots, N$ can be eliminated from equation (3.2.1) to obtain a description for the whole system as

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k)) \qquad (2.4)$$

where the function $\underline{f}$ is defined in an obvious manner.

The description in terms of equations (2.1) and (2.2) is preferred here to display the coupled nature of the system. Note that even though $\underline{x}(k)$ can be regarded as the state of the system if the driving noise is absent, $\underline{x}_i(k)$ is, strictly speaking, not a state for the ith subsystem since knowledge of $\underline{x}_i(k)$, together with all the control $\underline{u}_i(j)$, $j \geq k$ is not sufficient to determine the future behavior of the ith subsystem.

The cost functional for the whole system is a sum of cost functionals for the individual subsystems, i.e.,

$$J = \sum_{i=1}^{N} J_i \qquad (2.5)$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\underline{u}_i(k))\} \qquad (2.6)$$

It is required to minimize J. The expectation is taken with respect to all the primitive random variables.

The problem is not yet well defined because we have not specified the information pattern of the system.

Let $$\underline{y}_i(k) = \underline{h}_i(\underline{x}_i(k), \underline{\theta}_i(k)) \qquad i=1,\ldots\ldots, N \qquad (2.7)$$

$\underline{y}_i(k) \in R^{m_i}$ is the measurement on the ith subsystem by the ith controller.

$\underline{\theta}_i(k) \in R^{m_i}$ is the noise corrupting the measurement.

Let $$Y(k) = \{\underline{y}_i(s) \;\; 0 \leq s \leq k, \quad i=1,\ldots\ldots, N\} \qquad (2.8)$$

$$U(k) = \{\underline{u}_i(s);\quad 0 \leq s \leq k,\quad i=1,\ldots\ldots,N\} \qquad (2.9)$$
$$k=0,\ldots,T-1$$

Let $(Y_i(k), U_i(k-1), I_i)$ be the information available to the ith controller at time k.

$$Y_i(k) \subset Y(k)\ ;\ U_i(k-1) \subset U(k-1) \qquad (2.10)$$

I_i is the a priori information of the entire system available to the ith controller.

Then $\underline{u}_i(k)$ is required to be a measurable function of $Y_i(k)$ and $U_i(k-1)$ which can be generated from I_i, i.e.,

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k),\ U_i(k-1);\ I_i) \qquad (2.11)$$

I_i is introduced to take into consideration structural information of the system. The information available to the ith controller thus consists of two kinds: a priori (structural) information of the system and a posteriori (measurement) information. I_i essentially specifies the complexity of the control strategy. In the system given, if $I_i = \{\underline{f}_i, J_i, \underline{h}_i\}$, then as far as each controller is concerned, he is controlling an uncoupled system with an unknown input $\underline{v}_i(k)$. His control law $\underline{\gamma}_i^{\,k}$ would thus depend only on the parameters of his subsystem. This control law is thus "simpler", although a "loss" in mathematical optimality results. In most of the work done thus far, [3, 4, 7] decentralization refers mainly to measurements, i.e., a posteriori information. The structure of the whole system is assumed known to each controller. With this a priori information, decentralized a posteriori information almost inevitably gives rise to a more complicated control strategy than centralized a posteriori information because each controller tries to generate the missing measurements using the common a priori information. The amount of on-line computation involved always increases, as well as the amount of off-line computation. Even when the on-line computation is constrained by choosing suboptimal control structures, as in Chong and Athans [3], the off-line computation required is still tremendous. In the implementation of control laws, computation considerations are as important as information considerations. This leads us to consider decentralized a priori information, at a sacrifice in overall optimality.

There is some work in the control literature which is vaguely related to decentralized a priori information. This is found in Ref. [10] and [13] and can be essentially illustrated by the following theorem for deterministic systems.

