Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 28

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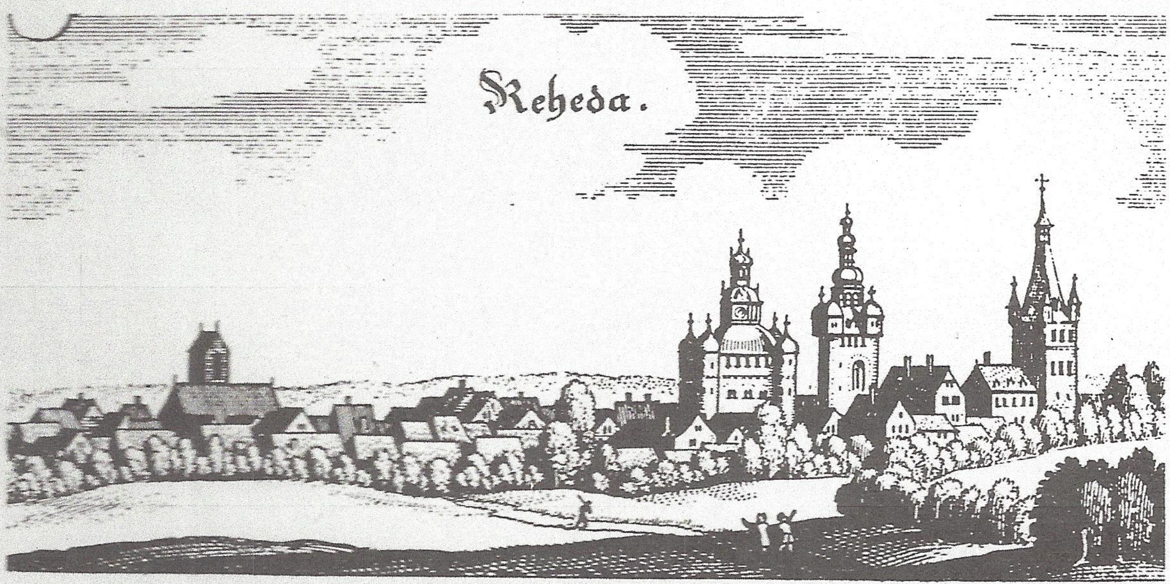
Real Income, The Cost-of-Living and Consumer Surplus: A Unified Approach to their Conceptual Foundation

April 1975



Anno 1600 nach Merian

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REAL INCOME, THE COST-OF-LIVING AND CONSUMER SURPLUS: A UNIFIED APPROACH TO THEIR CONCEPTUAL FOUNDATION

ABSTRACT

It is argued that the problems dealt with in consumer surplus and index number theories are the same. Real income and the cost-of-living are the basic constructs; their increments are the compensating and equivalent variations. Continuing controversies regarding the foundations of cost-benefit analysis are shown to be the consequence of conceptual confusions. An implication of the analysis which is relevant for applications, is that the traditional consumer surplus formula is not a valid quadratic approximation to either the compensating of the equivalent variation.

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1. Two bodies of literature have evolved dealing with consumer surplus and with consumption (or cost-of-living) index numbers. I shall argue that the basic problems considered under these distinct headings are identical, and that they can be given a definitive solution. The continuing controversies, particularly relating to consumer surplus, are due essentially to conceptual confusion and to the lack of a clear understanding regarding the purpose of the measures constructed. Consequently, the analysis of this paper will be mainly conceptual; once clarity is achieved at this level, the derivation of analytical results is simple.

There are two generic problems of which the applications of consumer surplus and index numbers are particular instances; I call these the evaluation problem and the standard-of-living problem. In the evaluation problem a number of alternative price,income combinations are possible; the most preferred of these is to be determined. In the standard-of-living problem there is a set of alternative price vectors and a fixed utility level u'. For each price vector a level of money income is to be determined such that the household can just attain u'.

