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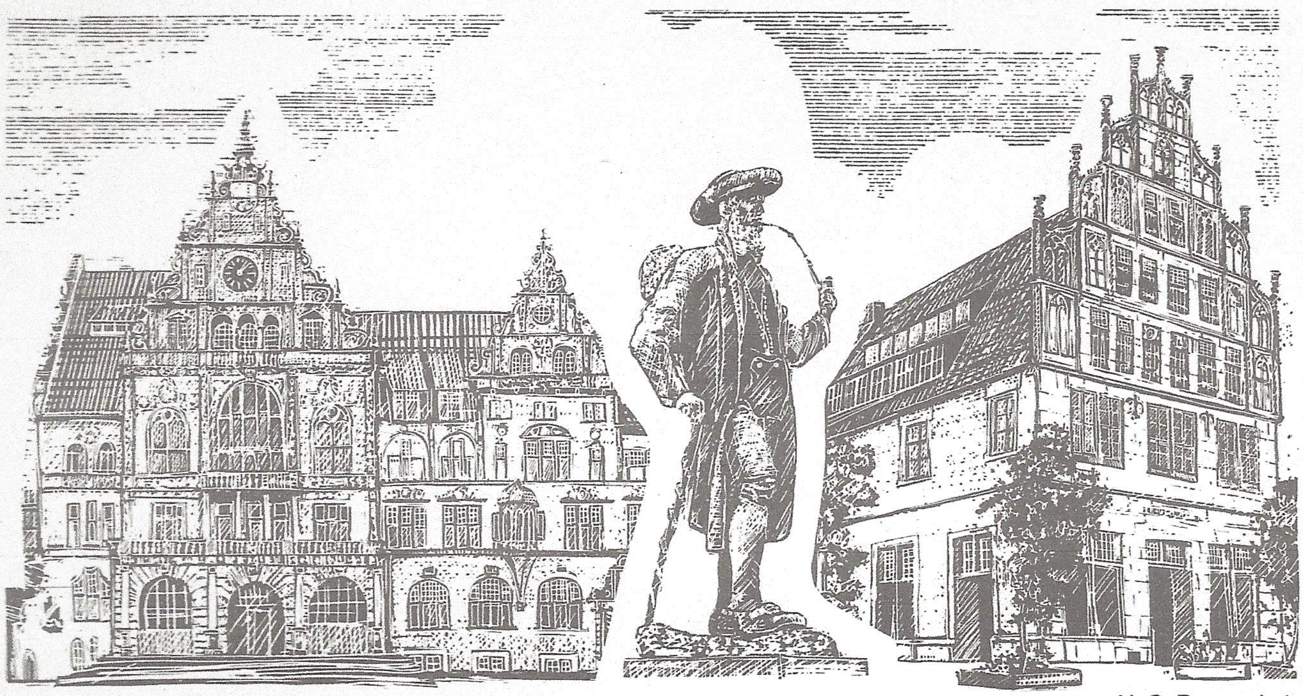

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ABSTRACT

DECOMPOSITION FOR STOCHASTIC DYNAMIC SYSTEMS

Hans W. Gottinger

In this paper we will be applying some of the methods of decomposition as proposed in Part I but we will concentrate on one particularly interesting case of a stochastic dynamic system, e.g. that of a linear-quadratic Gaussian dynamic team. In fact, this is a more complicated analogue of the familiar linear system with a quadratic criterion. Despite partially successful attempts the complete solution of a linear-quadratic Gaussian dynamic team is not yet known - mainly because of a counterexample by Witsenhausen, who showed in this case that the optimal decision rule need not be linear.

DECOMPOSITION FOR STOCHASTIC DYNAMIC SYSTEMS - PART II

1. Introduction

In this paper we will be applying some of the methods of decomposition as proposed in Part I [9] but we will concentrate on one particularly interesting case of a stochastic dynamic system, e.g. that of a linear-quadratic Gaussian dynamic team. In fact, this is a more complicated analogue of the familiar linear system with a quadratic criterion as discussed in Falb [5] . A linear quadratic Gaussian dynamic team is a dynamic or sequential team (see [7]), linear in the information structures of its members, quadratic in its payoffs and nature's variables distributed according to a Gaussian distribution. It is an obvious extension of a static team of that sort introduced by J. Marschak and R. Radner [11] in which it is shown that the optimal decision rule in a linear-quadratic Gaussian static team is linear. As pointed out in the previous paper [9] - despite partially successful attempts in [4, 10] - the complete solution of a linear-quadratic Gaussian dynamic team is not yet known - mainly because of a counterexample by Witsenhausen [18] , who showed in this case that the optimal decision rule need not be linear.

Since the report of this 'counter-intuitive' result some effort has been spent in searching for information structures that imply linearity of the optimal decision rule. Ho and Chu [10] have found that information structures with the <u>sequential nesting property</u> constitute a case where linear optimal decision rules do exist, however, by showing (in the proof) the equivalence with the corresponding static team their result does not constitute a general solution.

Even if the general solution is found the computational effort involved would be beyond any reasonable bound. So from the point of view of the design of a system it might be preferable to adopt a centralized decision-making system. However, if one is forced to cope with institutional or physical constraints then for computational purposes one has to structure the team decision problem in terms of hierarchical computation. In fact, to put it in a different way, computational considerations require to decompose a complex (unstructured) team decision problem in a hierarchical fashion.

Such a hierarchical structuring is of practical importance for team-like organizational forms like a macro-economic control design proposed by D. McFadden [12] and also for large-scale economic planning systems [15] .

The philosophy behind hierarchical decomposition for large-scale (complex) systems has been convincingly argued by H. Simon [16] . In much of the literature on the economic theory of organization it is implicity asserted that the computational requirements of large-scale systems suggest decentralized organizational forms. The availability of high-speed digital computers has imposed a centralizing tendency on many organizational forms, e.g. air-traffic control, air-line reservations and inventory control systems, also 'internalization' of 'externalities' in large-scale economic systems has similar effects. However, these systems which could have been modeled as team decision problems have been restructured in a hierarchical design, and it can be shown here that both off-line and on-line computations will be decreasing with a slight sacrifice in optimality. Of course, decentralized design structures have also been used in large scale mathematical programming but apart from other peculiarities they mostly pertain to static organizational forms (see [6]), except for [17] . The relation of decomposition theory to hierarchical designs is briefly discussed in [14] .

In the next section, we formulate the linear-quadratic Gaussian problem and decompose it into two levels. The equations needed by the lower level controllers and the coordinator are given in Section 3. The lower level problem with a linear term in the cost functional is solved in Section 4. In Section 5 the corresponding higher level problem is solved.

2. Statement of the Problem

Consider the linear dynamic system

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \quad i=1,\ldots,N \qquad (2.1)$$

$$\underline{v}_i(k) = \sum_{j\neq i} \underline{A}_{ij}\underline{x}_j(k) \qquad (2.2)$$

The cost functional is quadratic.

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)\} \qquad (2.3)$$

where $\underline{F}_i$, $\underline{Q}_i$, $\underline{R}_i$ are positive definite matrices.