Theorem 2.1: Consider the optimal control problem given by

System:
$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\underline{v}_i(k),\underline{u}_i(k)) \quad i=1,\ldots,N \qquad (2.12)$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k)) \qquad \underline{x}(0) \text{ given}$$

Cost functional:
$$J = \sum_{i=1}^{N} J_i$$

$$J_i = K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\underline{u}_i(k)) \qquad (2.13)$$

Suppose there exists a constrained saddle-point $(\underline{x}^*,\underline{u}^*,\underline{v}^*,\underline{p}^*)$ to the problem

$$L(\underline{x}^*,\underline{u}^*,\underline{v}^*,\underline{p}) \leq L(\underline{x}^*,\underline{u}^*,\underline{v}^*,\underline{p}^*) \leq L(\underline{x},\underline{u},\underline{v},\underline{p}^*) \qquad (2.14)$$

where $\underline{x} = \{\underline{x}_i(k);\quad k=1,\ldots,T;\quad i=1,\ldots,N\}$

$\underline{u} = \{\underline{u}_i(k),\quad k=0,\ldots,T-1;\ i=1,\ldots,N\}$

$\underline{v} = \{\underline{v}_i(k);\quad k=0,\ldots,T-1;\ i=1,\ldots,N\}$

$\underline{p} = \{\underline{p}_i(k);\quad k=0,\ldots,T-1;\ i=1,\ldots,N\}$

$\underline{x},\underline{u},\underline{v}$ are constrained by equation (2.12)

$$L(\underline{x},\underline{u},\underline{v},\underline{p}) = J + \sum_{i=1}^{N} \sum_{k=0}^{T-1} \underline{p}_i'(k)(\underline{v}_i(k) - \sum_{j \neq i} \underline{g}_{ij}(\underline{x}_j(k))) \qquad (2.15)$$

Then the optimal control problem can be solved as a two-level problem.

Lower Level: Minimize $\tilde{J}_i(\underline{u}_i,\underline{v}_i,\underline{p})$

$\underline{u}_i,\underline{v}_i$

$$\tilde{J}_i(\underline{u}_i,\underline{v}_i,\underline{p}) = K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k),\underline{u}_i(k)) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$- \sum_{j \neq i} \underline{p}_j'(k)\underline{g}_{ji}(\underline{x}_i(k)) \qquad (2.16)$$

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\underline{v}_i(k),\underline{u}_i(k)) \qquad (2.17)$$

Higher Level: $\max_{\underline{p}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p})$ (2.18)

where $\tilde{J}_i*(\underline{p})$ is the minimum obtained in equation (2.16).

Proof: The results in Section 2.3 are used. $L(\underline{x},\underline{u},\underline{v},\underline{p})$ is split up into uncoupled $\tilde{J}_i$'s by collecting all the terms involving $\underline{x}_i,\underline{v}_i$ and $\underline{u}_i$.

If the optimal $\underline{p}^*$ is given, then the lower level control problems are all uncoupled. The optimal control $\underline{u}^*_i$ can be found using only the structure of the ith system (its system dynamics and cost functional) plus the interconnection functions $g_{ji}(\cdot)$, $j \neq i$. The structural information of the other subsystems are not required. On the other hand, $\underline{p}^*$ is determined using all the optimal $\tilde{J}_i*(\underline{p})$'s. Although algorithms can be devised making use of the special two-level structure of the optimization problem, the convergence to the optimal solution is not accomplished in real time [13]. Thus the decomposition achieved is really with respect to the off-line computation. In the deterministic problem given above, this corresponds to finding the open-loop control functions in some decentralized manner. In the next section we shall show that this philosophy can be extended to the stochastic case.

3. Formulation of the Stochastic Decomposition Problem

In the deterministic case given above, $\underline{v}_i$ is the action of the other subsystems on the ith subsystem, a quantity which is needed for the optimal control of the ith subsystem but is not itself optimized. However, if the constraint $\underline{v}_i(k) = \sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k))$ is satisfied exactly, optimizing with respect to $\underline{u}_i$ and $\underline{v}_i$ simultaneously is equivalent to solving the original optimal control problem with $\underline{u}_i$ as the only control to be optimized. In the actual implementation of the control, only $\underline{u}_i$ is used.

For each lower level problem, $\underline{v}_i$ can be regarded as an estimate of the interaction given $\underline{p}$. If the optimal $\underline{p}^*$ is used, then $\underline{v}_i$ is equal to the action of the other subsystems exactly.