The measures which allow these problems to be unambiguously solved are real income and the cost-of-living. The changes in these are the equivalent and compensating variations introduced by Hicks into consumer surplus theory. The connection with consumer surplus is that the compensating variation is equal to the relevant area under the constant utility demand curve.

The choice of bases for these measures will be clarified and it will be shown,that with the proper choice,the change in money income can always be decomposed into a change in real income and a change in the cost-of-living.

Real income and the cost-of-living can be approximated quadratically without any special restrictions. In particular, there is no need for the assumption of a "constant" or "nearly constant" marginal utility of income which has led many theorists to question the validity of consumer surplus analysis. A principal result of the paper is that the correct quadratic approximations do not coincide with those commonly employed.

The interest in real income and the cost-of-living derives in large part from the fact that they have the dimension of money and can be added interpersonally. The justification of the use of these aggregative measures in making collective choices requires normative assumptions and in most cases also the analysis of a general equilibrium system. The difficult problems which arise in these contexts are beyond the scope of this paper.

The relation of the analysis to the literature is sketched in section 6.

2. In this section we define the basis concepts with which we are concerned and derive some of their properties. Let p be the price vector facing an individual, x his consumption vector and $m = px$ his income. The indirect utility function is

$$u = u(m,p) \tag{1}$$

which can be solved for the expenditure function

$$m = m(u,p) \tag{2}$$

giving the minimum expenditure required to reach the utility level u at prices p.[1] Let u', p' be fixed base period values. Using these, two further functions can be defined:

$$r = r(u|p') = m(u,p') \qquad (3)$$

is the real income function which enables one to compare the cost of different utility levels at constant base period prices.

$$c = c(p|u') = m(u',p) \qquad (4)$$

is the cost-of-living function which measures the cost of attaining a fixed utility level at alternative prices. It is often useful to substitute (1) into (3) and (4) leading to

$$r[u(m,p)|p'] = R(m,p|p') \qquad (5)$$

$$c[p|u(m',p')] = C(p|m',p') \qquad (6)$$

The geometry of the R function is illustrated in Figure 1.A. An expansion path E corresponding to p' is shown. Let each indifference curve be numbered by the expenditure along E at the point of intersection. $R(m,p|p')$ maps each pair (m,p) into the number of the indifference curve to which the budget line $m = px$ is tangent. Real income is thus a normalization of the utility function. In Figure 1.B the budget line is rotated about the base indifference curve I' corresponding to the utility level $u' = u(m',p')$. The function $C(p|m',p')$ is obtained by associating with each p the expenditure along the budget line characterized by p and tangent to I'.

Consider the consumer in two situations parametrized by (m_0, p_0) and (m_1, p_1). There are two ways to decompose the change in money income:

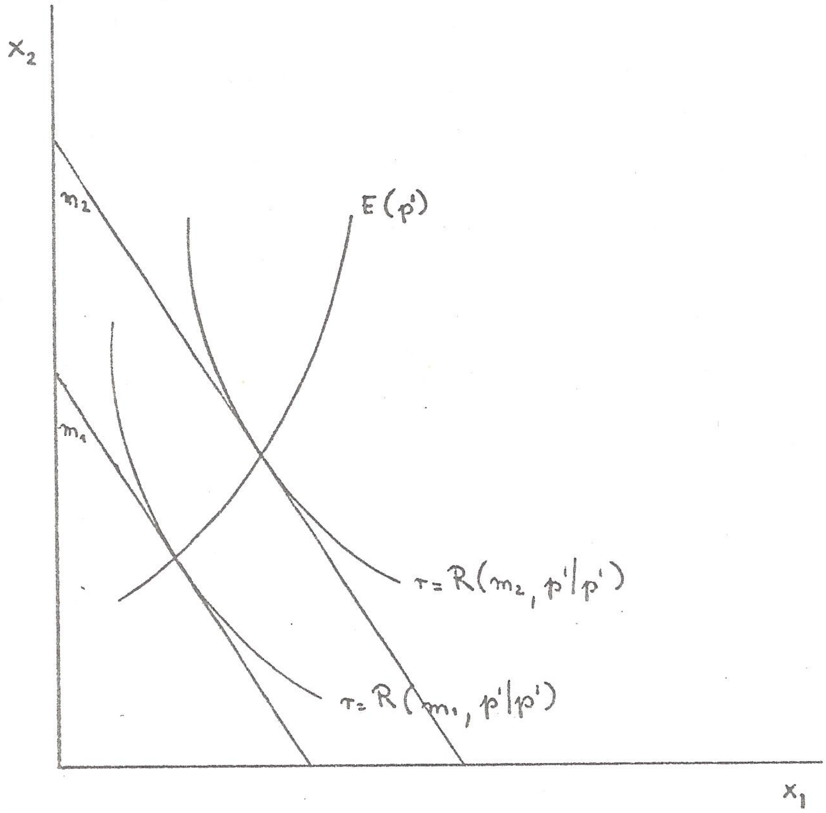
$$m_1 - m_0 = r(u_1|p_0) - r(u_0|p_0) + c(p_1|u_1) - c(p_0|u_1) \qquad (7.a)$$