The measurements are given by

$$\underline{y}_i(k) = \underline{C}_i\underline{x}_i(k) + \underline{\theta}_i(k) \quad i=1,\ldots,N \qquad (3.4)$$

Each controller is allowed only to use his past measurements to find the controls, i.e.,

$$\underline{u}_i(k) = \underline{\gamma}_i^{k}(Y_i(k),\ U_i(k-1)) \qquad (2.5)$$

where

$$Y_i(k) = \{\underline{y}_i(0),\ldots,\underline{y}_i(k)\} \qquad (2.6)$$

$$U_i(k) = \{\underline{u}_i(0),\ldots,\underline{u}_i(k)\} \qquad (2.7)$$

It is required to find optimal control strategies $\underline{\gamma}_i^{k}$ such that J is minimized.

$\underline{\xi}_i(k)$, k=0,...,T-1 are independent Gaussian variables with zero mean and covariance $\underline{\Xi}_i(k)$.

$\underline{\theta}_i(k)$, k=0,...,T-1 are independent Gaussian variables with zero mean and covariance $\underline{\Theta}_i(k)$.

$\underline{x}_i(0)$ is Gaussian with mean $\bar{\underline{x}}_i(0)$ and covariance $\underline{\Sigma}_i(0)$.

$\underline{\xi}_i(k)$, $\underline{\theta}_j(k)$, $\underline{x}_h(0)$, i, j, h = 1,...,N are all mutually independent.

The matrices $\underline{A}_{ii}$, $\underline{A}_{ij}$, $\underline{B}_i$, $\underline{C}_i$, $\underline{Q}_i$, $\underline{R}_i$ can be time-varying but for simplicity of notation, the dependence on k has been omitted.

The general solution to this problem, assuming no communication of the a posteriori information between the controllers, is not known, although several particular cases have been considered [1, 3]. We propose to solve this problem using the approach suggested in the previous paper by defining a new kind of optimality.

Problem 1:

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k) \qquad (2.8)$$

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)\} \qquad (2.9)$$

$$E\{\underline{v}_i(k) - \sum_{j\neq i} A_{ij}\underline{x}_j(k)\} = 0 \qquad (2.10)$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k),\ U_i(k-1);\ \tilde{I}) \qquad (2.11)$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I}) \qquad (2.12)$$

$\tilde{I}$ consists of the a priori information contained in this model. It is required to find $\underline{\gamma}_i^k$ and $\underline{\eta}_i^k$ such that J is minimized.

Using the results of Section 5 [9] we obtain the following two-level problem.

Lower Level: (Problem 2)

$$\underline{x}_i(k+1) = \underline{A}_{ii}\underline{x}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k) + \underline{\xi}_i(k)$$

$$\underline{u}_i(k) = \underline{\gamma}_i^k(Y_i(k),\ U_i(k-1);\ \tilde{I})$$

$$\underline{v}_i(k) = \underline{\eta}_i^k(\tilde{I})$$

$$\tilde{J}_i = E\{\underline{x}_i'(T)\underline{F}_i\underline{x}_i(T) + \sum_{k=0}^{T-1} \underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) + \underline{u}'(k)\underline{R}_i\underline{u}_i(k) \qquad (2.13)$$

$$+ \underline{p}_i'(k)\underline{v}_i(k) - \underline{\tilde{p}}_i'(k)\underline{x}_i(k)\}$$

$$\underline{\tilde{p}}_i(k) = \sum_{j\neq i} \underline{A}_{ji}'\ \underline{p}_j(k) \qquad (2.14)$$

It is desired to find $\underline{\gamma}_i^k$, $\underline{\eta}_i^k$ to minimize $\tilde{J}_i(\underline{p})$, i=1,...,N.

<u>Higher Level: (Problem 3)</u>

$$\underset{\underline{p}}{\text{Maximize}} \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p}) \qquad (2.15)$$

where $\tilde{J}^*(\underline{p})$ is the optimal cost in Problem 2 for a particular $\underline{p}$.

3. Structure of the Decomposition

In this section we summarize the relevant equations needed by the lower level controllers and the coordinator.

The optimal control of the ith controller is given by

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \bar{\underline{x}}_i(k)) - \underline{E}_i(k+1)\ \underline{p}_i(k) \qquad (3.1)$$

The gain matrices are given by:

$$\underline{D}_i(k+1) = \underline{T}_i^{-1}(k+1)\ \underline{B}_i'\underline{K}_i(k+1)\ \underline{A}_{ii} \qquad (3.2)$$

$$\underline{T}_i(k+1) = \underline{R}_i + \underline{B}_i'\underline{K}_i(k+1)\ \underline{B}_i \qquad (3.3)$$

$$\underline{K}_i(k) = \underline{Q}_i + \underline{A}_{ii}'\underline{K}_i(k+1)\underline{A}_{ii} - \underline{A}_{ii}'\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}$$

$$\underline{K}_i(T) = \underline{F}_i \qquad (3.4)$$

$$\underline{E}_i(k+1) = \frac{1}{2}\underline{R}_i^{-1}(k)\ \underline{B}_i'(k) \qquad (3.5)$$

$$\underline{S}_i(k+1) = \underline{K}_i(k+1) - \underline{K}_i(k+1)\ \underline{B}_i\underline{T}_i^{-1}(k+1)\ \underline{B}_i'\underline{K}_i(k+1) \qquad (3.6)$$

The estimates $\hat{\underline{x}}_i(k)$ and $\bar{\underline{x}}_i(k)$ are generated as follows.

$$\hat{\underline{x}}_i(k+1) \triangleq E\{\underline{x}_i(k+1)|Y_i(k+1),U_i(k)\}$$

$$= \underline{A}_{ii}\hat{\underline{x}}_i(k)+\underline{v}_i^*(k)+\underline{B}_i\underline{u}_i^*(k)+\underline{G}_i(k+1)\,[\underline{y}_i(k+1)-\underline{C}_i(\underline{A}_{ii}\hat{\underline{x}}_i(k)+\underline{v}_i^*(k)+\underline{B}_i\underline{u}_i(k))]$$

$$\hat{\underline{x}}_i(0) = \bar{\underline{x}}_i(0) \qquad (3.7)$$

where

$$\underline{G}_i(k+1) = \underline{\Sigma}_i(k+1|k)\underline{C}_i'\,[\underline{C}_i\underline{\Sigma}_i(k+1|k)\underline{C}_i' + \underline{\Theta}_i(k+1)]^{-1} \qquad (3.8)$$