We now extend this philosophy to the stochastic case. Instead of solving for the problem described by equations (2.1), (2.2), (2.5), (2.6) and (2.11) we shall exploit the coupled nature of the system. Since $\underline{x}(k)$ given the control strategy is a random vector, it is no longer possible to choose $\underline{v}_i(k)$ such that it equals $\sum_{j\neq i} \underline{g}_{ij}(\underline{x}_j(k))$ exactly. Rather $\underline{v}_i(k)$ is only required to be an estimate of the interaction and this is the job of the coordinator. We thus have the following formulation.

Problem 1:

Given $$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k),\underline{v}_i(k),\underline{u}_i(k),\underline{\xi}_i(k)) \qquad (3.1)$$

$$i=1,\ldots, N$$

$$J = \sum_{i=1}^{N} J_i$$

$$J_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k))\} \qquad (3.2)$$

$$E\{\underline{v}_i(k) - \sum_{j \neq i}^{N} \underline{g}_{ij}(\underline{x}_j(k))\} = 0 \qquad i=1,\ldots, N \qquad (3.3)$$

$$k=0,\ldots T-1$$

$$\underline{u}_i(k) = \underline{\gamma}_i^{\,k}(Y_i(k), U_i(k-1); \tilde{I}) \qquad (3.4)$$

$$\underline{v}_i(k) = \underline{\eta}_i^{\,k}(\tilde{I}) \qquad (3.5)$$

$$Y_i(k) = \{\underline{y}_i(s); 0 \leq s \leq k\} \qquad (3.6)$$

$$U_i(k) = \{\underline{u}_i(s); 0 \leq s \leq k\} \qquad (3.7)$$

Find $\underline{\gamma}_i^{\,k}$ and $\underline{\eta}_i^{\,h}$, i=1,...., N; k=0,....T-1 such that J is minimized. $\tilde{I}$ consists of the a priori information contained in the model and the cost functional.

The original stochastic control problem has been modified in the following manner. The subsystems are all assumed to be uncoupled. The interaction of the other subsystems is represented by $\underline{v}_i(k)$ which is to be optimized. $\underline{v}_i(k)$ is chosen, however, so that constraint (3.3) is satisfied; thus it is an unbiased a priori estimate of the interaction of the other subsystems. The control problem then consists of finding the optimal control strategies $\underline{\gamma}$ and the optimal estimates of the interactions such that the cost functional is minimized.

Although this problem is very similar to the deterministic problem given in Section 2 of this paper , the results of decomposition in mathematical programming cannot be applied directly since closed loop control strategies $\underline{\gamma}_i$ are required. In the next section, we show how the stochastic control problem can be reformulated so as to lead to a constrained optimization problem.

4. A Constrained Stochastic Optimal Control Problem

Consider the following stochastic control problem.

Problem 2:

System: $\underline{x}(k+1) = \underline{f}(\underline{x}(k),\underline{u}(k),\underline{\xi}(k)) \quad \underline{x}(k) \in R^n$ (4.1)

Measurement: $y(k) = \underline{h}(\underline{x}(k),\underline{\theta}(k)) \quad \underline{u}(k) \in R^p$ (4.2)

Cost functional: $J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k))\}$ (4.3)

$\underline{\xi}(k),\underline{\theta}(k)$, $k=0,\ldots, T-1$ and $\underline{x}(0)$ are random vectors with known statistics.

$$Y(k) \subseteq \{\underline{y}(0),\ldots,\underline{y}(k);\ \underline{u}(0),\ldots,\underline{u}(k-1)\} \quad (4.4)$$

$\underline{u}(k)$ is constrained to be an admissible function of $Y(k)$, i.e.,

$$\underline{u}(k) = \gamma^k(Y(k)) \quad (4.5)$$

$\gamma \in \Gamma$

It is required to choose $\gamma^* \in \Gamma$ such that

$$J(\gamma^*) = \min_{\gamma \in \Gamma} J(\gamma) \quad (4.6)$$

In the problem stated above, the minimization is only over the strategy space Γ. We can transform this to a minimization over random sequences subject to certain constraints.