$$= r(u_1|p_1) - r(u_0|p_1) + c(p_1|u_0) - c(p_0|u_0) \qquad (7.b)$$

where as bases we must use (u_1, p_0) or (u_0, p_1). The two decompositions are illustrated in Figure 2. The total movement of

Figure 1

A



B

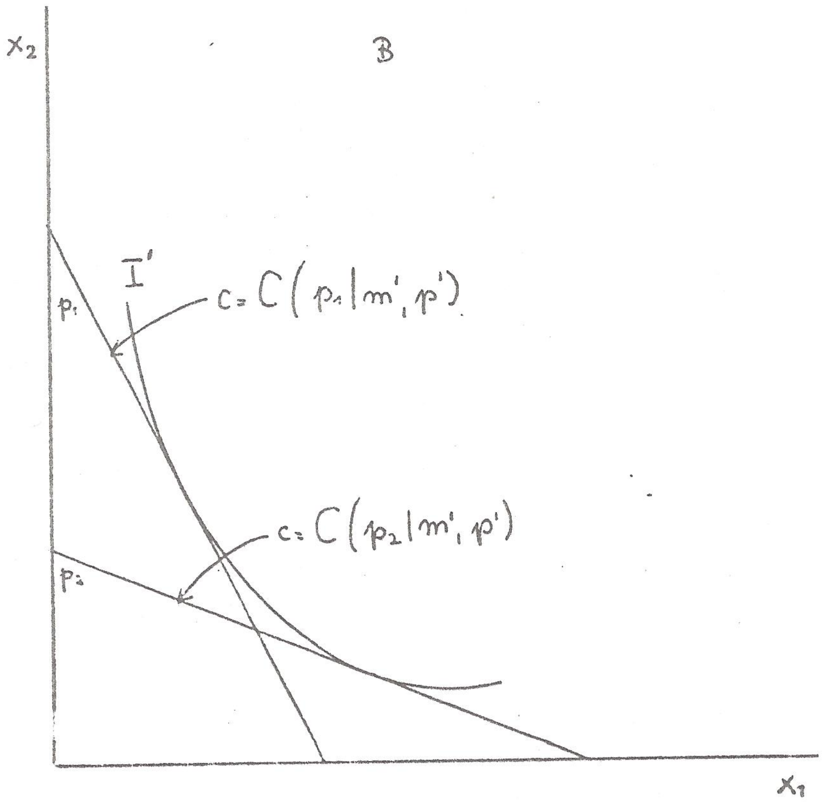
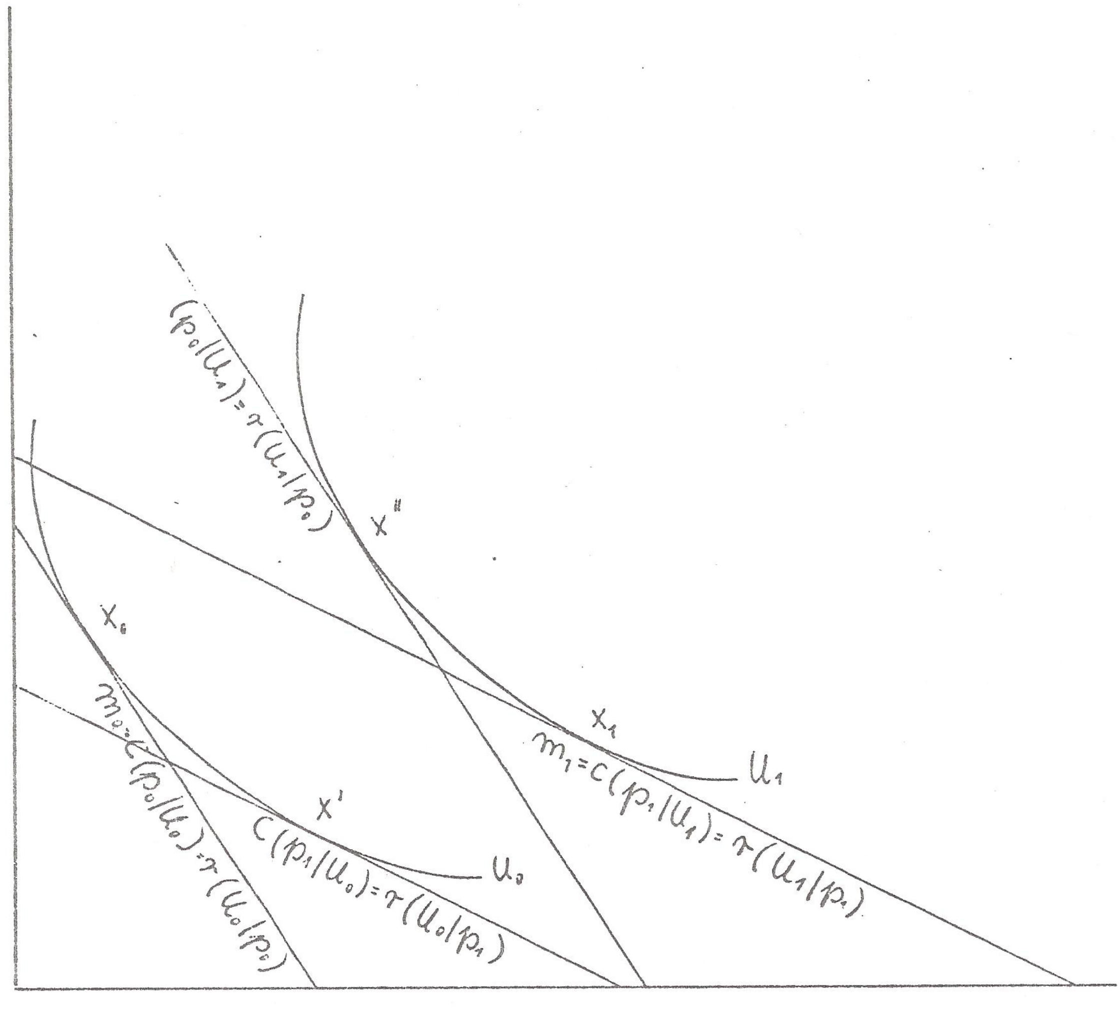


Figure 2



$c(p_0/u_1) = r(u_1/p_0)$
x''
x_0
x_1
$m_1 = c(p_1/u_1) = r(u_1/p_1)$
u_1
x'
$m_0 = c(p_0/u_0) = r(u_0/p_0)$
$c(p_1/u_0) = r(u_0/p_1)$
u_0

the consumer is from x_0 to x_1. The decomposition can be made in two ways. Moving from x_0 to x' we have a substitution effect along u_0 which costs $c(p_1 | u_0) - c(p_0 | u_0)$. The income effect from x' to x_1 costs $r(u_1 | p_1) - r(u_0 | p_1)$. Equation (7.a) states that the sum of these changes is $m_1 - m_0$, the cost of moving from x_0 to x_1. Similarly, (7.b) corresponds to a decomposition in which the first move is from x_0 to x'' at p_0 prices, followed by a move from x'' to x_1 along the u_1 indifference curve.