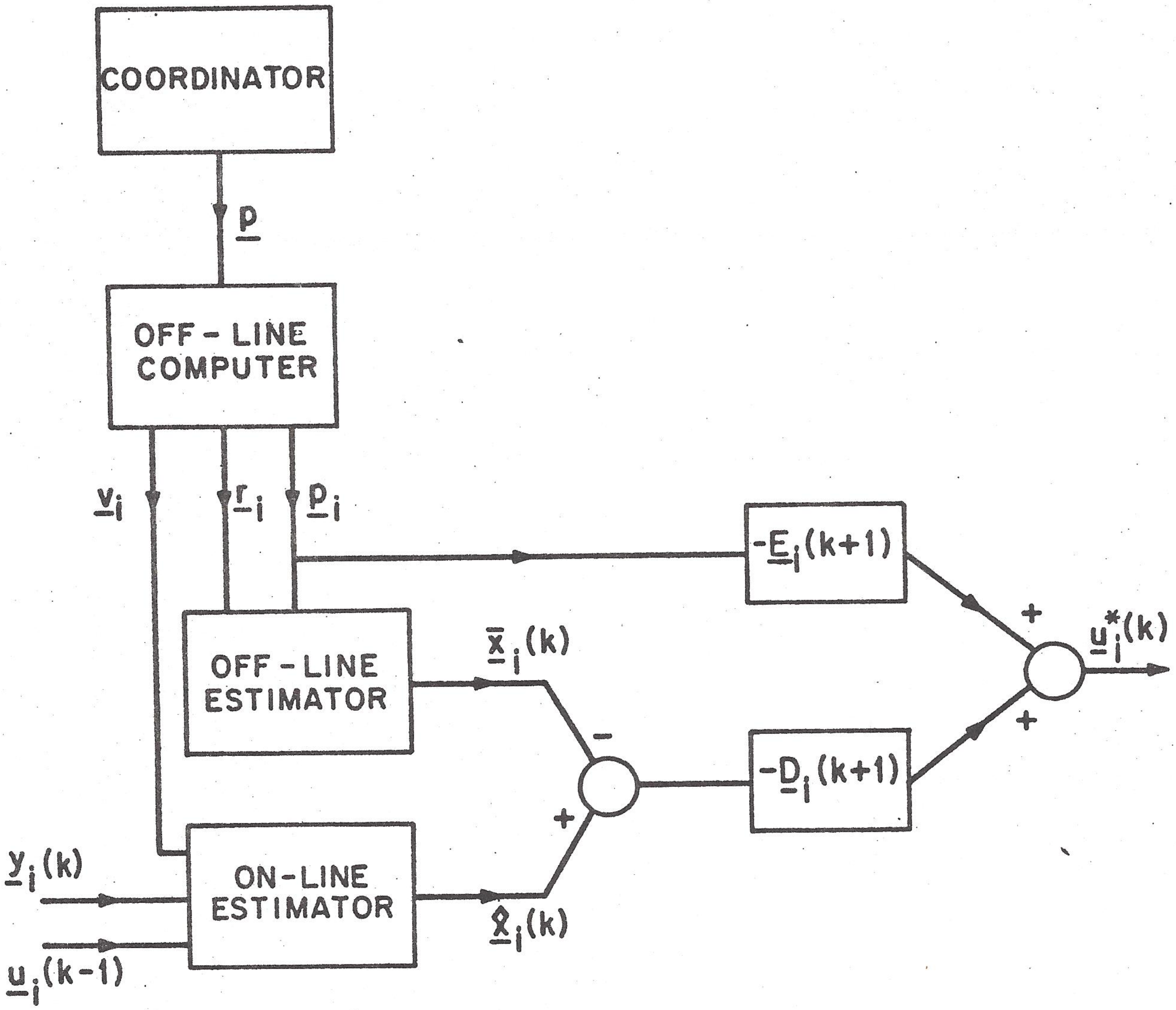

Fig. 1 Structure of Control for i^{TH} Controller

$$\underline{\Sigma}_i(k+1|k) = \underline{\Xi}_i(k)+\underline{A}_{ii}[\underline{\Sigma}_i(k|k-1)-\underline{\Sigma}_i(k|k-1)\underline{C}_i'(\underline{C}_i\underline{\Sigma}_i(k|k-1)\underline{C}_i'+\underline{\Theta}_i(k))^{-1}$$

$$\underline{C}_i\Sigma_i(k|k-1)]\underline{A}_{ii}'$$

$$\underline{\Sigma}_i(0|-1) = \underline{\Sigma}_i(0) \qquad (3.9)$$

$$\overline{\underline{x}}_i(k+1) = E\{\underline{x}_i(k+1)\}$$

$$= -\underline{K}_i^{-1}(k+1)[\underline{r}_i(k+1) + \frac{1}{2}\underline{p}_i(k)] \qquad (3.10)$$

$\underline{v}_i^*(k)$ and $\underline{r}_i(k)$ are given by

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\overline{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) - \frac{1}{2}\underline{S}_i^{-1}(1)\underline{p}_i(0) \qquad (3.11)$$

$$\underline{v}_i^*(k) = \underline{A}_{ii}\underline{K}_i^{-1}(k)[\underline{r}_i(k) + \frac{1}{2}\underline{p}_i(k-1)] - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1)$$

$$- \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \qquad k=1,\ldots,T-1 \qquad (3.12)$$

$$\underline{r}_i(0) = -\frac{1}{2}\tilde{\underline{p}}_i(0) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(0) - \underline{A}_{ii}'\underline{S}_i(1)\underline{A}_{ii}\overline{\underline{x}}_i(0) \qquad (3.13)$$

$$\underline{Q}_i\underline{K}_i^{-1}(k)\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(k) + \frac{1}{2}\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)\underline{p}_i(k-1)$$

$$k=1,\ldots,T-1 \qquad (3.14)$$

$$\underline{r}_i(T) = \underline{0} \qquad (3.15)$$

The structure of the control mechanism is illustrated in Fig. 1. The gain matrices can all be computed off-line, along with $\underline{r}_i(k)$ and $\underline{v}_i^*(k)$, which depend on $\underline{p}(k)$. $\underline{K}_i(k)$ is the solution of the Riccati [5, p. 84] equation assuming the systems are uncoupled and $\underline{D}_i(k)$ is the optimal gain matrix for each of the uncoupled deterministic optimal problems.

$\bar{x}_i(k)$ is the unconditional mean of $x_i(k)$ by the ith controller given only his a priori information. It can be computed off-line given $r_i(k)$ and $p_i(k)$.

$\hat{x}_i(k)$ is the best estimate of $x_i(k)$ given the measurements of the ith controller and his a priori information. It is generated using $v_i^*(k)$ calculated off-line and the on-line measurements $u_i^*(k)$ and $y_i(k)$.

The coordinator finds the optimal $p^*(k)$'s by solving the following deterministic two-point boundary value problem

$$\bar{x}(k+1) = A\,\bar{x}(k) - \frac{1}{2} B\, R^{-1} B'\lambda(k+1) \qquad (3.16)$$

$$\lambda(k) = A'\lambda(k+1) + 2Q\,\bar{x}(k) \qquad (3.17)$$

$$\bar{x}(0) \text{ given}$$

$$\lambda(T) = 2F\,x(T) \qquad (3.18)$$

$$p^*(k) = -\lambda(k+1) \qquad k=0,\dots,T-1 \qquad (3.19)$$

The matrices A, B and Q are as defined in Section 5. R and F are given by

$$R \triangleq \begin{bmatrix} R_1 & 0 & \cdots & \\ 0 & R_2 & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & R_N \end{bmatrix} \qquad F \triangleq \begin{bmatrix} F_1 & 0 & 0 \cdots & \\ 0 & F_2 & \cdots & \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & F_N \end{bmatrix}$$

Alternatively, $p^*(k)$ can be expressed as follows.

$$p^*(k) = -2K(k+1)\bar{x}(k+1) \qquad (3.20)$$

where

$$\bar{x}(k+1) = (A - B\;T^{-1}(k+1)B'K(k+1)A)\bar{x}(k)$$

$$\bar{x}(0) \text{ given} \qquad (3.21)$$

$$\underline{K}(k) = \underline{Q} + \underline{A}'\underline{K}(k+1)\,[\underline{I}-\underline{B}\;\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)]\underline{A}$$

$$\underline{K}(T) = \underline{F} \qquad (3.22)$$

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \qquad (3.23)$$

4. Solution of the Lower Level Problem

Since each controller knows the structure of his system as defined in Problem 2 we shall not include the a priori information in specifying the information structure of the controller. Thus $\underline{u}_i(k)$ would depend on $Y_i(k)$ and $U_i(k-1)$ while $\underline{v}_i(k)$ is allowed to depend on the a priori information only.