Let the underlying probability spaces be $(\Omega,\mathcal{B},\mu)$. $\underline{\theta}(k)$, $\underline{\xi}(k)$, $\underline{x}(0)$ are random vectors over Ω.

Let $\underline{x}(\omega) = (\underline{x}(1,\omega),\ldots,\underline{x}(T,\omega))$ be a $\mathcal{B}$-measurable L^2 function over Ω into k^{nT}, i.e., $\underline{x} \in L^2(\Omega, R^{nT})$

Let $\underline{u}(\omega) = (\underline{u}(0,\omega),\ldots,\underline{u}(T-1,\omega)) \in L^2(\Omega,R^{pT})$.

Let

$$S_1 = \{\underline{x} \in L^2(\Omega,T^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \,|\, \underline{x}(k+1,\omega) = \underline{f}(\underline{x}(k,\omega),\ \underline{u}(k,\omega))\ \text{a.e.}\}$$

= set of $\underline{x},\underline{u}$ which correspond to the given dynamic system (4.7)

$$S_2 = \{\underline{x} \in L^2(\Omega,R^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \,|\, \exists\, \gamma \in \Gamma \text{ such that}$$

$$\underline{u}(k,\omega) = \gamma^k(Y(k,\omega)) \quad \text{a.e.}\}$$

$$= \{\underline{x} \in L^2(\Omega,R^{nT}),\ \underline{u} \in L^2(\Omega,R^{pT}) \,|\, \exists\, \gamma \in \Gamma \text{ such that}$$

$$\underline{u}(k,\omega) = \gamma^k(\underline{h}(\underline{x}(0,\omega),\ \underline{\theta}(0,\omega)),\ \underline{h}(\underline{x}(1,\omega),\ \underline{\theta}(1,\omega)),\ldots,$$

$$\underline{h}(\underline{x}(k,\omega),\ \underline{\theta}(k,\omega));\ \underline{u}(0,\omega),\ldots,\underline{u}(k-1,\omega))\ \text{a.e.}\}$$

= set of $\underline{x},\underline{u}$ which can be generated from the given information structure and admissible control strategy. (4.8)

Let $G : \Gamma \to L^2(\Omega,R^{nT}) \times L^2(\Omega,R^{pT})$ be defined as

$$G(\gamma) = (\underline{x}(\gamma),\ \underline{u}(\gamma)) \qquad (4.9)$$

Then by the definition of S_1 and S_2,

$$\text{Range } G = S_1 \cap S_2 \qquad (4.10)$$

Therefore

$$\min_{\gamma \in \Gamma} J(\gamma) = \min_{\gamma \in \Gamma} J(\underline{x}(\gamma),\ \underline{u}(\gamma))$$

$$= \min_{G(\gamma) \in S_1 \cap S_2} J(\underline{x}(\gamma),\ \underline{u}(\gamma))$$

$$= \min_{(\underline{x},\underline{u}) \in S_1 \cap S_2} J(\underline{x},\underline{u}) \qquad (4.11)$$

Note that the minimization is now over random sequence $\underline{x},\underline{u}$. The dynamics of the system, the constraint on the control strategy and the information structure allowed have been incorporated into the constraint set $S_1 \cap S_2$.

We next consider the constrained stochastic control problem.

Problem 3:

System: $\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k))$ (4.12)

Measurement: $\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k))$ (4.13)

Cost Functional: $J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k))\}$ (4.14)

$$\underline{u}(k) = \gamma^k(Y(k)) \quad (4.15)$$

$$E\{\underline{H}(\underline{x}(k), \underline{u}(k))\} = \underline{0} \in R^q \quad k = 0,\ldots T-1 \quad (4.16)$$

It is required to choose $\gamma^* \in \Gamma$ such that

$$J(\gamma^*) = \min_{\gamma \in \Gamma} J(\gamma)$$

and the constraint (4.16) is satisfied. $\underline{H}$ is a vector-valued function.

This constraint is only required to be satisfied on the average. Problem 3 can be transformed into the following unconstrained stochastic control problem.