3. How is the choice of bases u', p' for the measures r, c to be made? In the literature it has usually been held that in a comparison of two situations parametrized by $(m_0, p_0), (m_1, p_1)$ we may arbitrarily use p_0 or p_1 or some mean of these as a base for c and similarly u_0, u_1 or and average of these as the base for r. The view taken here is that the choice of a base cannot be determined in the abstract but only in relation to a concrete application. This is illustrated by the following example: A union contract signed at time t specifies that in periods $t_1, \ldots, t_N$ a representative worker is to receive a nominal wage of $m_1, \ldots, m_N$ subject to a cost-of-living adjustment. Let the price vectors which are realized in these periods be $p_0, p_1, \ldots, p_N$. The contract implicitly stipulates the payment of real wages $r_1 = R(m_1, p_0 | p_0), \ldots, r_N = R(m_N, p_0 | p_0)$. The actual money wages which must be paid are $c_1 = C(p_1 | m_1, p_0), \ldots, c_N = C(p_N | m_N, p_0)$. In this case the choice of a base is completely determined in each instance by the features of the problem.

The matter is different for an evaluation problem. Let the alternatives be characterized by the income-price combinations $(m_1, p_1), \ldots, (m_N, p_N)$. These can be compared by means of a real income function giving $r_1 = R(m_1, p_1 | p')$, ..., $r_N = R(m_N, p_N | p')$. In this case the choice of p' is immaterial; the base vector p' could be set equal to any of the vectors $p_1, \ldots, p_N$ or it may be some other arbitrary vector. The real income functions generated will all be monotone transformations of each other and the ranking of alternatives will not be affected.

4. According to Hicks the fundamental concepts in consumer surplus theory are the compensating variation- the change in money income which would just restore the individual to the original utility level-and the equivalent variation-the change in money income which, from the initial position, would produce the same utility increment as a given price change. These are simply the increments in the cost-of-living and real income:

$$\text{compensating variation} = c(p_1 | u') - c(p_0 | u') \tag{8.a}$$

$$\text{equivalent varaition} = r(u_1 | p') - r(u_0 | p') \tag{8.b}$$

The link with the traditional definition of consumer surplus is provided by the fact that the compensating variation associated with the change in a particular price is the relevant area under compensated (constant utility) demand curve for that good. This follows from the property of the indirect utility function that

$$\frac{\partial u}{\partial p^i} = -\lambda(m,p)x^i(m,p) \tag{9}$$

where λ is the marginal utility of income $\partial u/\partial m$. From the definition of c

$$u' = u(c,p) \tag{10}$$

Implicit differentiation of (10) using (9) gives

$$\frac{\partial c}{\partial p^i} = x^i(p|u') = x^i[c(p|u'),p] \tag{11}$$

If only the ith price changes

$$c(p_1|u') - c(p_0|u') = \int_{p_0}^{p_1} x^i(p|u')dp^i \tag{12}$$

More generally

$$c(p_1|u') - c(p_0|u') = \sum_i \int_{p_0^i}^{p_1^i} x^i(p|u')dp^i \tag{13}$$

Since the total differential of a function is being integrated, the integrability conditions are satisfied.