The problem as stated has a nonquadratic cost functional and controls which depend on different information sets. However, the information of $\underline{v}_i(k)$ consists of a priori information only and thus is included in that of $\underline{u}_i(k)$. This makes things easier than the general dynamic team problem and the following theorem can be used.

Theorem 4.1:

Consider the system

$$\underline{x}(k+1) = \underline{f}(\underline{x}(k), \underline{u}(k), \underline{v}(k), \underline{\xi}(k)) \qquad (4.1)$$

$$\underline{u}(k) = \underline{\gamma}^k(Y(k)) \qquad (4.2)$$

$$\underline{v}(k) = \underline{\eta}^k(Z(k)) \qquad (4.3)$$

$$Z(k) \subset Y(k) \qquad (4.4)$$

$\underline{\xi}(k)$ is a white noise process driving the system and $Y(k)$, $Z(k)$ are information available to the controller.

$$Y(k) = \{\underline{y}(0), \ldots, \underline{y}(k);\ \underline{u}(0), \ldots, \underline{u}(k-1);\ \underline{v}(0), \ldots, \underline{v}(k-1)\} \qquad (4.5)$$

$$\underline{y}(k) = \underline{h}(\underline{x}(k), \underline{\theta}(k)) \qquad (4.6)$$

$\underline{\theta}(k)$ is a white noise process [13,Ch. 3].

$$J = E\{K(\underline{x}(T)) + \sum_{k=0}^{T-1} L(\underline{x}(k),\ \underline{u}(k),\ \underline{v}(k))\} \qquad (4.7)$$

Then the optimal cost is given by

$$E\{V(Y(0),0)\} \qquad (4.8)$$

where V(Y(k),k)) satisfies the functional equation

$$V(Y(k),k) = \min_{\substack{\underline{u}(k)\\ \underline{\eta}^k}} E\{L(\underline{x}(k),\ \underline{u}(k),\ \underline{v}(k)) + V(Y(k+1),k+1)\,|\,Y(k)\} \qquad (4.9)$$

$$V(Y(T),T) = E\{K(x(T))\,|\,Y(T)\} \qquad (4.10)$$

<u>Proof:</u>

Define

$$V(Y(k),k) = \min_{\substack{\underline{u}(k),\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{k},\underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}}} E\{\sum_{t=k}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|\,Y(k)\}$$

$$= \min_{\substack{\underline{u}(k)\\ \underline{\eta}^k}} \{E\{L(\underline{x}(k),\underline{u}(k),\underline{v}(k))\,|\,Y(k)\}$$

$$+ \min_{\substack{\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|\,Y(k)\}\} \qquad (4.11)$$

Note that the minimization is done with respect to $\underline{u}(k)$, and the control strategies $\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}$, $\underline{\eta}^{k},\ldots,\underline{\eta}^{T-1}$. The first term in the minimization is separated from the rest because it does not depend on $\underline{\gamma}^{k+1},\ldots,\underline{\gamma}^{T-1}$, $\underline{\eta}^{k+1},\ldots,\underline{\eta}^{T-1}$.

Using Lemma A.3 (in Appendix A) of [8] :

$$\min_{\substack{\underline{\gamma}^{k+1},\dots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{k+1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(\underline{x}(T))\,|Y(k)\}$$

$$= E\{\min_{\substack{\underline{u}(k+1),\underline{\gamma}^{k+2},\dots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{k+1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{t=k+1}^{T-1} L(\underline{x}(t),\underline{u}(t),\underline{v}(t)) + K(x(T))|Y(k+1)\}|Y(k)\}$$

$$= E\{V(Y(k+1),k+1)\,|Y(k)\} \qquad (4.12)$$

From this and equation (4.11) we obtain equation (4.9) and further

$$V(Y(0),0) = \min_{\substack{\underline{u}(0),\underline{\gamma}^{1},\dots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{0},\underline{\eta}^{1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\,|Y(0)\}$$

Again by Lemma A.3

$$\min_{\substack{\underline{\gamma}^{0},\dots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{0},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\}$$

$$= E\{\min_{\substack{\underline{u}(0),\underline{\gamma}^{1},\dots,\underline{\gamma}^{T-1}\\ \underline{\eta}^{0},\underline{\eta}^{1},\dots,\underline{\eta}^{T-1}}} E\{\sum_{k=0}^{T-1} L(\underline{x}(k),\underline{u}(k),\underline{v}(k)) + K(\underline{x}(T))\,|Y(0)\}\}$$

$$= E\{V(T(0),0)\}. \qquad (4.14)$$

We can then apply this theorem to solve the lower-level problem. This will be stated in the following theorem. Since we have a linear system, with Gaussian driving and observation noises, the information Y(k) can be replaced by the sufficient statistics $\hat{\underline{x}}_i(k)$. From now on we would deal with $V_i(\hat{\underline{x}}_i(k),k)$ instead of $V_i(Y_i(k),U_i(k-1),k)$.

Theorem 4.2:

The solution to the lower level problem is given by

$$\underline{u}_i^*(k) = -\underline{D}_i(k+1)(\hat{\underline{x}}_i(k) - \bar{\underline{x}}_i(k)) - \underline{E}_i(k+1)\ \underline{p}_i(k) \qquad (4.15)$$

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\bar{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) - \frac{1}{2}\underline{S}_i^{-1}(1)\underline{p}_i(0) \qquad (4.16)$$

$$\underline{v}_i^*(k) = \underline{A}_{ii}\underline{K}_i^{-1}(k)[\underline{r}_i(k) + \frac{1}{2}\underline{p}_i(k-1)] - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1) - \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \qquad k=1,\ldots,T-1 \qquad (4.17)$$

where $\underline{D}_i(k)$, $\underline{E}_i(k)$, $\hat{\underline{x}}_i(k)$, $\bar{\underline{x}}_i(k)$, $\underline{r}_i(k)$, $\underline{K}_i(k)$ and $\underline{S}_i(k)$ are as given in Section 3. Moreover, the optimal cost is given by $E\{V_i(\hat{\underline{x}}_i(0),0)\}$ where

$$V_i(\hat{\underline{x}}_i(k),k) = \hat{\underline{x}}_i'(k)\underline{K}_i(k)\hat{\underline{x}}_i(k) + 2\underline{r}_i'(k)\hat{\underline{x}}_i(k) + s_i(k) \qquad (4.18)$$

with

$$\begin{aligned} s_i(k) = {} & s_i(k+1) + 2\underline{r}_i'(k+1)\underline{v}_i^*(k) + \underline{v}_i^{*\prime}(k)\underline{K}_i(k+1)\underline{v}_i^*(k) \\ & - [\underline{K}_i(k+1)\underline{v}_i^*(k) + \underline{r}_i(k+1)]'\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{v}_i^*(k) + \underline{r}_i(k+1)] \\ & + \underline{p}_i'(k)\underline{v}_i^*(k) + \mathrm{tr}\underline{Q}_i\underline{\Sigma}_i(k|k) + \mathrm{tr}\underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1)) \end{aligned}$$

$$s_i(T) = \mathrm{tr}\underline{F}_i\underline{\Sigma}_i(T|T) \qquad (4.19)$$

$$\underline{\Sigma}_i(k|k) = \underline{\Sigma}_i(k|k-1) - \underline{\Sigma}_i(k|k-1)\underline{C}_i'[\underline{C}_i\underline{\Sigma}_i(k|k-1)\underline{C}_i' + \underline{\Theta}_i]^{-1}\underline{C}_i\underline{\Sigma}_i(k|k-1) \qquad (4.20)$$

$$\underline{\Sigma}_i(k+1|k) = \underline{A}_{ii}\underline{\Sigma}_i(k|k)\underline{A}_{ii}' + \Xi(k)$$

$$\underline{\Sigma}_i(0|-1) = \underline{\Sigma}_i(0) \qquad (4.21)$$

Proof:

The functional equation corresponding to this problem is

$$V_i(\hat{\underline{x}}_i(k),k) = \underset{\substack{\underline{u}_i(k)\\ \underline{v}_i(k)}}{\text{Min}} \; E\{\underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k)$$

$$+ \underline{p}_i'(k)\underline{v}_i(k) + V_i(\hat{\underline{x}}_i(k+1),k+1)\,|\,\hat{\underline{x}}_i(k)\} \qquad (4.22)$$

where $\underline{v}_i(k)$ is to be independent of any a posteriori information.

If we let $V_i(\hat{\underline{x}}_i(k),k)$ to be of the form given by equation (4.18), the right-hand side of (4.22) becomes

$$E\{\underline{x}_i'(k)\underline{Q}_i\underline{x}_i(k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$+ \hat{\underline{x}}_i'(k+1)\underline{K}_i(k+1)\hat{\underline{x}}_i(k+1) + 2\underline{r}_i'(k+1)\hat{\underline{x}}_i(k+1) + s_i(k+1)\,|\,\hat{\underline{x}}_i(k)\}$$

$$= \hat{\underline{x}}_i'(k)\underline{Q}_i\hat{\underline{x}}_i(k) + \mathrm{tr}\underline{Q}_i\underline{\Sigma}_i(k|k) - \tilde{\underline{p}}_i'(k)\underline{x}_i(k) + \underline{u}_i'(k)\underline{R}_i\underline{u}_i(k) + \underline{p}_i'(k)\underline{v}_i(k)$$

$$+ [\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]'\underline{K}_i(k+1)[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]$$

$$+ \mathrm{tr}\underline{K}_i(k+1)\underline{\Sigma}_i(k+1|k) + 2\underline{r}_i'(k+1)[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)] + s_i(k+1)$$

$$- \mathrm{tr}\underline{K}_i(k+1)\underline{\Sigma}_i(k+1|k+1) \qquad (4.23)$$

where we have used the fact that

$$E\{\hat{\underline{x}}_i'(k+1)\underline{K}_i(k+1)\hat{\underline{x}}_i(k+1)\,|\,\hat{\underline{x}}_i(k)\} = [\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)]'\underline{K}_i(k+1)$$

$$[\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{v}_i(k) + \underline{B}_i\underline{u}_i(k)] + \mathrm{tr}\underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1)) \qquad (4.24)$$

Given $\underline{v}_i(k)$, minimizing (4.4.23) with respect to $\underline{u}_i(k)$ gives

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}\hat{\underline{x}}_i(k) + \underline{K}_i(k+1)\underline{v}_i(k) + \underline{r}_i(k+1)] \qquad (4.25)$$

Denote (4.23) with $\underline{u}_i^*$ substituted in by $\underline{W}_i(\hat{\underline{x}}_i(k),k)$. To minimize with respect to $\underline{v}_i(k)$ we minimize $E\{\underline{W}_i(\hat{\underline{x}}_i(k),k\}$. This gives

$$\underline{p}_i(k) + 2\underline{K}_i(k+1)[\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{B}_i\overline{\underline{u}}_i^*(k)] + 2\underline{K}_i(k+1)\underline{v}_i^*(k) + 2\underline{r}_i(k+1) = 0 \qquad (4.26)$$

where

$$\overline{\underline{u}}_i^*(k) = E\{\underline{u}_i^*(k)\}$$

$$= -\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{K}_i(k+1)\underline{v}_i(k) + \underline{r}_i(k+1)] \qquad (4.27)$$

Substituting equation 4.27 into equation 4.26 we have

$$[\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i']\underline{K}_i(k+1)\underline{v}_i^*(k)$$

$$= -[\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'][\underline{K}_i(k+1)\underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{r}_i(k+1)] - \frac{1}{2}\underline{p}_i(k) \qquad (4.28)$$

Since $\underline{K}_i(k+1)$ is invertible (see Appendix)

$$\underline{S}_i(k+1) = \underline{K}_i(k+1) - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)$$

$$= [\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']^{-1} \qquad (4.29)$$

$[\underline{I}-\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'] = \underline{S}_i(k+1)\underline{K}_i^{-1}(k+1)$ is then invertible. Thus

$$\underline{v}_i^*(k) = -\underline{A}_{ii}\overline{\underline{x}}_i(k) - \underline{K}_i^{-1}(k+1)\underline{r}_i(k+1) - \frac{1}{2}\underline{S}_i^{-1}(k+1)\underline{p}_i(k) \qquad (4.30)$$

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'[\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k))$$

$$-\frac{1}{2}(\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1}\underline{p}_i(k)] \qquad (4.31)$$

It can be shown that (see Appendix)

$$\underline{T}_i^{-1}(k+1)\underline{B}_i'(\underline{I} - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1} = \underline{R}_i^{-1}\underline{B}_i' \qquad (4.32)$$

Thus

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k) - \overline{\underline{x}}_i(k)) + \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{p}_i(k) \qquad (4.33)$$

By substitution into equation 4.22 and identifying the terms quadratic in $\hat{\underline{x}}_i(k)$, linear in $\hat{\underline{x}}_i(k)$ and independent of $\hat{\underline{x}}_i(k)$ we obtain equations for $\underline{K}_i(k)$, $S_i(k)$ as well as

$$\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) + \underline{A}_{ii}[\underline{I}_i - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i']\underline{r}_i(k+1)$$
$$+ \underline{A}_{ii}'\underline{S}_i(k+1)\underline{v}_i^*(k)$$
$$\underline{r}_i(T) = \underline{0} \qquad (4.34)$$

To find the optimal controls $\underline{u}_i^*(k)$ and the optimal "estimates" $\underline{v}_i^*(k)$, a two-point boundary value problem has to be solved. This involves equations 4.30, 4.34, and the following equation

$$\overline{\underline{x}}_i(k+1) = \underline{A}_{ii}\overline{\underline{x}}_i(k) + \underline{v}_i^*(k) + \underline{B}_i\overline{\underline{u}}_i^*(k) \qquad (4.35)$$

$$\overline{\underline{x}}_i(0) \text{ given}$$

From equation 4.33

$$\overline{\underline{u}}_i^*(k) = \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{p}_i(k) \qquad (4.36)$$