Problem 4:

System: $\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{\xi}(k))$ (4.17)

Measurement: $\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k))$ (4.18)

Cost Functional:
$$\tilde{J}(\gamma,\underline{p}) = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k), \underline{u}(k))\} \quad (4.19)$$

$$\underline{u}(k) = \gamma^*(Y(k)) \quad (4.20)$$

It is required to find γ^* such that $\tilde{J}(\gamma,\underline{p})$ is minimized.

Theorem 4.1: Suppose a saddle point exists for the stochastic control problem 4, i.e. there exist $\gamma^*,\underline{p}^*$ such that

$$\tilde{J}(\gamma^*,\underline{p}) \leq \tilde{J}(\gamma^*,\underline{p}^*) \leq \tilde{J}(\gamma,\underline{p}^*) \qquad (4.21)$$

Then γ^* is the solution to Problem 3.

Proof: The constraint (4.16) can be written as

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \in R^{qT} \qquad (4.22)$$

where

$$\underline{x} \in L^2(\Omega,R^{nT}), \; \underline{u} \in L^2(\Omega,R^{pT}) .$$

Problem 3 is then equivalent to

$$\min_{\underline{x},\underline{u} \in S_1 \cap S_2} J(\underline{x},\underline{u})$$

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \qquad (4.23)$$

$$\tilde{J}(\gamma,\underline{p}) = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k),\underline{u}(k))\}$$

$$= E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k))\} + \sum_{k=0}^{T-1} \underline{p}'(k)\, E\{\underline{H}(\underline{x}(k),\underline{u}(k))\}$$

$$= J(\gamma) + \underline{p}'\,\tilde{H}(\underline{x}(\gamma),\underline{u}(\gamma)) \qquad (4.24)$$

If $\tilde{J}(\gamma,\underline{p})$ has a saddle point $(\gamma^*,\underline{p}^*)$, then $(G(\underline{\gamma}^*),\underline{p}^*)$ is a saddle point for the function $J(\underline{x},\underline{u}) + \underline{p}'\underline{H}(\underline{x},\underline{u})$.

By Theorem 3.1 [6], $G(\gamma^*) = (\underline{x}^*,\underline{u}^*)$ solves

$$\underset{(\underline{x},\underline{u}) \in S_1 \cap S_2}{\text{Min}} J(\underline{x},\underline{u})$$

such that

$$\tilde{H}(\underline{x},\underline{u}) = \underline{0} \tag{4.25}$$

or γ^* solves Problem 3. Q.E.D.

The following corollary follows immediately.

<u>Corollary 4.2:</u> If a saddle point $(\gamma^*,\underline{p}^*)$ exists for Problem 4, then the optimal strategy γ^* can be found by

$$\underset{\underline{p}}{\text{Max}}\ \underset{\gamma}{\text{Min}}\ E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k), \underline{u}(k))\} \tag{4.26}$$

<u>Proof:</u> We need only the fact that if a saddle point $(\gamma^*,\underline{p}^*)$ exists for the function $L(\gamma,\underline{p})$, then

$$\underset{\gamma}{\text{Min}}\ \underset{\underline{p}}{\text{Max}}\ L(\gamma,\underline{p}) = \underset{\underline{p}}{\text{Max}}\ \underset{\gamma}{\text{Min}}\ L(\gamma,\underline{p}) = L(\gamma^*,\underline{p}^*) \tag{4.27}$$

To check for the saddle point, we need to verify the condition directly or use condition (4.27). The following condition is sometimes more convenient.

<u>Lemma 4.3:</u> Consider the problem

$$\underset{x}{\text{Min}}\ f(x)$$

$$g(x) = \underline{0} \quad x \in C \tag{4.28}$$

If

(1) $\underset{\underline{p}}{\text{Max}}\ \underset{x \in C}{\text{Min}}\ f(x) + \underline{p}'g(x) \triangleq f(x^*) + \underline{p}^{*\prime}g(x^*)$ exists

(2) $g(x^*) = \underline{0}$

then x^* minimizes $f(x)$ such that $g(x) = \underline{0}, x \in C$.