The attempt to obtain a similar result using ordinary demand function leads to the heart of the controversy regarding the assumption of a constant marginal utility of income. From the definition of r,

$$u(m,p) = u(r,p') \tag{14}$$

Implicit differentiation of (14), using (9), leads to

$$\frac{\partial r}{\partial m} = \frac{\lambda(m,p)}{\lambda(r,p')} \tag{15.a}$$

$$\frac{\partial r}{\partial p^i} = -\frac{\lambda(m,p)x^i(m,p)}{\lambda(r,p')} \tag{15.b}$$

From (15.b) it is evident that the equivalent variation would (in absolute value) correspond to the area under the ordinary demand curve only if λ were constant. That this cannot be the case for arbitrary income-price variations is evident from the fact that for any utility function, the marginal utility of income function $\lambda(m,p)$ is homogeneous of degree minus one.

We are led to the conclusion that the valid substance of the consumer surplus argument is embodied in equation (13).

The importance of consumer surplus viewed as the area under a demand curve appears to be mainly historical since it is easier to deal directly with the functions c( ) and r( ) than with the integral representation.

5. This section deals with approximations to the compensating and equivalent variations.

From (11) we see that

$$\frac{\partial c}{\partial p^j \partial p^i} = \frac{\partial x^i(p|u')}{\partial p^j} = s^i_j \qquad (16)$$

where s^i_j is the substitution term of the Slutsky equation

$$x^i_j = s^i_j - x^j x^i_m \qquad (17)$$

and x^i_j, x^i_m are the slopes $\partial x^i/\partial p^j$, $\partial x^i/\partial m$.

From (11) and (16) it follows that to a quadratic approximation

$$dc = \sum_i x^i dp^i + \frac{1}{2}\sum_{i,j} s^i_j dp^i dp^j \qquad (18)$$

Therefore using (7.a) or (7.b)

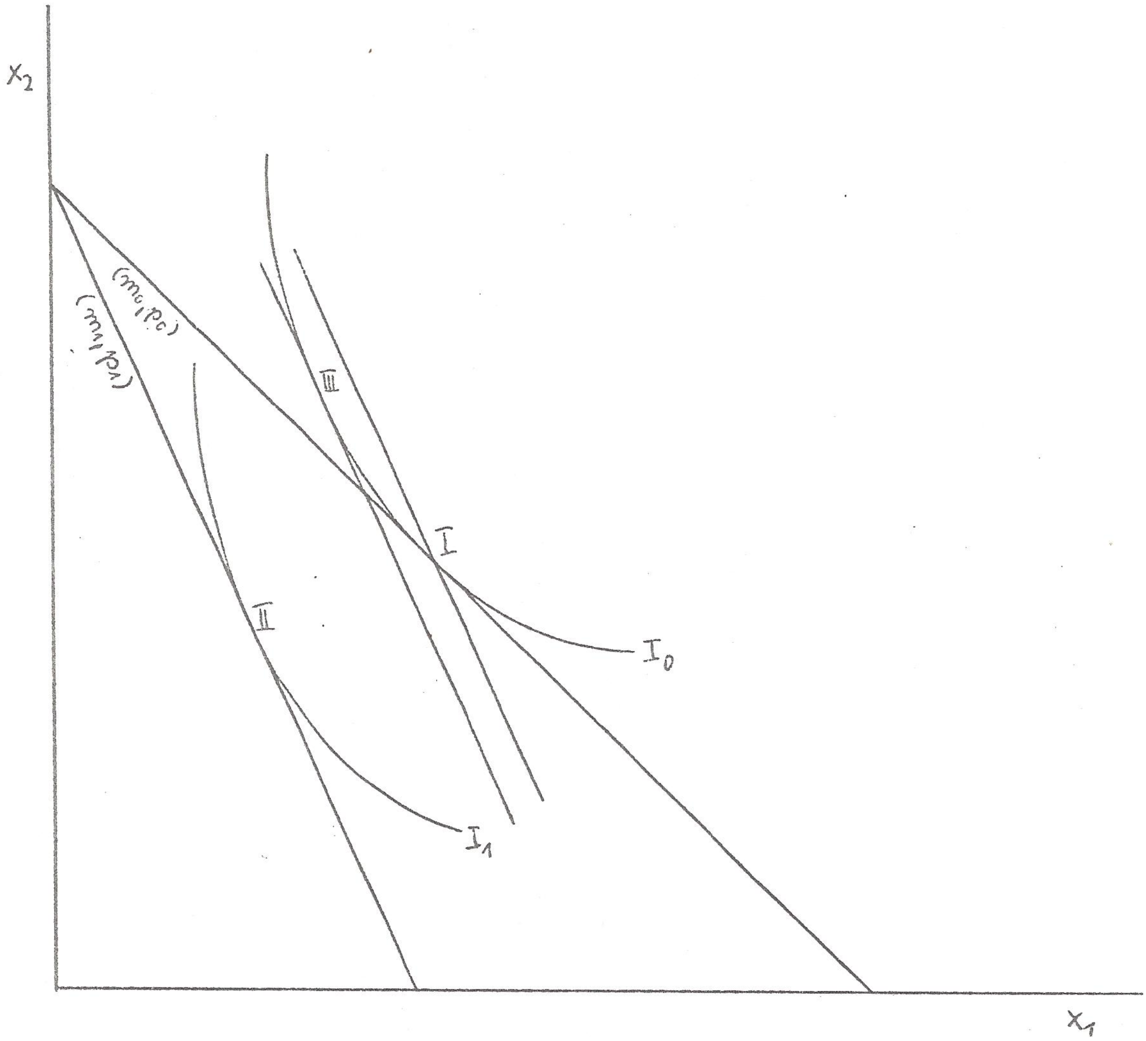
$$dr = dm - \sum_i x^i dp^i - \frac{1}{2}\sum_{i,j} s^i_j dp^i dp^j \qquad (19)$$