Substitution of (4.30) and (4.36) into (4.35) yields

$$\overline{\underline{x}}_i(k+1) = -\underline{K}_i^{-1}(k+1)[\underline{r}_i(k+1) + \frac{1}{2}\underline{p}_i(k)] \qquad (4.37)$$

From these we obtain equations 4.16 and 4.17. Substitution of (4.16) into (4.34) yields (3.13). Substitution of (4.17) into (4.34) gives

$$[\underline{I}_i - \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)]\underline{r}_i(k) = -\frac{1}{2}\tilde{\underline{p}}_i(k) - \frac{1}{2}\underline{A}_{ii}'\underline{p}_i(k)$$
$$+ \frac{1}{2}\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k)\ \underline{p}_i(k-1) \qquad (4.38)$$

Since from the Riccati equation

$$\underline{I}_i - \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) = \underline{Q}_i\underline{K}_i^{-1}(k) \qquad (4.39)$$

we obtain equation 3.14. Essentially, the two-point boundary value problem is uncoupled and becomes a single equation in $\underline{r}_i(k)$. $\underline{r}_i(k)$ is uniquely defined when $\underline{Q}_i$ is positive definite. This is a sufficient condition for the positive definiteness of $\underline{K}_i(k)$, k=0,...,T-1. Q.E.D.

The control $\underline{u}_i^*(k)$ which is actually applied by the ith controller consists of two parts: a closed loop part which depends on the measurements and an open loop part which does not. The closed loop part can be written to depend on the difference between the a priori and the a posteriori estimates of the ith controller about the state of the ith subsystem. It looks like the solution of a tracking problem with $\bar{\underline{x}}_i(k)$ as the reference state. In fact, the optimal cost to go $V_i(\hat{\underline{x}}_i(k),k)$ has a form similar to that of the tracking problem. The open loop part depends only on $\underline{p}$, the coordinating signals received from the higher level. When the a priori and a posteriori estimates of the local controllers are the same, as in the case of no measurements, the closed loop part disappears and only the open loop control remains. In the next section we will find out what the open loop part really is.

5. Solution of the Higher Level Problem

The higher level problem is choosing the optimal $\underline{p}^*$ to maximize

$$\tilde{J}^*(\underline{p}) = \sum_{i=1}^{N} \tilde{J}_i^*(\underline{p})$$

From Section 4,

$$\tilde{J}_i^*(\underline{p}) = \overline{\underline{x}}_i'(0)\underline{K}_i(0)\overline{\underline{x}}_i(0) + 2\underline{r}_i'(0)\overline{\underline{x}}_i(0) + s_i(0) \qquad (5.1)$$

where $\underline{r}_i(0)$ satisfies equation 3.13 and $s_i(0)$ is given by equation 4.19.

Since $\overline{\underline{x}}_i'(0)\underline{K}_i(0)\overline{\underline{x}}_i(0)$ is independent of $\underline{p}$, the higher level problem is

$$\underset{\underline{p}}{\text{Max}} \sum_{i=1}^{N} 2\underline{r}_i'(0)\overline{\underline{x}}_i(0) + s_i(0) \qquad (5.2)$$

Let

$$\underline{r}(k) \triangleq \begin{bmatrix} \underline{r}_1(k) \\ \cdot \\ \cdot \\ \cdot \\ \underline{r}_N(k) \end{bmatrix} \qquad \underline{Q} \triangleq \begin{bmatrix} \underline{Q}_1 & \underline{0} & \cdots \\ \underline{0} & \underline{Q}_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \underline{Q}_N \end{bmatrix}$$

$$\tilde{\underline{K}}(k) \triangleq \begin{bmatrix} \underline{K}_1(k) & \underline{0} & \cdots \\ \underline{0} & \underline{K}_2(k) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \underline{K}_N(k) \end{bmatrix} \qquad \underline{A} \triangleq \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} & \cdots \\ \underline{A}_{21} & \underline{A}_{22} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \underline{A}_{NN} \end{bmatrix} \qquad (5.3)$$

$$\underline{B} \triangleq \begin{bmatrix} \underline{B}_1 & \underline{0} & \cdots \\ \underline{0} & \underline{B}_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \underline{B}_N \end{bmatrix} \qquad \tilde{\underline{S}}(k) \triangleq \begin{bmatrix} \underline{S}_1(k) & \underline{0} & \cdots \\ \underline{0} & \underline{S}_2(k) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \underline{S}_N(k) \end{bmatrix}$$

Then equations 3.13 and 3.14 become

$$\underline{r}(0) = -\frac{1}{2}\underline{A}'\underline{p}(0) - (\underline{I} - \underline{Q}\,\tilde{\underline{K}}^{-1}(0))\tilde{\underline{K}}(0)\overline{\underline{x}}(0) \qquad (5.4)$$

$$\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) = -\frac{1}{2}\underline{A}'\underline{p}(k) + \frac{1}{2}(\underline{I} - \underline{Q}\,\underline{\tilde{K}}^{-1}(k))\underline{p}(k-1) \qquad (5.5)$$

$$k=1,\ldots,T-1$$

$$s_i(0) = \sum_{k=0}^{T-1}(s_i(k) - s_i(k+1)) + s_i(T)$$

$$= \underline{\bar{x}}_i'(0)\underline{A}_{ii}'\underline{S}_i(1)\underline{A}_{ii}\underline{\bar{x}}_i(0) + \mathrm{tr}\,\underline{K}_i(T)\underline{\Sigma}_i(T|T)$$

$$+ \sum_{k=0}^{T-1}\{\mathrm{tr}\,\underline{Q}_i\underline{\Sigma}_i(k|k) + \mathrm{tr}\,\underline{K}_i(k+1)(\underline{\Sigma}_i(k+1|k) - \underline{\Sigma}_i(k+1|k+1))\}$$

$$+ \sum_{k=1}^{T-1}\{\underline{r}_i'(k)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{K}_i^{-1}(k)]\underline{r}_i(k)$$

$$+ \underline{p}_i'(k-1)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{K}_i^{-1}(k)]\underline{r}_i(k)$$

$$+ \frac{1}{4}\underline{p}_i'(k-1)[\underline{K}_i^{-1}(k)\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii}\underline{K}_i^{-1}(k) - \underline{S}_i^{-1}(k)]\underline{p}_i(k-1)\}$$

$$- \underline{r}_i'(T)\underline{K}_i^{-1}(T)\underline{r}_i(T) - \underline{p}_i'(T-1)\underline{K}_i^{-1}(T)\underline{r}_i(T)$$

$$- \frac{1}{4}\underline{p}_i'(T-1)\underline{S}_i^{-1}(T)\underline{p}_i(T-1) \qquad (5.6)$$