Proof:

$$f(x^*) + \underline{p}'g(x^*) = f(x^*) = f(x^*) + \underline{p}^{*\prime}g(x^*) \qquad (4.29)$$

$$f(x) + \underline{p}^{*\prime}\, g(x) \geq \underset{x \in C}{\text{Min}}\ f(x) + \underline{p}^{*\prime}g(x) \qquad (4.30)$$

$$\underset{x \in C}{\text{Min}}\ f(x) + \underline{p}^{*\prime}\, g(x) = f(x^*(\underline{p}^*)) + \underline{p}^{*\prime}\, g(\underline{x}^*(\underline{p}^*)) \qquad (4.31)$$

where $x^*(\underline{p})$ minimizes $f(x) + \underline{p}'g(x)$, $x \in C$.

Thus

$$\underset{\underline{p}}{\text{Max}}\ \underset{x \in C}{\text{Min}}\ f(x) + \underline{p}'g(x) = \underset{\underline{p}}{\text{Max}}\ f(x^*(\underline{p})) + \underline{p}'g(x^*(\underline{p}))$$

$$= f(x^*(\underline{p}^*)) + \underline{p}^{*\prime}g(\underline{x}^*(\underline{p}^*)) \text{ by definition}$$

$$= \underset{x \in C}{\text{Min}}\ f(x) + \underline{p}^{*\prime}g(x) \qquad (4.32)$$

Then

$$f(x^*) + \underline{p}'g(x^*) \leq f(x^*) + \underline{p}^{*\prime}g(x^*) \leq f(x) + \underline{p}^{*\prime}g(x)$$

$$\text{for all } x \in C \text{ and } \underline{p} \qquad (4.33)$$

$(x^*, \underline{p}^*)$ is a saddle point and x^* minimizes $f(x)$ such that $g(x) = \underline{0}, x \in C$. Q.E.D.

Theorem 4.1 can then be restated in the following form.

Theorem 4.4: Suppose

$$\underset{\underline{p}}{\text{Max}}\ \underset{\gamma}{\text{Min}}\ E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k), \underline{u}(k)) + \underline{p}'(k)\underline{H}(\underline{x}(k), \underline{u}(k))\}$$

exists for the system described in Problem 4, and further

$$E\{\underline{H}(\underline{x}^*(k),\ \underline{u}^*(k))\} = \underline{0} \qquad k = 0,\ldots,T-1$$

where $\underline{x}^*, \underline{u}^*$ are the optimal trajectory and control using γ^*. Then γ^* is the optimal strategy for Problem 3.

5. Decomposition of the Stochastic Control Problem

We now apply the results of the last section to Problem 1 and transform it to an unconstrained problem.

Theorem 5.1: Consider the system

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \underline{\xi}_i(k)) \quad i=1,\ldots,N \tag{5.1}$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k), U_i(k-1);\tilde{I}) \tag{5.2}$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I}) \tag{5.3}$$

$$\tilde{J} = \sum_{i=1}^{N} \tilde{J}_i \tag{5.4}$$

$$\tilde{J}_i = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)\, \underline{v}_i(k) - \sum_{j\neq i} \underline{p}_j'(k)\, \underline{g}_{ji}(\underline{x}_i(k))\} \tag{5.5}$$

If $\underset{\underline{p}}{\text{Max}}\ \underset{\underline{\gamma},\underline{\eta}}{\text{Min}}\ \tilde{J}$ exists and

$$E\{\underline{v}_i^*(k) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j^*(k))\} = \underline{0} \quad i=1,\ldots,N; \quad k=0,\ldots,T-1 \tag{5.6}$$

then $\underline{\gamma}^*,\underline{\eta}^*$ are the optimal strategies for Problem 1.