For the economic interpretation we note that

$$\sum_{i,j} s^i_j dp^i dp^j = \sum_i dx^i dp^i < 0 \qquad (20)$$

where the dx^i are taken along an indifference surface, is the generalized substitution effect. The linear part of the approximation in (18) gives the change in money income required to allow the individual to buy the old set of quantities at the new prices. In Figure 3, which is drawn for $dp^1 > 0$, it corresponds to the parallel shift from point II to point I. This

Figure 3



amount is larger than required since by substituting, the household can reach the old indifference curve more cheaply at III. The expenditure reduction in going from I to III is estimated by the quadratic term.

From (18) and (20) it is seen that the quadratic term is a linear approximation to the relevant area under the compensated demand curve. This differs from the traditional approximation which corresponds to a linearization of the ordinary demand curve. To show how the two approximations are related we use (17) and the total differential

$$dx^i = x^i_m\, dm + \sum_j x^i_j\, dp^j \tag{21}$$

From these it follows that

$$\sum_{i,j} s^i_j\, dp^i dp^j = \sum_i dx^i dp^i - \sum_i x^i_m\, dm\, dp^i + \sum_{i,j} x^j x^i_m\, dp^i dp^j \tag{22}$$

It is seen that the traditional approximation would be valid if $x^i_m = 0$ for all i. This would be equivalent to a constant marginal utility of income as can be seen from the first order conditions $\partial u/\partial x^i = \lambda p^i$. There seems to be no reason for imposing on the approximation a restriction which is incompatible with the utility maximization hypothesis. The traditional approximation has a systematic bias the direction of which depends on whether we are dealing with normal or inferior goods. I believe that either the valid quadratic approximations (18),(19) should be used, or else, simply the linear parts of these approximations.