The terms involving $\underline{\Sigma}_i(k|k)$ and $\underline{\Sigma}_i(k+1|k)$ are independent of $\underline{p}$. Thus the quantity to be maximized is

$$2\underline{r}(0)\underline{\bar{x}}(0) + \sum_{k=1}^{T-1} - \underline{r}'(k)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k) - \underline{p}'(k-1)\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k)\underline{r}(k)$$

$$- \frac{1}{4}\underline{p}'(k-1)[\underline{\tilde{K}}^{-1}(k)\underline{Q}\,\underline{\tilde{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}\underline{B}']\,\underline{p}(k-1) - \frac{1}{4}\underline{p}(T-1)\underline{\tilde{S}}^{-1}(T)\underline{p}(T-1)$$

$$(5.7)$$

Redefining

$$\underline{\lambda}(k) = -\underline{p}(k-1) \qquad k=1,\ldots,T$$

we have

$$\text{Max } \underline{\lambda}'(1)\underline{A}\,\bar{\underline{x}}(0) + \sum_{k=1}^{T-1} -\underline{r}'(k)\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k) + \underline{\lambda}'(k)\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k)$$

$$-\frac{1}{4}\underline{\lambda}'(k)[\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k) + \underline{B}\,\underline{R}^{-1}\underline{B}']\underline{\lambda}(k) - \frac{1}{4}\underline{\lambda}'(T)\tilde{\underline{S}}^{-1}(T)\underline{\lambda}(T) \qquad (5.8)$$

with respect to

$\underline{r}(k);\quad k=1,\ldots,T-1$

$\underline{\lambda}(k);\quad k=1,\ldots,T$

such that

$$\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k) = \frac{1}{2}\underline{A}'\underline{\lambda}(k+1) - \frac{1}{2}[\underline{I} - \underline{Q}\,\tilde{\underline{K}}^{-1}(k)]\,\underline{\lambda}(k)$$

$$k=1,\ldots,T-1 \qquad (5.9)$$

<u>Theorem 5.1</u>:

The optimal solution $\underline{\lambda}^*(k)$ to equations 5.8 and 5.9 corresponds to the costates of the deterministic linear regulator problem for the entire system. Minimize

$$\underline{x}'(T)\underline{F}\,\underline{x}(T) + \sum_{k=0}^{T-1} \underline{x}'(k)\,\underline{Q}\,\underline{x}(k) + \underline{u}'(k)\,\underline{R}\,\underline{u}(k) \qquad (5.10)$$

subject to

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) + \underline{B}\,\underline{u}(k)$$

$$\underline{x}(0) = \bar{\underline{x}}(0) \text{ given} \qquad (5.11)$$

<u>Proof</u>:

We form the Lagrangian $H(\underline{\lambda},\underline{r},\underline{\alpha})$ given by

$$H(\underline{\lambda},\underline{r},\underline{\alpha}) = \underline{\lambda}'(1)\underline{A}\,\bar{\underline{x}}(0) + \sum_{k=0}^{T-1} -\underline{r}'(k)\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k)$$

$$+ \underline{\lambda}'(k)\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k) - \frac{1}{4}\underline{\lambda}'(k)[\tilde{\underline{K}}^{-1}(k)\underline{Q}\,\tilde{\underline{K}}^{-1}(k)+\underline{B}\,\underline{R}^{-1}\underline{B}']\,\underline{\lambda}(k)$$

$$- \underline{\alpha}'(k)[\underline{Q}\,\tilde{\underline{K}}^{-1}(k)\underline{r}(k) - \frac{1}{2}\underline{A}'\underline{\lambda}(k+1) + \frac{1}{2}(\underline{I} - \underline{Q}\,\tilde{\underline{K}}^{-1}(k))\underline{\lambda}(k)]$$

$$- \frac{1}{4}\underline{\lambda}'(T)\tilde{\underline{S}}^{-1}(T)\underline{\lambda}(T) \qquad (5.12)$$

Using the necessary conditions for optimality we obtain

$$\frac{\partial H}{\partial \lambda(1)} = A\,\bar{x}(0) + \tilde{K}^{-1}(1)Q\,\tilde{K}^{-1}(1)r(1) - \frac{1}{2}[\tilde{K}^{-1}(1)Q\,\tilde{K}^{-1}(1) + B\,R^{-1}B']\,\lambda(1)$$
$$- \frac{1}{2}[I - Q\,\tilde{K}^{-1}(1)]'\,\alpha(1) = 0 \qquad (5.13)$$

$$\frac{\partial H}{\partial \lambda(k)} = \tilde{K}^{-1}(k)Q\,\tilde{K}^{-1}(k)r(k) - \frac{1}{2}[\tilde{K}^{-1}(k)Q\,\tilde{K}^{-1}(k) + B\,R^{-1}B']\lambda(k) + \frac{1}{2}A\,\alpha(k-1)$$
$$- \frac{1}{2}[I - Q\,\tilde{K}^{-1}(k)]'\alpha(k) = 0 \qquad k=2,\dots,T-1 \qquad (5.14)$$

$$\frac{\partial H}{\partial \lambda(T)} = \frac{1}{2}A\,\alpha(T-1) - \frac{1}{2}\tilde{S}^{-1}(T)\,\lambda(T) = 0 \qquad (5.15)$$

$$\frac{\partial H}{\partial r(k)} = -2\tilde{K}^{-1}(k)Q\,\tilde{K}^{-1}(k)r(k) + \tilde{K}^{-1}(k)Q\,\tilde{K}^{-1}(k)\lambda(k) - \tilde{K}^{-1}(k)Q\,\alpha(k) = 0$$
$$k=1,\dots,T-1 \qquad (5.16)$$

$$\frac{\partial H}{\partial \alpha(k)} = -\,Q\,\tilde{K}^{-1}(k)r(k) + \frac{1}{2}A'\,\lambda(k+1) - \frac{1}{2}[I - Q\,\tilde{K}^{-1}(k)]\,\lambda(k) = 0$$
$$k=1,\dots,T-1 \qquad (6.17)$$

From (5.13) and (5.16), we obtain

$$\alpha(1) = 2\,A\,\bar{x}(0) - B\,R^{-1}B'\lambda(1) \qquad (5.18)$$

From (5.14) and (5.16), we obtain

$$\alpha(k) = A\,\alpha(k-1) - \frac{1}{2}B\,R^{-1}B'\lambda(k) \quad k=2,\dots,T-1 \qquad (5.19)$$

Since

$$\tilde{S}^{-1}(T) = F^{-1} + B\,R^{-1}B' \qquad (5.20)$$

equation 5.15 becomes

$$(F^{-1} + B\,R^{-1}B')\lambda(T) = A\,\alpha(T-1)$$
$$\lambda(T) = F[A\,\alpha(T-1) - B\,R^{-1}B'\lambda(T)] \qquad (5.21)$$

From (5.17) and (5.16), we obtain

$$\underline{\lambda}(k) = \underline{A}'\underline{\lambda}(k+1) + \underline{Q}\,\underline{\alpha}(k) \qquad (5.22)$$

Let

$$\underline{\alpha}(k) \triangleq 2\,\underline{x}(k)$$

Then we have

$$\underline{x}(k+1) = \underline{A}\,\underline{x}(k) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}(k+1) \quad k=0,\dots,T-1 \qquad (5.23)$$

$$\underline{\lambda}(k) = \underline{A}'\underline{\lambda}(k+1) + 2\,\underline{Q}\,\underline{x}(k) \qquad (5.24)$$

$$\underline{x}(0) = \overline{\underline{x}}(0)$$

$$\underline{\lambda}(T) = 2\,\underline{F}\,\underline{x}(T) \qquad (5.25)$$

This is the two-point boundary value problem associated with the optimal control problem (5.10) and (5.11) [2]. Q.E.D.