Proof: This problem can be cast into the form of Problem 3 by identifying

$$\underline{u}(k) \quad \text{with} \quad \{\underline{u}_i(k), \underline{v}_i(k);\ i=1,\ldots,N\} \tag{5.7}$$

$$\gamma^k \quad \text{with} \quad \{\underline{\gamma}_i^k, \underline{\eta}_i^k;\ i=1,\ldots,N\} \tag{5.8}$$

$$\underline{H}(\underline{x}(k), \underline{u}(k)) = \begin{bmatrix} \underline{v}_1(k) - \sum_{j\neq 1} g_{1j}(\underline{x}_j(k)) \\ \vdots \\ \underline{v}_N(k) - \sum_{j\neq N} g_{Nj}(\underline{x}_j(k)) \end{bmatrix} \tag{5.9}$$

$$\sum_{i=1}^{N} E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)(\underline{v}_i(k) - \sum_{j\neq i}^{N} \underline{g}_{ij}(\underline{x}_j(k))\}$$

$$= \sum_{i=1}^{N} E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)\ \underline{v}_i(k)$$

$$- \sum_{j\neq i} \underline{p}_j'(k)\ \underline{g}_{ji}(\underline{x}_i(k))\}$$

$$= \sum_{i=1}^{N} \tilde{J}_i \qquad (5.10)$$

Theorem 4.4 can then be applied in a straight forward manner. Q.E.D.

Note that given any $\underline{p}$, the minimization problem is separated into N uncoupled stochastic control problems. The ith controller needs only the structure of his own system as his a priori information. Thus there is decentralization of a priori as well as a posteriori information.

A two-level hierarchical decomposition for finding the optimal control strategy is possible.

Lower Level:

$$\underline{x}_i(k+1) = \underline{f}_i(\underline{x}_i(k), \underline{v}_i(k), \underline{u}_i(k), \underline{\xi}_i(k))$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k), U_i(k-1); \tilde{I})$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I})$$

$$\tilde{J}_i(\underline{p}) = E\{K_i(\underline{x}_i(T)) + \sum_{k=0}^{T-1} L_i(\underline{x}_i(k), \underline{u}_i(k)) + \underline{p}_i'(k)\ \underline{v}_i(k)$$

$$- \sum_{j\neq i} \underline{p}_j'(k)\ \underline{g}_{ji}(\underline{x}_i(k))\} \qquad (5.11)$$

Find $\underline{\gamma}_i^k$ and $\underline{\eta}_i^k$ such that $\tilde{J}_i(\underline{p})$ is minimized, $i=1,\ldots,N$. Let $\tilde{J}_i^*(\underline{p})$ be the optimal cost associated with a particular $\underline{p}$.

<u>Higher Level</u>: Maximize$_{\underline{p}}$ $\sum_{i=1}^{N} \tilde{J}_i^*(\underline{p})$ (5.12)

<u>Remark</u>: The higher level problem is deterministic and static in nature whereas the lower level problems are stochastic and dynamic, although uncoupled. The decentralized a priori information allows the off-line computation to be done in an algorithmic manner. Typically, the higher level coordinator will choose a $\underline{p}$, the lower level controllers then compute the optimal cost associated with this $\underline{p}$. The coordinator then chooses another $\underline{p}$ to increase the optimal cost of the lower level systems. The decomposition is <u>off-line</u> because it is done before the system starts using only a priori information. The advantages of this approach are the following:

(1) The overall stochastic control problem is split up into N stochastic control problems with lower dimension. Each of these can be solved more easily.

(2) Although the value of $\underline{p}$ may change, the structure of the lower level problems remains the same, and hence essentially the stochastic control problems need only to be solved once.

6. Discussion

We have considered the stochastic control of N coupled systems with decentralized information structure. By defining a new kind of optimality, it is found that the optimal control strategies can be found in a decentralized manner. Moreover, given the optimal coordinating parameters, the control problems of the N subsystems are uncoupled. Thus the control strategies using decentralized a posteriori information can be computed with decentralized a priori information. Although this scheme is suboptimal with respect to the ordinary stochastic control problem, computationally it is more efficient.

Because of the nonlinear nature of the problem we cannot say much about the detailed computations involved. However, it is obvious that instead of one high dimensional stochastic control problem we now have N lower dimensional stochastic control problems and one extra deterministic optimization problem to be solved by the coordinator. In the next chapter, we shall look at the linear-quadratic-Gausian problem in detail and obtain explicit solutions for these lower and higher level problems.

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