6. This section explores the relationship of the analysis to the literature.[2]

The idea that the area under a demand curve can be used to estimate the monetary equivalent of a price change goes back to

Dupuit. The importance that this notion has gained in economics is due to its being championed by Alfred Marshall. Interpreters of Marshall have not been able to agree on whether he worked with compensated or uncompensated demand curves. His stress on constancy of the marginal utility of income indicates that his analysis was not entirely satisfactory. This feature has been at the heart of subsequent controversies regarding the validity of consumer surplus analysis.[3]

Decisive advances in consumer surplus analysis were made by Hicks, especially in [6] and in [7, pp. 330-33]. He stressed that the theoretically important magnitudes are the compensating and equivalent variations. He also recognized that the compensating variation is the change in the cost-of-living and the equivalent variation the change in real income. A limitation is that he did not define the cost-of-living and real income explicitly and that he only considered the effects of price changes unaccompanied by income changes. Hicks also demonstrated that the compensating variation is exactly measured by the area under the compensated demand curve. Further, he solved the problem of quadratic approximation and obtained our equation (18). It is seen that nearly all of the elements of the complete solution which we have presented in this paper were already present in his work. On one issue Hicks did not see clearly and this led him into errors which have perpetuated the indecisive consumer surplus controversy. As we have seen, there are four measures that enter into the comparison of two alternative situations: the compensating and equivalent variations, where either (u_1, p_0) or (u_0, p_1) can serve as bases.

Hicks did not recognize that the choice between these depends on the concrete application and cannot be determined in the abstract. Since in his analysis the two situations differ only in prices, not in money income, the compensating variation to one base is (after reversing the sign) equal to the equivalent variation to the other base. There are thus only two independent magnitudes which we may identify with the two compensating variations. Since he saw no reason for preferring one base as against the other, it was natural for him to think of taking an average of the two. For straight line approximations he showed that this average corresponds to the relevant area under the ordinary demand curve. Thus he came in the end to champion the traditional approximation. Unfortunately, by averaging Hicks sacrifices most of the analytical gains he himself had made. There is no theoretical concept which corresponds to an average of the two variations. Further, the traditional formula is not a quadratic approximation to either variation.

The traditional approximation was also obtained by Hotelling [9]. In contrast to the verbal and diagrammatic argument of Hicks, Hotelling gives a purely formal derivation. Implicit in his proof is the assumption of a constant marginal utility of income.[4]

Turning to some recent contributions, we find the traditional approximation being championed by Harberger [5] and Burns [2]. To these articles our criticism of this approximation applies.

The argument of Silberberg [15] is in substantial agreement with our findings. He rejects the area under the ordinary demand curve as a meaningful measure since it is path dependent. He also notes that the compensating variation is a well defined measure which may be estimated without running into this problem.

Three approaches to index number theory have been distinguished by Samuelson and Swamy [14]. The earliest of these, which may be called the statistical approach, regarded a price index as some sort of measure of central tendency for the distribution of prices. This was followed by the test approach, most fully elaborated in the writings of Irving Fisher. The modern approach is the economic theory of index numbers in which the indexes of real income and the cost-of-living are related to individual preferences.

Writers on index numbers have been primarily concerned with price indexes. The first explicit formal definition of real income appears to have been given surprisingly recently by Kloek [12]. The concept appears also in the work of Goldberger [4]and Theil [16], [17].

The duality of real income and the cost-of-living are stressed by Samuelson and Swamy. Their analysis focuses on the case of homogeneous utility functions for which they show that proportional changes in the ratios $c(p_1|u')/c(p_o|u')$ and $r(u_1|p')/r(u_o|p')$ are independent of the choice of bases.

7. In a recent article Boadway [1] states that the measurement problems associated with consumer surplus analysis appear insoluble and he proceeds to attack as well the normative foundations. The analysis of this paper indicates that the measurement problem can be regarded as solved. The normative aspect has been left out of consideration, but I am hopeful that once the confusions regarding definition and measurement are cleared up, that progress will be possible here also.

NOTES

1. For properties of the indirect utility and expenditure functions see [10], [11].
2. No attempt is made at providing a general survey of the vast literature on consumer surplus and index numbers. The interested reader may consult [14] for references to survey articles and individual contributions to index number theory. A survey of consumer and producer surplus literature with many references is [3].
3. Cf. the analysis of Samuelson [13].
4. Without this assumption it is impossible to derive his equation (18) from (16).

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