Since $\underline{\lambda}(k) = 2\,\underline{K}(k)\underline{x}(k)$ where $\underline{K}(k)$ is the solution of the Riccati equation for the whole system

$$\underline{K}(k) = \underline{Q} + \underline{A}'\,\underline{K}(k+1)[\underline{I} - \underline{B}\,\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)]\underline{A}$$

$$\underline{K}(T) = \underline{F} \qquad (5.26)$$

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \qquad (5.27)$$

the optimal control $\underline{u}_i^*$ given the optimal coordinating signal $\underline{p}^*(k)$ is

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\overline{\underline{x}}_i(k)) - \frac{1}{2}\underline{R}_i^{-1}\underline{B}_i'\underline{\lambda}_i^*(k+1)$$

$$= -\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\overline{\underline{x}}_i(k)) - \frac{1}{2}[\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(k+1)]_i \qquad (5.28)$$

where $[\underline{a}]_i$ corresponds to the ith component of vector $\underline{a}$.

We now show how $\bar{\underline{x}}_i(k)$ is related to the solution of the deterministic linear regulator problem of the entire system.

Theorem 5.2:

Given the optimal coordinating parameters, the unconditional estimates $\bar{\underline{x}}(k)$ of the state of the system by the lower level (given by equation (4.35)) are equal to the unconditional estimate of the coordinator, i.e.,

$$\bar{\underline{x}}(k+1) = \underline{A}\,\bar{\underline{x}}(k) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\,\underline{\lambda}^*(k+1)$$

$$\bar{\underline{x}}(0) \text{ given} \qquad (5.29)$$

Proof:

By equation 4.30

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\bar{\underline{x}}_i(0) - \underline{K}_i^{-1}(1)\underline{r}_i(1) + \frac{1}{2}(\underline{K}_i^{-1}(1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i')\underline{\lambda}_i^*(1) \qquad (5.30)$$

By equation 5.16

$$-\underline{K}_i^{-1}(1)\underline{r}_i(1) + \frac{1}{2}\underline{K}_i^{-1}(1)\underline{\lambda}_i^*(1) - \bar{\underline{x}}_i(1) = \underline{0} \qquad (5.31)$$

Thus

$$\underline{v}_i^*(0) = -\underline{A}_{ii}\bar{\underline{x}}_i(0) + \underline{x}_i(1) + \frac{1}{2}\underline{B}_i\underline{R}_i^{-1}\underline{B}_i'\underline{\lambda}_i^*(1)$$

$$= -\underline{A}_{ii}\bar{\underline{x}}_i(0) + \sum_{j=1}^{N}\underline{A}_{ij}\bar{\underline{x}}_j(0)$$

$$= \sum_{j\neq i}\underline{A}_{ij}\bar{\underline{x}}_j(0) \qquad (5.32)$$

We then have

$$\bar{\underline{x}}(1) = \underline{A}\,\bar{\underline{x}}(0) - \frac{1}{2}\underline{B}\,\underline{R}^{-1}\underline{B}'\underline{\lambda}^*(1) \qquad (5.33)$$

By induction, we can easily show that

$$\underline{v}_i^*(k) = \sum_{j\neq i} \underline{A}_{ij}\bar{\underline{x}}_j(k) \qquad (5.34)$$

and hence equation 5.29. Q.E.D.

We have thus verified constraint (2.10). Moreover, we have shown that the unconditional mean (a priori estimate) of the ith controller given the optimal coordinating parameter and the uncoupled subsystem is the same as the a priori estimate obtained by the coordinator. The optimal control $\underline{u}_i^*(k)$ is given by

$$\underline{u}_i^*(k) = -\underline{T}_i^{-1}(k+1)\underline{B}_i\underline{K}_i(k+1)\underline{A}_{ii}(\hat{\underline{x}}_i(k)-\bar{\underline{x}}_i(k)) - [\underline{T}^{-1}(k+1)\underline{B}'\underline{K}(k+1)\underline{A}\,\bar{\underline{x}}(k)]_i \qquad (5.35)$$

where

$$\underline{T}(k+1) = \underline{R} + \underline{B}'\underline{K}(k+1)\underline{B} \qquad (5.36)$$

This optimal control consists of two parts, a closed loop part which has been discussed before and an open loop part. The open loop part is the optimal deterministic control for the whole system assuming no measurements are made. Thus the optimal control $\underline{u}_i^*(k)$ has a deterministic component which takes into account the effect of the coupling and a closed-loop part which utilizes the local information available. The closed loop part resembles the solution of a tracking problem where the a priori estimate by the coordinator is the reference state.

6. Discussion

We have obtained an off-line decomposition of the linear-quadratic-Gaussian problem. It is found that the optimal control strategy consists of two parts: a closed-loop part which can be generated by the lower level controller himself and an open-loop part which depends on the coordinating parameter $\underline{p}$. The closed-loop part consists of the optimal deterministic gain for the ith subsystem acting on the difference of two estimates. The optimal coordinating parameter $\underline{p}$ is essentially the costate corresponding to the optimal deterministic control of the entire system using the mean of $\underline{x}(0)$ as its initial state. Then the open loop part is the optimal deterministic control of the whole system. The scheme of control is simpler than the solution to the optimal dynamic team since it requires less on-line and off-line computation. Compared with the centralized case, when there is communication among all the controllers, it is also simpler since a full dimensional Kalman-Bucy filter has been replaced by N local filters. The decrease in computation and communication is accompanied by a loss in mathematical optimality.

APPENDIX

INVERTIBILITY OF $\underline{K}_i(k)$ AND VERIFICATION OF EQUATION (4.32)

1. <u>Invertibility of $\underline{K}_i(k)$</u>

$$\underline{K}_i(k) = \underline{Q}_i + \underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii} \;), \quad \underline{K}_i(T) = \underline{F}_i > \underline{0} \tag{1}$$

where

$$\underline{S}_i(k+1) = \underline{K}_i(k+1) - \underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1). \tag{2}$$

If $\underline{K}_i(k+1) > \underline{0}$, then $\underline{K}_i^{-1}(k+1)$ exists.

$$\underline{S}_i(k+1) = [\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']^{-1} \tag{3}$$

$$\underline{A}_{ii}'\underline{S}_i(k+1)\underline{A}_{ii} \geq \underline{0} \tag{4}$$

If $\underline{Q}_i > \underline{0}$, then $\underline{K}_i(k) > \underline{0}$.

Therefore by induction $\underline{K}_i(k)$ is invertible.

Remark: $\underline{Q}_i > \underline{0}$ is sufficient but not necessary. If $\underline{A}_{ii} > \underline{0}$ then $\underline{K}_i(k) > \underline{0}$.

2. <u>Verification of Equation (4.32)</u>

$$\underline{T}_i^{-1}(k+1)\underline{B}_i'(\underline{I}-\underline{K}_i(k+1)\underline{B}_i\underline{T}_i^{-1}(k+1)\underline{B}_i')^{-1}$$

$$= \underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)\underline{S}_i^{-1}(k+1)$$

$$= \underline{T}_i^{-1}(k+1)\underline{B}_i'\underline{K}_i(k+1)[\underline{K}_i^{-1}(k+1) + \underline{B}_i\underline{R}_i^{-1}\underline{B}_i']$$

$$= \underline{T}_i^{-1}(k+1)[\underline{R}_i + \underline{B}_i'\underline{K}_i(k+1)\underline{B}_i]\underline{R}_i^{-1}\underline{B}_i'$$

$$= \underline{R}_i^{-1}\underline{B}_i' \tag{5}$$

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